Spectral decimation and its implications
(including the problem session)

Alexander Teplyaev
University of Connecticut

September 27 — October 2, 2015
Casa Matematica Oaxaca – Workshop 15w5083 Spectral Properties of Quasicrystals via Analysis, Dynamics, and Geometric Measure Theory

Banff International Research Station
for Mathematical Innovation and Discovery

Some relevant problems

- Answer the 1988 question of **Bellissard** about the spectrum of the **Sierpinski lattice with a uniform magnetic field** (published in 1992 in *Renormalization group analysis and quasicrystals*)

Some relevant problems

- Answer the 1988 question of **Bellissard** about the spectrum of the **Sierpinski lattice with a uniform magnetic field** (published in 1992 in *Renormalization group analysis and quasicrystals*)
- Try to **approximate the spectrum** of the Fibonacci Schroedinger operator and similar operators **inductively**, repeatedly solving simple algebraic equations. Try to employ interval arithmetic to obtain theoretical bounds (see the talk of Mark Embree for motivation).

Some relevant problems

- Answer the 1988 question of **Bellissard** about the spectrum of the **Sierpinski lattice with a uniform magnetic field** (published in 1992 in *Renormalization group analysis and quasicrystals*)
- Try to **approximate the spectrum** of the Fibonacci Schroedinger operator and similar operators **inductively**, repeatedly solving simple algebraic equations. Try to employ interval arithmetic to obtain theoretical bounds (see the talk of Mark Embree for motivation).
- Conjecture: **the spectrum of the Penrose lattice is absolutely continuous** (see the talk of Siegfried Beckus for useful techniques and the talk of Mark Embree for motivation).

Some relevant problems

- Answer the 1988 question of **Bellissard** about the spectrum of the **Sierpinski lattice with a uniform magnetic field** (published in 1992 in *Renormalization group analysis and quasicrystals*)
- Try to **approximate the spectrum** of the Fibonacci Schroedinger operator and similar operators **inductively**, repeatedly solving simple algebraic equations. Try to employ interval arithmetic to obtain theoretical bounds (see the talk of Mark Embree for motivation).
- Conjecture: **the spectrum of the Penrose lattice is absolutely continuous** (see the talk of Siegfried Beckus for useful techniques and the talk of Mark Embree for motivation).
- Is there a version of **random walks for alternating current electrical networks ?** See the talk of Jake Fillman on Quantum Random Walks; also see preprint `http://arxiv.org/abs/1507.05682` Fractal AC circuits and propagating waves on fractals by Eric Akkermans, Joe P. Chen, Gerald Dunne, Luke G. Rogers, T.; Random walks and electric networks by Doyle and Snell; `http://arxiv.org/abs/1505.03161` by Dual graphs and modified Barlow–Bass resistance estimates for repeated barycentric subdivisions by Antoni Brzoska, Daniel Kelleher, Hugo Panzo, T.

Some relevant problems

- Answer the 1988 question of **Bellissard** about the spectrum of the **Sierpinski lattice with a uniform magnetic field** (published in 1992 in *Renormalization group analysis and quasicrystals*)
- Try to **approximate the spectrum** of the Fibonacci Schroedinger operator and similar operators **inductively**, repeatedly solving simple algebraic equations. Try to employ interval arithmetic to obtain theoretical bounds (see the talk of Mark Embree for motivation).
- Conjecture: **the spectrum of the Penrose lattice is absolutely continuous** (see the talk of Siegfried Beckus for useful techniques and the talk of Mark Embree for motivation).
- Is there a version of **random walks for alternating current electrical networks ?** See the talk of Jake Fillman on Quantum Random Walks; also see preprint `http://arxiv.org/abs/1507.05682` Fractal AC circuits and propagating waves on fractals by Eric Akkermans, Joe P. Chen, Gerald Dunne, Luke G. Rogers, T.; Random walks and electric networks by Doyle and Snell; `http://arxiv.org/abs/1505.03161` by Dual graphs and modified Barlow–Bass resistance estimates for repeated barycentric subdivisions by Antoni Brzoska, Daniel Kelleher, Hugo Panzo, T.
- Try to prove the **conjectures of Strichartz** (published in 2012 in *Spectral asymptotics revisited*) for quasicrystals, Fibonacci Schroedinger operators etc.

3.SIERPINSKY LATTICE IN A MAGNETIC FIELD

3.1 The 2D Sierpinsky Lattice [Alexander(83,84), Rammal(84)]

The Sierpinsky lattice S in 2D is usually constructed according to the fig.1 below. Namely, let e_1,e_2 be two unit vectors making an angle of 60°. Then S is contained in the set Ne_1+Ne_2 . Let then S_k be the subset of points $x\in S$ with $x= me_1+ne_2$ and $0 < m+n \leq 2^k$. S_k is recursively constructed as $S_1 = \{me_1+ne_2 ; 0 \leq m+n \leq 2\}$, $S_{k+1}= S_k\cup\{S_k+2^ke_1\}\cup\{S_k+2^ke_2\}$ for $k \geq 1$, and $S = \cup_{k\geq 1}S_k$.

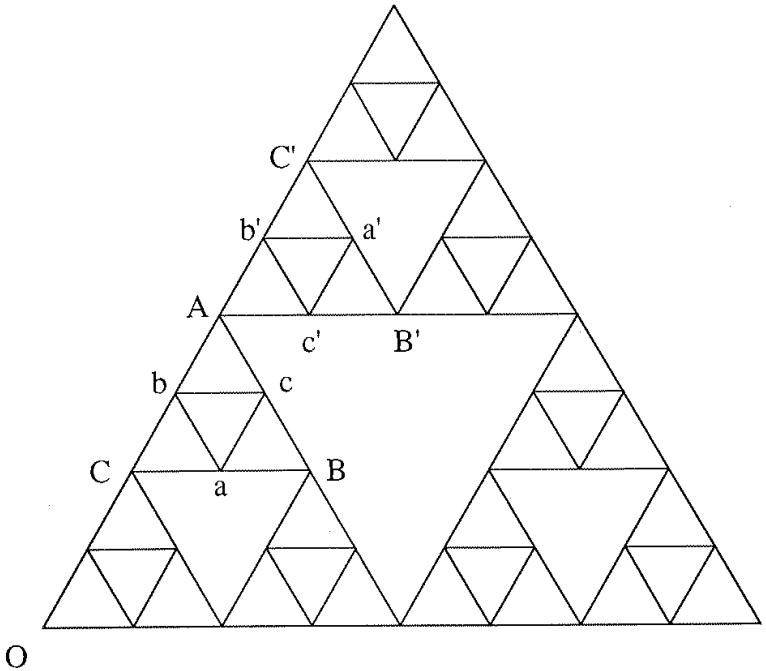


Fig.1- The subset S_3 of the Sierpinsky lattice in 2D -

From this construction it follows that $2S$ is included in S . A site in $2S$ is called "even", the others "odd". Any odd site admits the decomposition $2x+y$ where $x\in S$ and $y\in T=\{e_1,e_2, e_1+e_2\}$. The subsets $T(x) = T+2x$ are called "blocks". If $x\in S$, its nearest neigbours are all points in S within a distance 1 of x .

3.2 The Laplacean on S

The Laplace operators Δ_+ and Δ_- are defined on the Hilbert spaces $l^2(S)$ and $l^2(S\backslash\{0\})$ respectively by:

$$\Delta_+\phi\ (0) = \sqrt{2}\sum_{x';|x'|=1} \phi(x') , \qquad \Delta_+\phi\ (x) = \sqrt{2}\psi(0)+\sum_{x'\neq 0;|x'-x|=1} \phi(x') ,$$

if $|x| =1$, and

$$\Delta_+\phi\ (x) = \sum_{x';|x'-x|=1} \phi(x') , \qquad \text{if } |x| >1,\ \phi\in l^2(S) , \tag{11a}$$

$$\Delta_-\psi\ (x) = \sum_{x';|x'-x|=1} \psi(x') , \qquad x\in S\backslash\{0\} ,\ \psi\in l^2(S\backslash\{0\}) . \tag{11b}$$

Our goal is to compute the spectrum of Δ . In order to do so we will use the scale invariance of the Sierpinsky lattice. The main result is the following [Rammal(84), Bellissard(85)]

Theorem 3: The spectrum of $\Delta_\pm$ is made of two infinite sequences of eigenvalues of infinite multiplicity accumulating on the Julia set of the polynomial $P(z) = z(z-3)$. The first sequence consists of one isolated eigenvalue in each gap of $J(P)$, whereas the other consists of one edge of each gap of $J(P)$. ◊

Proof: Let us introduce the dilation operator D defined by

$$D\psi\ (x) = \psi(2x) \qquad x\in S, \qquad \psi\in l^2(S) \tag{12}$$

It is a partial isometry such that $DD^*=1$. Then we claim that $\Delta_\pm$ are solutions of the following RG equation [Bellissard(85)]:

$$D\{z\mathbf{1}-\Delta\}^{-1}D^* = (z-2)(z+1)/(z+2)\ \{P(z)\mathbf{1}-\Delta\}^{-1} , \qquad P(z) = z(z-3) \tag{13}$$

It follows immediately that if $z\in\rho(\Delta)$ then $z\neq -2$, otherwise the right hand side of (13) would be unbounded. Thus $z=-2$ belongs to the spectrum. Let χ_{-2} be the corresponding eigenprojection. From (13), it follows that $D\,\chi_{-2}\,D^* = 4\{10-\Delta\}^{-1}$. Note that by (11), Δ satisfies the a priori bound $||\Delta|| \leq 4$ so that χ_{-2} is a non zero projection. Now if $z\in\rho(\Delta)$ and $z\neq -1,2$, then (13) shows that $P(z)\in\rho(\Delta)$.

Hence the spectrum is "essentially" P^{-1} -invariant in that there are only two possible exceptions. Thus we expect that the Julia set $J(P)$ of P plays a role. It is easy to check that the convex hull of $J(P)$ is the interval $[-1,4]$. By the Julia-Fatou theorem, it is completely disconnected for its critical point $z=3/2$ satisfies $P(3/2)=9/4\notin[-1,4]$, and does not belong to $J(P)$. In particular, -2 does not belong to $J(P)$, nor does its inverse images $z=1, 2$.

From (13), $z=2$ requires a special treatment. An elementary calculation then shows that if χ_2 is the corresponding eigenprojection, then $D\chi_2 D^*=0$. This means that any eigenvector ψ of Δ with eigenvalue 2 is localized on the odd sites of S . From the eigenvalue equation it follows that $\psi(a)=\psi(b)=\psi(c)$ on each block (see fig.1). Applying the eigenvalue equation at each even site, we can check that on each triangle $T_2(x) = 4x+S_2$ the eigenvector must vanish, showing that $\chi_2 =0$. Thus $2\notin Sp(\Delta)$. On the other hand one can check that each inverse image of 1 is actually an eigenvalue, giving then one isolated eigenvalue in each gap of $J(P)$. In addition, if χ_ζ is the eigenprojection corresponding to the eigenvalue ζ , then

$$D\chi_\zeta D^*=c(\zeta)\chi_{P(\zeta)}\,, \qquad c(\zeta)=(\zeta-2)(\zeta+1)/(\zeta+2)P'(\zeta)\,. \qquad (14)$$

The value $z=-1$ also requires a special treatment. By the same argument if χ_{-1} is the corresponding eigenprojection, then $D\chi_{-1}D^*=0$. Again, any eigenvector ψ of Δ with eigenvalue -1 is localized on the odd sites of S . From the eigenvalue equation it follows that $\psi(a)+\psi(b)+\psi(c)=0$ on each block (see fig.1). But now it is a simple matter of calculation to check that such a vector indeed exists, showing that χ_{-1} is non zero. Thus any inverse image of -1 is also an eigenvalue with eigenprojector again satisfying (14). In this way we get another countable family of eigenvalues given by the gap edges of the Julia set of P .

Both families of eigenvalues actually accumulate on $J(P)$ showing that the spectrum contains the union of $J(P)$ and $\{-2\}\cup\{P^{-n}(1);n\geq 0\}$.

Would the dilation operator be Δ-cyclic, the argument in §2 could be used to conclude that

the rest of the spectrum is actually the entire Julia set. However here, the subspace $\mathcal{H}_\perp$ of vectors ϕ in the Hilbert space for which $D\Delta^n\phi = 0$ for all $n\in N$, is closed and not reduced to zero for it contains the image of the projection χ_{-1} . However, $\mathcal{H}_\perp$ is Δ-invariant, and if $\phi\in\mathcal{H}_\perp$, ϕ vanishes on the even sites. So that $\Delta\phi=\oplus_T\,\Delta_T\phi$ where the sum runs over the set of blocks, and Δ_T represents then restriction of the Laplacean on the block T . Clearly then, Δ_T is a 3x3 matrix with spectrum $\{2,-1\}$. We have seen that 2 is not in the spectrum of Δ whereas -1 is, showing that the restriction of Δ to $\mathcal{H}_\perp$ is exactly the projection χ_{-1} . With this last argument the proof is achieved. ◊

3.2 Effect of a Magnetic Field

The same kind of properties have been studied with the occurence of a uniform magnetic field perpendicular to the plane of the lattice [Alexander(83,84), Rammal(83), Ghez(87)]. The Laplace operator is then modified by means of a phase factor according to :

$$\Delta_-\psi\,(x) = \sum_{x';|x'-x|=1} e^{2\pi i e A(x,x')/h}\,\psi(x')\,, \qquad x\in S\backslash\{0\}\,,\ \psi\in l^2(S\backslash\{0\})\,, \qquad (15)$$

and a similar formula for Δ_+ , where h is the Planck constant, e is the electric charge of the particle involved, and $A(x,x')=\int_x^{x'} dl.A$ where $A=(A_1,A_2)$ is the vector potential defined as a solution of $\partial_1 A_2-\partial_2 A_1=B$. This Laplace operator does depend upon the specific solution chosen for A (choice of a gauge), but changing the gauge results in a unitary equivalent operator having thus the same spectrum. Thus the spectrum of $\Delta_\pm$ depends only upon the magnetic field B , namely upon the flux ϕ in a unit triangle. The normalized flux is $\alpha=e\phi/h$ and is a dimensionless parameter. We then remark that changing α in $\alpha+1$ does not change the spectrun, so that α is defined modulo 1 .

In much the same way as in the previous section, both $\Delta_\pm$ are solution of a RG equation. However, in order to get a closed relation we must introduce two normalized fluxes α, α' the first corresponding to elementary triangles like abc in fig.1, the second to triangles like Cab . The flux per unit cell is then $\alpha+\alpha'$. Then one gets:

$$D\{z\mathbf{1}-\Delta(\alpha,\alpha')\}^{-1}D^* = G(z,\alpha,\alpha')\,\{E\mathbf{1}-\Delta(\beta,\beta')\}^{-1} \qquad (16)$$

where [Ghez(87)]:

$$G(z,\alpha,\alpha') = \{z^3 - 3z - 2(XU+YV)\}/\{S^2 + C^2\}^{1/2}\,,$$

$$E = \{z^4 - 7z^2 - [2(XU+YV)+ 4X]z + 4(1-U)\}/\{S^2 + C^2\}^{1/2}, \tag{17}$$

$$\beta + \beta' = 4(\alpha + \alpha') \qquad \beta - \beta' = 2(\alpha - \alpha') - 3/\pi \text{ Arctan } (S/C)$$

with:

$$S = [(XV + YU) + 2Y]z + 2(V + XY), \quad C = z^2 + [(XY - YV) + 2X]z + 2(U - Y^2), \tag{18}$$

$$X = \cos 2\pi\alpha, Y = \sin 2\pi\alpha, \quad U = \cos 2\pi(\alpha+\alpha') \quad V = \sin 2\pi(\alpha+\alpha')$$

Following the intuition provided by the last section, the "dynamical spectrum" is defined as the invariant set of the map $F(z,\alpha,\alpha') = (E,\beta,\beta')$ of $\boldsymbol{R}x\boldsymbol{T}^2$. Since $\beta + \beta' = 4(\alpha + \alpha')$ in (17), only one of the two normalized fluxes is actually relevant, leading to an effective 2D map. *Is the dynamical spectrum equal to the actual spectrum of the original operator ? This is a question with no answer yet.* Nevertheless the numerical calculation of the dynamical spectrum given in fig.2 below [Ghez(87)], shows that it should be.

One should point out there that this calculation has been compared to an experiment performed in Grenoble, on a superconducting network designed according to fig. 1. Landau-Ginzburg's theory [de Gennes(81), Alexander(83)] shows that the transition between the normal metal and the super conducting phases occurs in the (T,B) plane (where T is the temperature) on a curve which is simply related to the edge of the dynamical spectrum as calculated above [Ghez(87)]. The comparison between theory and experiment is actually very accurate as shown in fig.3 below.

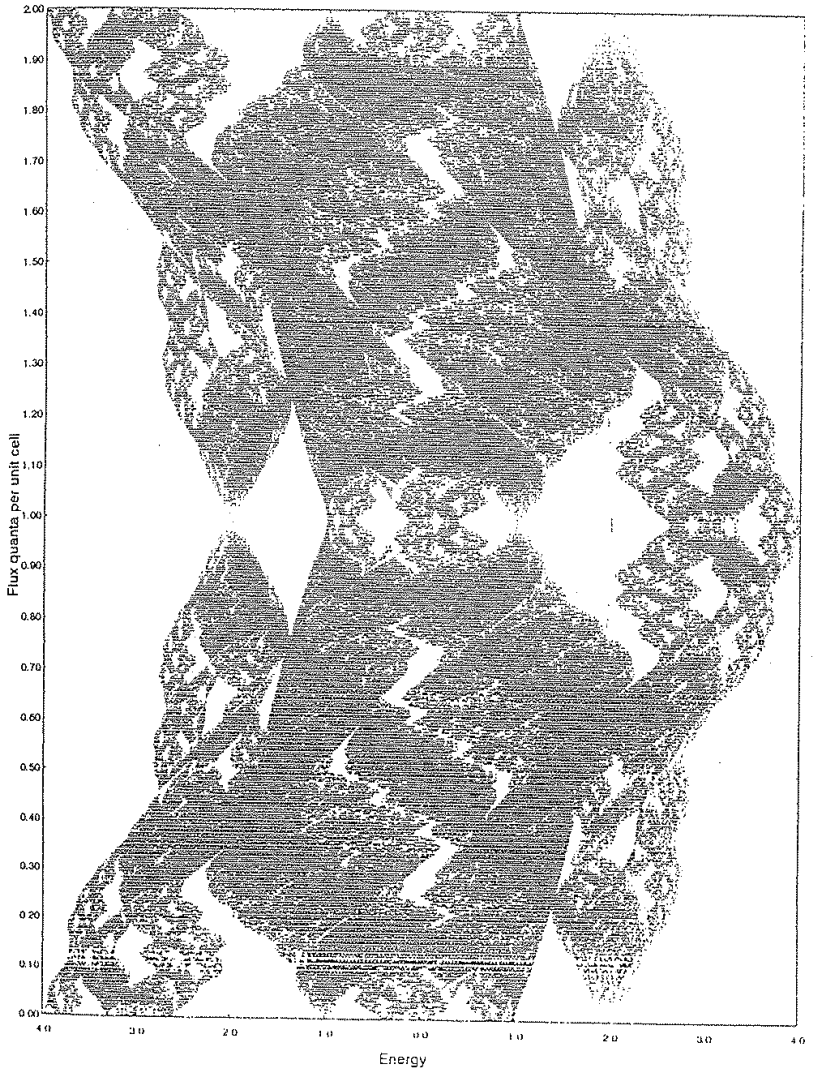


Fig.2- The dynamical spectrum of the Sierpinsky Laplacean in a magnetic field. The magnetic flux is represented on the horizontal axis, whereas the energy is represented on the vertical one. (Picture designed by C.Kreft) -

$$E = \{z^4 - 7z^2 - [2(XU+YV)+ 4X]z + 4(1-U)\}/\{S^2 + C^2\}^{1/2}, \tag{17}$$

$$\beta + \beta' = 4(\alpha + \alpha') \qquad \beta - \beta' = 2(\alpha - \alpha') - 3/\pi \text{ Arctan } (S/C)$$

with:

$$S = [(XV + YU) + 2Y]z + 2(V + XY)\,, \quad C = z^2 + [(XY - YV) + 2X]z + 2(U - Y^2), \tag{18}$$

$$X = \cos 2\pi\alpha\,, Y = \sin 2\pi\alpha\,, \qquad U = \cos 2\pi(\alpha+\alpha') \qquad V = \sin 2\pi\,(\alpha+\alpha')$$

Following the intuition provided by the last section, the "dynamical spectrum" is defined as the invariant set of the map $F(z,\alpha,\alpha') = (E,\beta,\beta')$ of $\boldsymbol{R}x\boldsymbol{T}^2$. Since $\beta + \beta' = 4(\alpha + \alpha')$ in (17), only one of the two normalized fluxes is actually relevant, leading to an effective 2D map. *Is the dynamical spectrum equal to the actual spectrum of the original operator ? This is a question with no answer yet.* Nevertheless the numerical calculation of the dynamical spectrum given in fig.2 below [Ghez(87)], shows that it should be.

Strichartz, Robert S. (1-CRNL)

Spectral asymptotics revisited. (English. English summary)

J. Fourier Anal. Appl. **18** (2012), *no.* 3, 626–659.

For a self-adjoint operator L with discrete spectrum, one often is led to consider the counting function

$$N(\lambda) = \#\{\lambda_j \in \text{spectrum}(L) \leq \lambda\}.$$

There are many classical results on the behavior of this function as $\lambda \to \infty$. Perhaps the most recognizable is Weyl's law, which concerns the Laplacian on a bounded domain $\Omega \subset \mathbf{R}^n$, and which states $N(\lambda) = c_n \text{Vol}(\Omega)\lambda^{n/2} + o(\lambda^{n/2})$. An appropriate notion of mass density function is $M(\lambda) = \frac{N(\Omega)}{\text{Vol}(\Omega)}$.

In order to generalize this to operators with continuous spectrum, say for L acting on $L^2(X, \mu)$, one is led to consider its spectral measure $E_\lambda = \chi_{[0,\lambda]}(L)$ and its representation

$$E_\lambda f(x) = \int_X K_\lambda(x, y) f(y) d\mu(y),$$

if it exists. Then, one may define (perhaps after appropriate justifications)

$$N(\lambda) := \int_X K_\lambda(x, x) d\mu(x), \quad M(\lambda) = \lim_{k\to\infty} \frac{1}{\mu(B_k)} \int_{B_k} K_\lambda(x, x) f d\mu(x),$$

where B_k is an appropriate "reasonable" sequence of sets B_k that exhausts X. The author considers several examples where this procedure can be applied (although for some of them, one can only hope to define lim inf and lim sup in the quantity for $M(\lambda)$). More precisely, he obtains that $M(\lambda) = c_n \lambda^{n/2} + o(\lambda^{n/2})$ for

(1) the classical Laplacian on $\mathbf{R}^n$,
(2) the hyperbolic Laplacian,
(3) the Heisenberg Laplacian and sub-Laplacian and
(4) the discrete Laplacian (and this is valid only close to $\lambda \sim 0$, due to the fact that the operator is bounded and thus the limit at infinity is trivial).

Abstract

I will describe the spectral decimation method developed by Rammal, Toulouse, Bellissard, Fukushima, Shima for the standard Laplacian on the Sierpinski gasket. Then I will show how this method can be generalized to other graphs and other Laplacians. In this way one can construct simple models with pure point or singularly continuous spectrum, and even understand the corresponding eigenprojectors. Moreover, combining this with the recent ideas of Strichartz concerning 'nicer Fourier series', one may be able to obtain results on fractals which are not even available in classical situations. As one of the applications we can discuss the uniform approximations to the wave equation solutions in a situation when the wave propagation speed is infinite.

For references, see the preprint
Singularly continuous spectrum of a self-similar Laplacian on the half-line
`http://arxiv.org/abs/1509.08875`
by Joe P. Chen and T.

Nuclear Physics B280 [FS 18] (1987) 147–180
North-Holland, Amsterdam

METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

F. ENGLERT, J.-M. FRÈRE[1] and M. ROOMAN[2]

Physique Théorique, C.P. 225, Université Libre de Bruxelles, 1050 Brussels, Belgium

Ph. SPINDEL

Faculté des Sciences, Université de l'Etat à Mons, 7000 Mons, Belgium

Received 19 February 1986

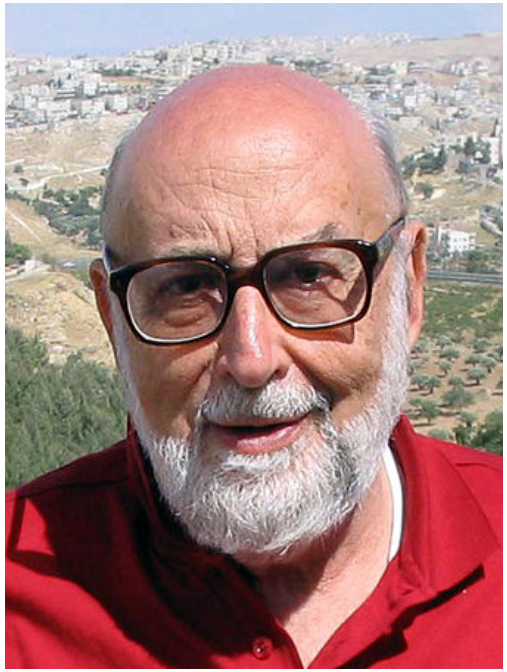
We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.

François Englert

From Wikipedia, the free encyclopedia

François Baron Englert (French: [ɑ̃glɛʁ]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.[4] He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research,

François Englert



François Englert in Israel, 2007

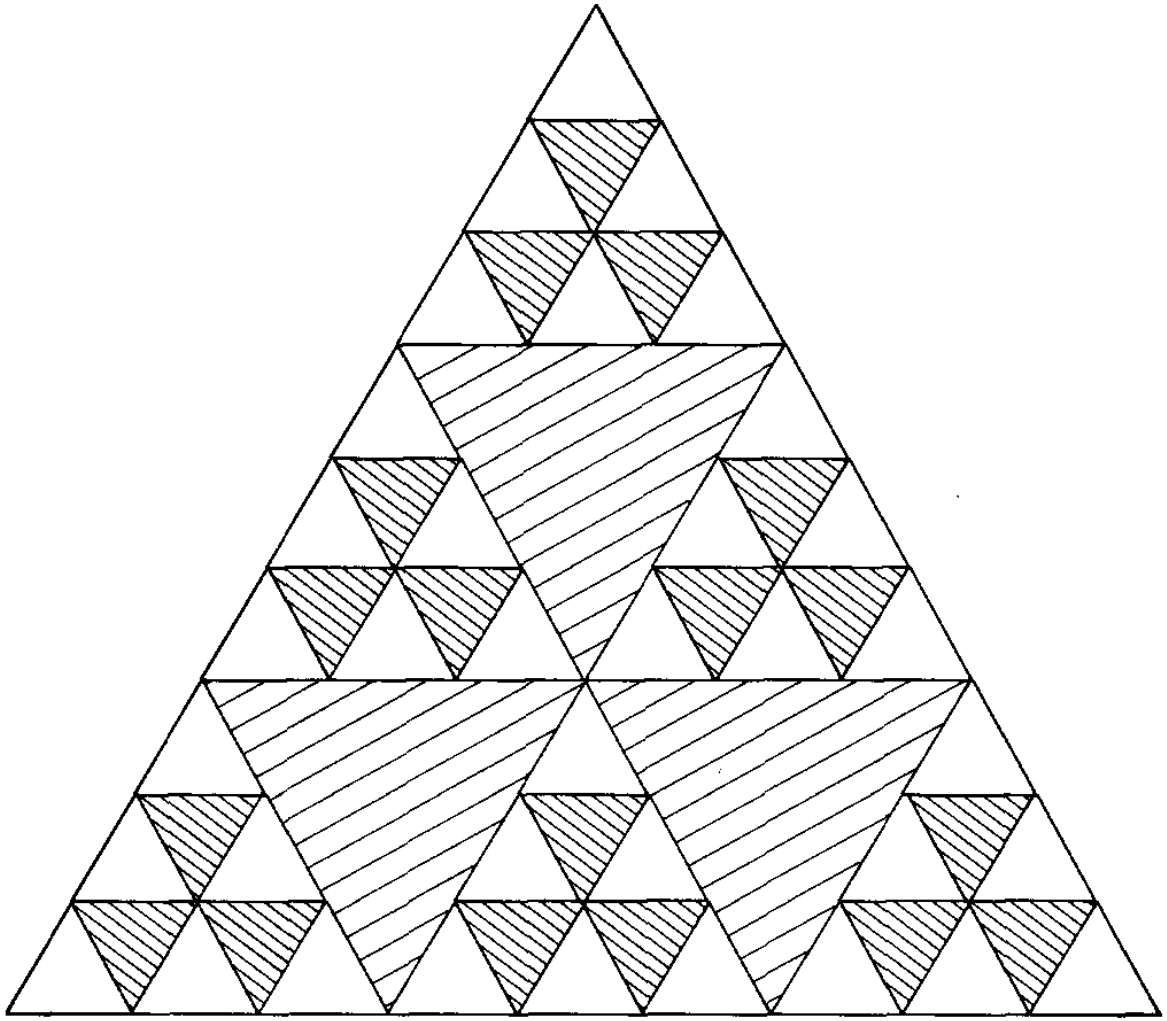


Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

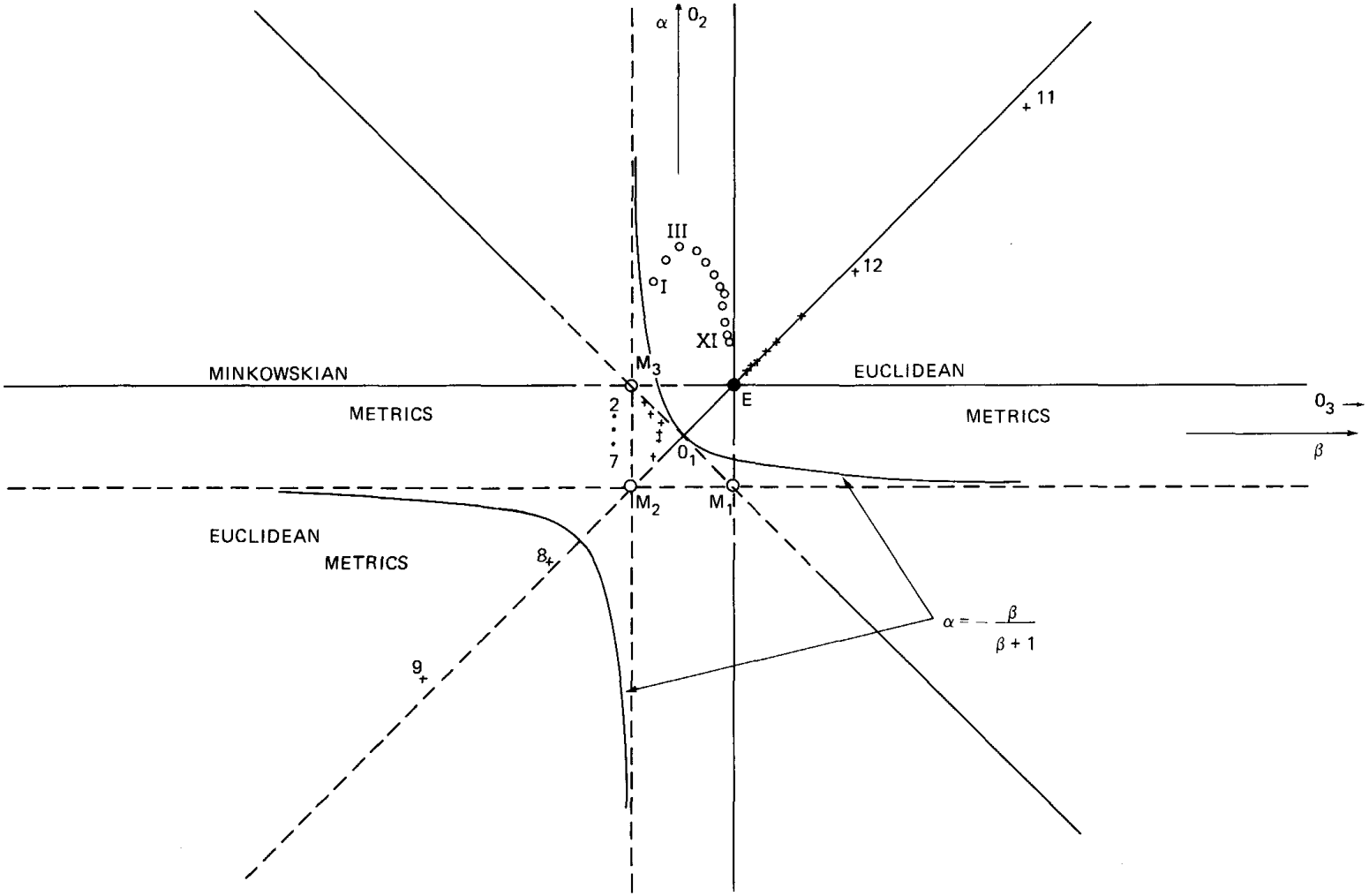


Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta/(\beta + 1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. M_1, M_2, M_3 denote unstable minkowskian fixed geometries while E corresponds to the stable euclidean fixed point. The unstable fixed points 0_1, 0_2 and 0_3 associated to 0-dimensional geometries are located at the origin and at infinity on the (α, β) coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ($\alpha = -56.4$, $\beta = -52.5$) is outside the frame of the figure.

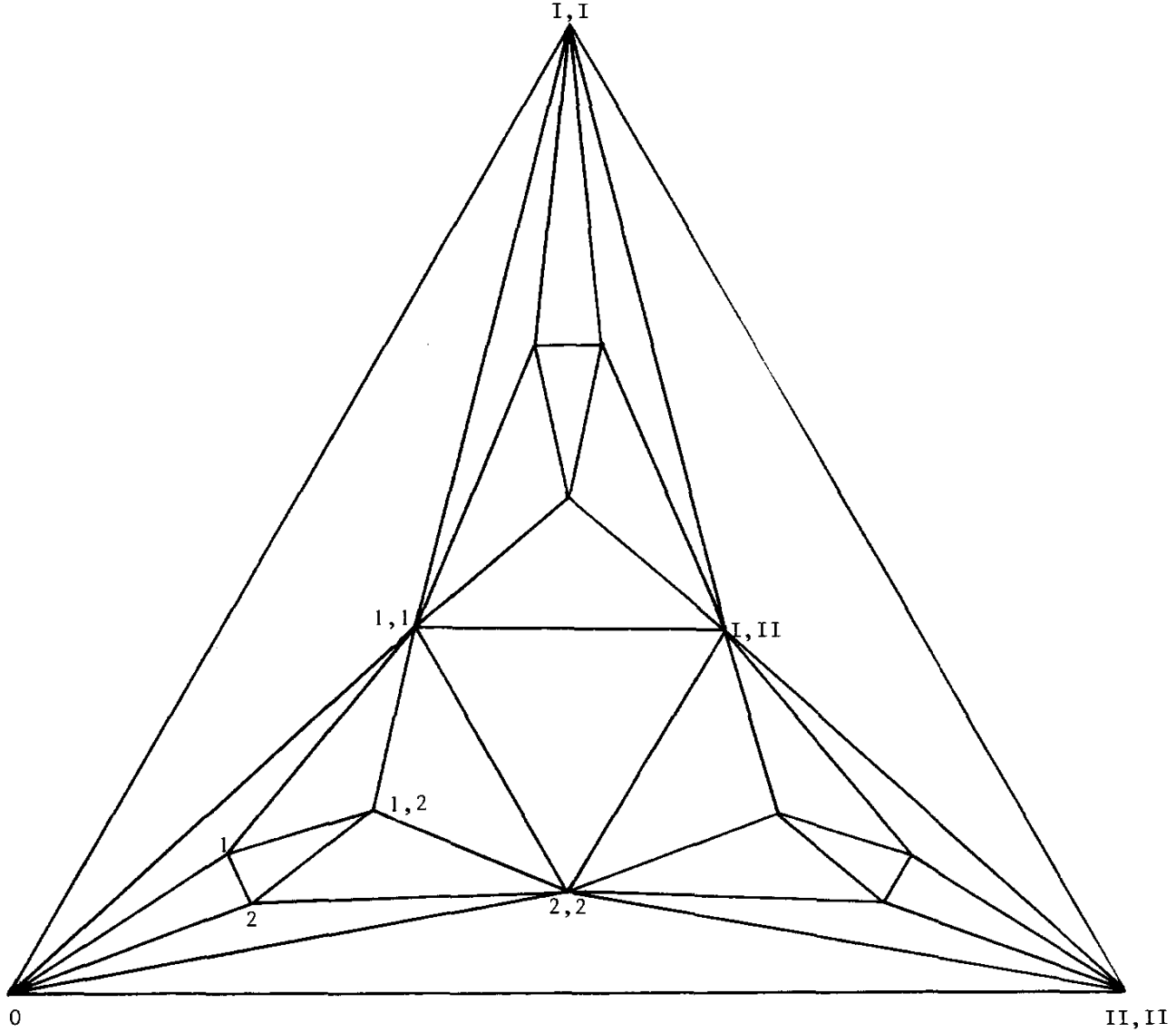


Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

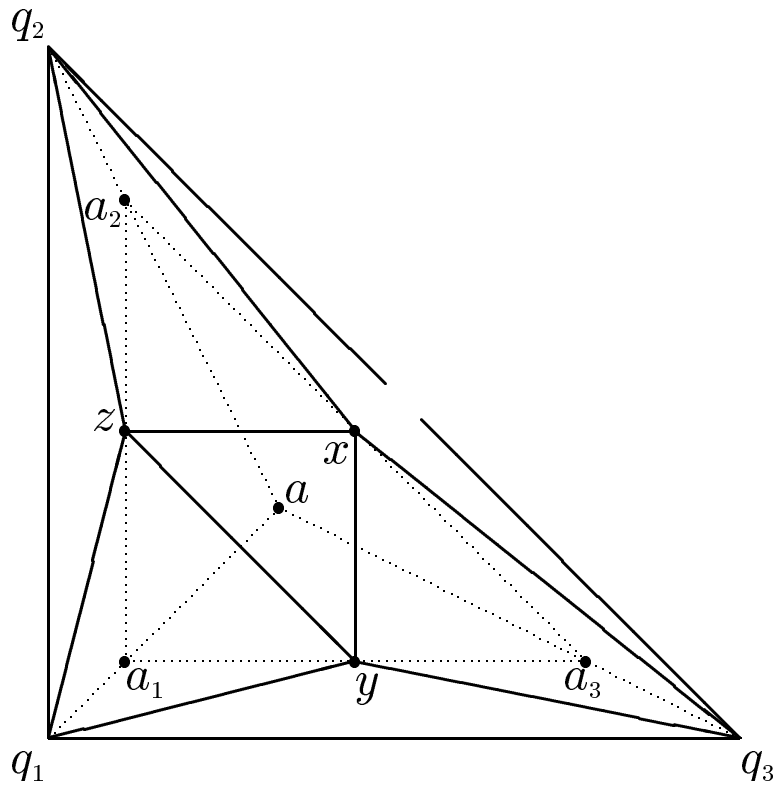


Figure 6.4. Geometric interpretation of Proposition 6.1.

7. Effective resistance metric, Green's function and capacity of points

We first recall from [**Ki4**] some facts about limits of resistance networks. Although we state all the results of this section for the Sierpiński gasket, they can be applied to general pcf fractals with only minor changes.

Let $\mathcal{E}(f,f)$ be defined by (1.2) for any function f on V_*, where $\mathcal{E}_n$ is a

PHYSICAL REVIEW B VOLUME 31, NUMBER 3 1 FEBRUARY 1985

Energy spectrum for a fractal lattice in a magnetic field

Jayanth R. Banavar
Schlumberger-Doll Research, Old Quarry Road, Ridgefield, Connecticut 06877-4108

Leo Kadanoff
Department of Physics, University of Chicago, Chicago, Illinois 60637

A. M. M. Pruisken*
Schlumberger-Doll Research, Old Quarry Road, Ridgefield, Connecticut 06877-4108
(Received 10 September 1984)

To simulate a kind of magnetic field in a fractal environment we study the tight-binding Schrödinger equation on a Sierpinski gasket. The magnetic field is represented by the introduction of a phase onto each hopping matrix element. The energy levels can then be determined by either direct diagonalization or recursive methods. The introduction of a phase breaks all the degeneracies which exist in and dominate the zero-field solution. The spectrum in the field may be viewed as considerably broader than the spectrum with no field. A novel feature of the recursion relations is that it leads to a power-law behavior of the escape rate. Green's-function arguments suggest that a majority of the eigenstates are truly extended despite the finite order of ramification of the fractal lattice.

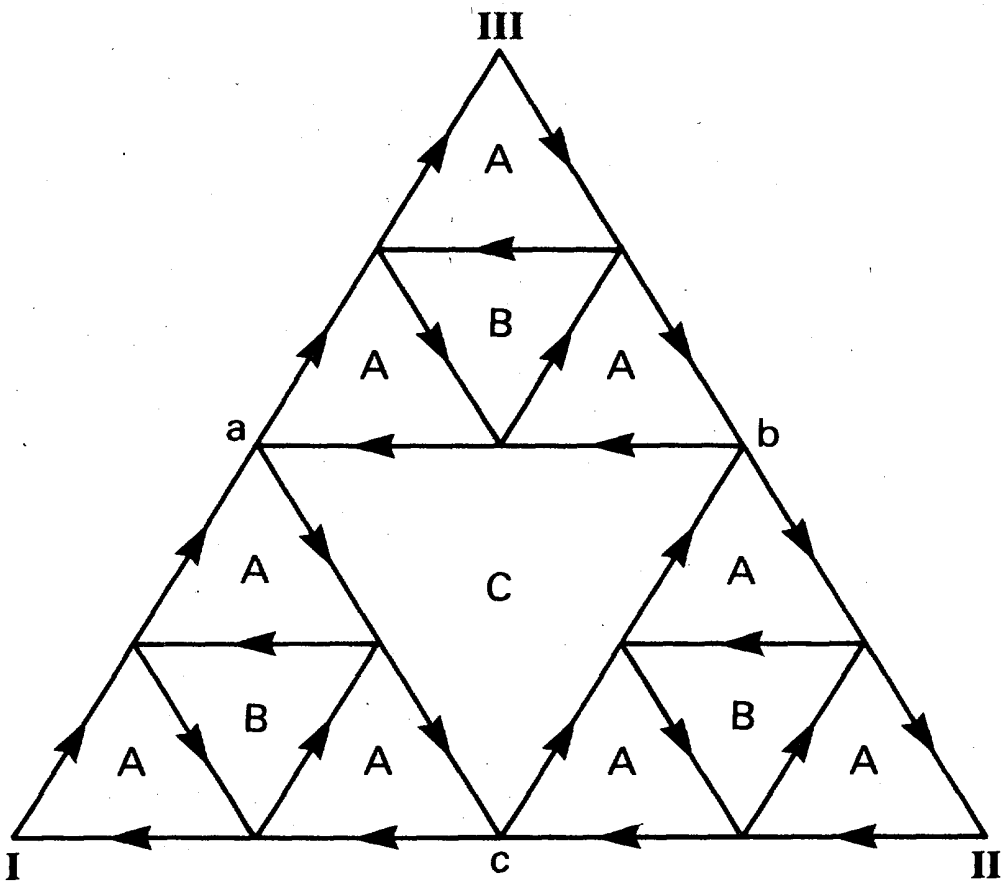


FIG. 1. Fragment of the Sierpinski gasket. The phase of the hopping matrix is equal to ϕ in the direction of the arrow and $-\phi$ otherwise.

©1985 The American Physical Society

Alexander[6] has studied more general flux patterns. We choose the pattern described above because it gives the simplest recursion equations and a maximum spatial homogeneity in f. Since the most interesting real magnetic field problem (the one in two dimensions) can be described as having a similarly spatially uniform f, we hope that our choice might lead to particularly interesting and perhaps extendable results.

This paper is organized as follows. In Sec. II we derive the recursion relations for the problem using a functional integral method. In Sec. III we study this map in a phenomenological manner. It turns out that the concept of "escape"[9,10] can be successfully employed in order to reveal the nature of the spectrum of eigenstates. A novel aspect is a power-law behavior of the escape, something which has not been observed before. This behavior is closely related to the occurrence of marginal behavior in the recursion relations in the neighborhood of a fixed line. On that fixed line the recursion relations reduce to a logistic map[11,12] with May-Feigenbaum parameter $\lambda=4$. From these notions the qualitative nature of the spectrum and of the eigenstates can be derived. The latter will be subject of Sec. IV (numerical studies).

In Sec. V we present arguments, using Green's functions, that a majority of the eigenstates are, indeed, truly extended. This is a quantum-mechanical result which is contrary to the "intuition" gained by studying classical problems on the gasket. Classically, the finite order of ramification of the gasket enables one to construct domains of arbitrary size having a constant domain-wall energy. As a result, for example, an Ising model on a gasket[13] has a T_c of zero. In contrast, in our quantum problem, resonance effects permit a few links to produce a large coherence length.

II. FORMULATION

A. Decimation

Consider the partition function

$$Z(\epsilon)=\int D[\psi]D[\bar{\psi}]\exp[\bar{\psi}(\epsilon\underline{1}-H)\psi]=\det(\epsilon\underline{1}-H)\ , \tag{1}$$

where $\bar{\psi}=i\psi^\dagger$ and the integration is over complex Grassmann fields. In Eq. (1), ϵ denotes a trial value of the energy. Each actual eigenenergy ϵ_α is a zero of the partition function. The partition function itself is simply a product of $\epsilon-\epsilon_\alpha$. For example, in the case of a trivial gasket with three sites (Fig. 2), we have

$$Z_0(\epsilon)=\det(\epsilon\underline{1}-H_0)\ , \tag{2}$$

where H_0 is given by

$$H_0=\begin{bmatrix} 0 & f & f^* \\ f^* & 0 & f \\ f & f^* & 0 \end{bmatrix}\ , \tag{3}$$

and $\underline{1}$ is a 3×3 unit matrix. This result is easily evaluated as

$$Z_0(\epsilon)=\epsilon^3-3\epsilon ff^*-f^3-(f^*)^3\ . \tag{4}$$

In our actual work we shall measure ϵ in units of the magnitude of the hopping, choosing as our variables ϕ and $x=\epsilon/|f|$. In terms of these variables, the basic partition function (4) becomes

$$D(x,\phi)=x^3-3x-2\cos3\phi\ . \tag{5}$$

On the trivial lattice, the eigenenergies are given in units of $|f|$ by the zeros of $D(x,\phi)$.

To elucidate the behavior of the nontrivial lattice, one performs a decimation procedure on it to eliminate sites until the trivial problem is reached. However, the coupling to the outside corners of the largest triangle may have to be handled separately since these sites are likely to have a special environment. We return to this point below, after the recursion relation for the other bonds is established.

The decimation is easily obtained as follows (see Fig. 3). The right-hand side of the figure is obtained from the left-hand side by eliminating sites a, b, and c, i.e., the fields ψ on sites a, b, and c are simply integrated out. The part of the Lagrangian corresponding to the left-hand side of Fig. 3 can be written as

$$L_{\text{block}}=[\bar{\psi}_i,\bar{\psi}_0]\begin{bmatrix} \epsilon-H_0 & -H_0 \\ -H_0 & \epsilon/2 \end{bmatrix}\begin{bmatrix} \psi_i \\ \psi_0 \end{bmatrix}\ , \tag{6}$$

where

$$[\psi_i,\psi_0]=[\psi_a\psi_b\psi_c,\psi_1\psi_2\psi_3]\ , \tag{7}$$

and H_0 is given by Eq. (3). The decimation, i.e., the integration over ψ_i, is easily performed, leading to a transformed Lagrangian for the block

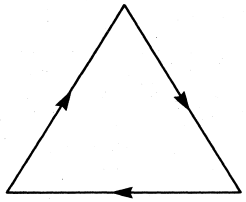


FIG. 2. The trivial lattice.

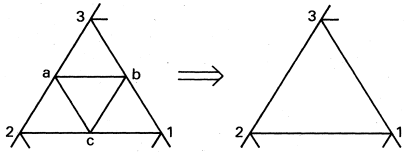


FIG. 3. Decimation consists of integration over Grassmann fields at sites a, b, and c, leading to effective hopping matrix elements between sites 1, 2, and 3.

phase difference of $2\pi l/3$, $l=0,1,2$. The eigenvalues thus obtained are exactly those for the larger gasket. In Sec. IV we exploit this property to obtain eigenenergies and functions for a gasket containing 1095 sites by diagonalizing the Hamiltonian on gaskets containing 366 sites.

One more boundary condition is of some interest. Following Ref. 1 once again, we can combine the triangles of Fig. 1 into a periodic lattice. The outside vertices of the triangles are special. Now they have six neighbors, while all other points have four. We describe these special points by an on-site energy y. The form of Eq. (17a) and the initial condition $y_n = x_n$ still hold, but now, instead of Eq. (17b), we have

$$g(y,x,\phi)=\frac{y(x^3-3x-2\cos 3\phi)-6x(x+\cos 3\phi)}{x(x^3-7x-6\cos 3\phi)} . \quad (18a)$$

Then, when the last stage is reached, the on-site energy is y_1, while the phase is ϕ_1. In this last stage the system lies upon a triangular lattice. A brief calculation shows that the allowed energies of the lattice are in the range

$$\min_{l=1,2,3} 6\cos\left[\phi+\frac{2\pi}{3}l\right] < x_1 < \max_{l=1,2,3} 6\cos\left[\phi+\frac{2\pi}{3}l\right] . \quad (18b)$$

Thus we find, with this boundary condition, that allowed energies occur in bands.

C. A note on the zero-field case

The approaches to finding the spectrum described in the preceding section must all be modified if the field is zero. The basic problem can be seen in specializing Eq. (10) to the case $\phi=0$. Then this equation reads

$$x'=\frac{x(x+1)(x+2)(x-3)}{\Gamma^{1/2}\mathrm{sgn}N} ,$$
$$N=(x+2)(x+1) , \quad (19)$$
$$\Gamma=(x+2)^2(x+1)^2 .$$

When x is -2 or -1, Eqs. (19) have an intractable 0/0 structure. That is why these values of x had to be handled specially in Ref. 1. At $\phi\to\pm 60°$, similar singularities appear at $x=2$ or 1.

However, this difficulty is peculiar to these three special cases of ϕ. For these ϕ values, special attention must be given to the values of x which produce the 0/0 structure. However, for all other ϕ values, Γ remains positive, no distinguished values of x arise, and the methods outlined in subsection B do give the spectrum.

III. THE SPECTRUM

A. Introduction

We now proceed in a semiphenomenological fashion and perform a kind of escape-rate experiment[10] in the x-ϕ plane. This procedure will help us understand the spectrum of eigenstates.

What we shall do is very simple: We initially distribute points uniformly in a region of the x,ϕ plane. We then perform i iterations and see how many and which of the original points remain in the region. In previous experiments of this kind[10] the number of points surviving after i iterations, $n_s(i)$, dropped off exponentially,

$$n_s(i)\sim e^{-\beta i}$$

for large i.

The measurement of $n_s(i)$ is interesting because n_s gives information about localization, densities of states, and the fractal character of the spectrum. For the case in which gaskets with the order of 3^i sites are combined into

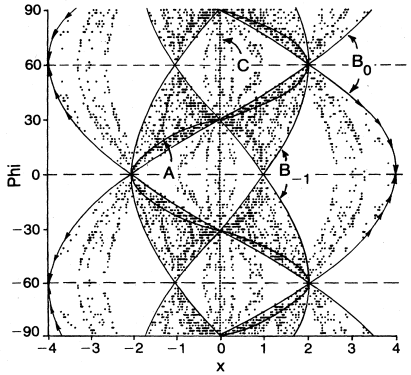


FIG. 4. Set of initial points that do not escape after 100 iterations. Line A corresponds to $x=-2\cos 3\phi$. Lines B and C are described in Sec. III C 2.

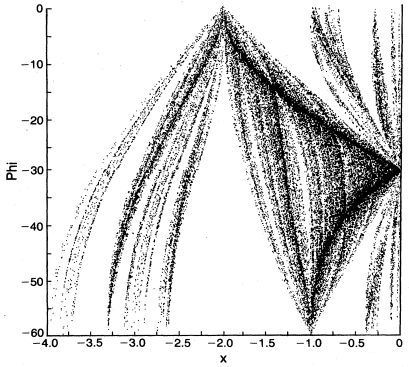


FIG. 5. Set of initial points that do not escape after 300 iterations.

Solid State Communications, Vol. 64, No. 10, pp. 1291–1294, 1987.
Printed in Great Britain.

0038–1098/87 $3.00 + .00
© 1987 Pergamon Journals Ltd.

BAND SPECTRUM FOR AN ELECTRON ON A SIERPINSKI GASKET IN A MAGNETIC FIELD

J.M. Ghez*

Centre de Physique Théorique, CNRS-Luminy, Case 907, F-13288, Marseille, Cedex 09, France

and

Yin Yu Wang, R. Rammal† and B. Pannetier

CRTBT, CNRS, BP 166X, Grenoble Cedex, France

and

J. Bellissard‡

Centre de Physique Théorique, CNRS-Luminy, Case 907, F-13288, Marseille, Cedex 09, France

(*Received* 20 *July* 1987 *by S. Alexander*)

We consider a quantum charged particle on a fractal lattice given by a Sierpinski gasket, submitted to a uniform magnetic field, in a tight binding approximation. Its band spectrum is numerically computed and exhibits a fractal structure. The groundstate energy is also compared to the superconductor transition curve measured for a Sierpinski lattice of superconducting material.

choose the gauge in such a way that H depends only upon α and α' in a periodic way with period one. We will denote by $H(\alpha, \alpha')$ this operator from now on.

We also introduce the dilation operator D defined as:

$$D\varphi(m) = \varphi(2m). \tag{2}$$

The scaling properties of this system are expressed in the following Renormalization Group Equation (RGE) [23]:

$$E\{E1 - H(\alpha, \alpha')\}^{-1}D = G\{E^*1 - H(\alpha,^*, \alpha'^*)\}^{-1}, \tag{3}$$

where [7, 16]:

(i) $$G = \{E^3 - 3E - 2(XU + YV)\}/(S^2 + C^2)^{1/2},$$

(ii) $$E^* = \{E^4 - 7E^2 - [2(XU + YV) + 4X]E + 4(1 - U)\}/(S^2 + C^2)^{1/2}, \tag{4}$$

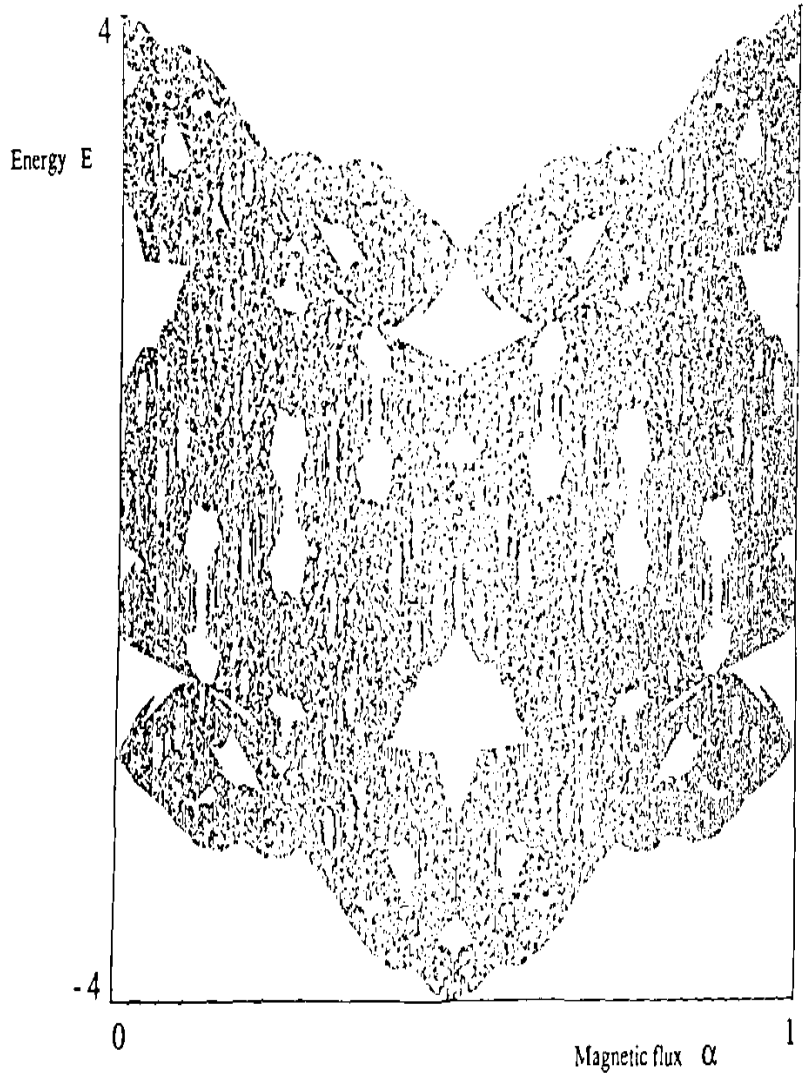


Fig. 2. Spectrum of $H(\alpha)$, computed by 10 iterations of F. α is the horizontal variable, ranging from 0 to 1. E is the vertical variable, ranging from -4 to 4.

These results have been compared with an experiment performed on an array of superconducting Al–wires shaped like a Sierpinski gasket with six levels of hierarchy. A description of this pattern generated by e-beam lithography has been given in [20]. More details will be published in a separate paper [21]. The transition curve in the parameter space (T, B), where observes experimentally the periodicity in the parameter α and also the scaling properties predicted by the RGE (equation 3). The plot in Fig. 4 shows the comparison between the experimental curve in log-log scale together with the theoretical results for the edge

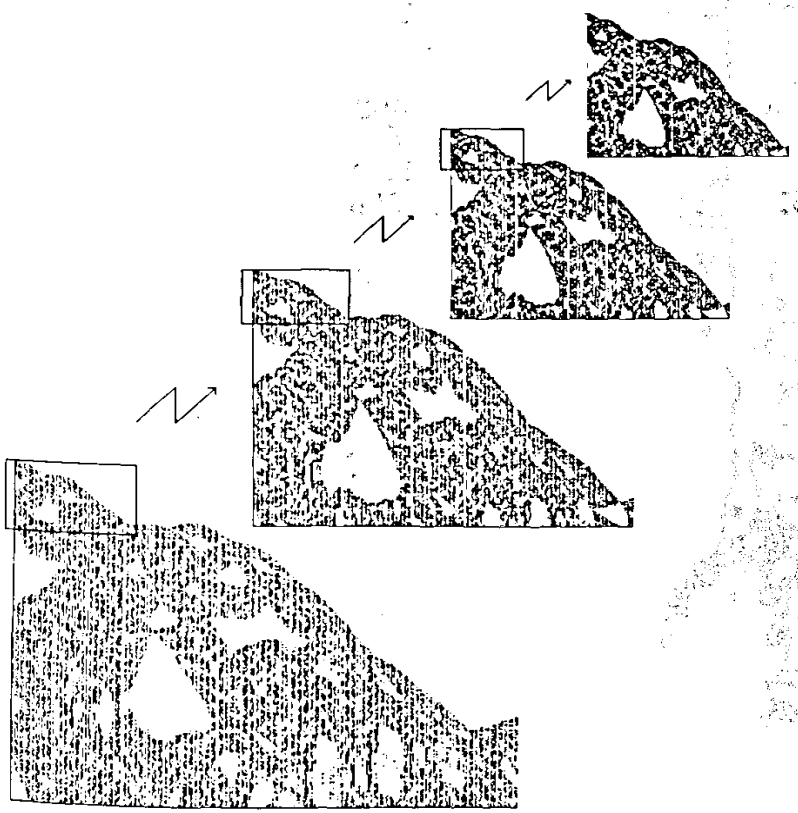


Fig. 3. Four enlargements of the upper left corner of Fig. 2, showing the fractal nature of the spectrum, with the approximate scaling law (7). α is the horizontal variable, ranging from 0 to 2^{-k}, $k = 2, 4, 6, 8$. E is the vertical variable, ranging from E_0 to 4, $E_0 = 2.4$, 3.68, 3.936, 3.9872.

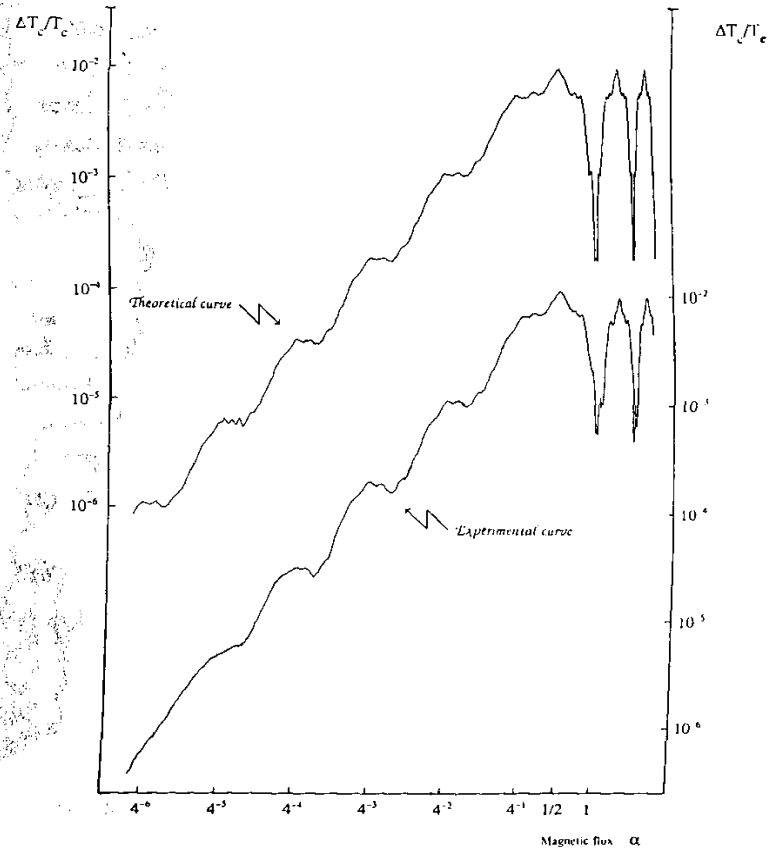


Fig. 4. Comparison between the calculated edge of the spectrum (left scale) with the experimental result (right scale) on the critical temperature of a superconducting gasket: $\Delta T_c/T_c$ vs α in log–log plot, where $\alpha = \Phi/\Phi_0$ is the reduced magnetic flux in the elementary triangle of the gasket: equation 8 has been used to calculate the theoretical curve using the best fit parameters as explained in the text. The two curves have been shifted for clarity.

The Spectral Dimension of the Universe is Scale Dependent

J. Ambjørn,[1,3,*] J. Jurkiewicz,[2,†] and R. Loll[3,‡]

[1]*The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*

[2]*Mark Kac Complex Systems Research Centre, Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, PL 30-059 Krakow, Poland*

[3]*Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands*

(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301 PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory.

tral dimension, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the

other hand, the "short-distance spectral dimension," obtained by extrapolating Eq. (12) to $\sigma \longrightarrow 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \tag{15}$$

and thus is compatible with the integer value two.

Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data

arXiv:1110.5224v1 [hep-th] 24 Oct 2011

Martin Reuter and Frank Saueressig

Institute of Physics, University of Mainz
Staudingerweg 7, D-55099 Mainz, Germany

reuter@thep.physik.uni-mainz.de

saueressig@thep.physik.uni-mainz.de

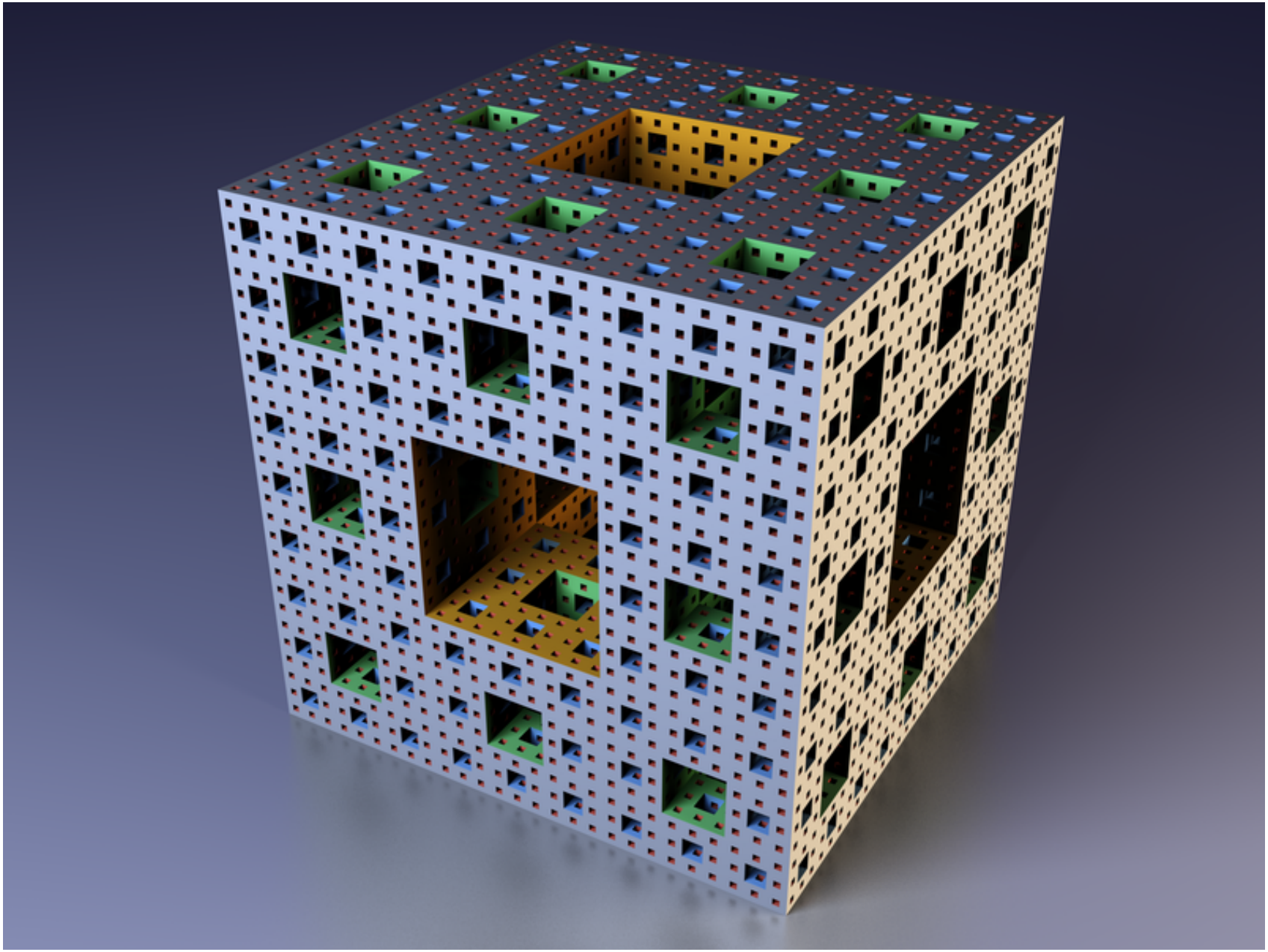
Abstract

The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension d_s and walk dimension d_w associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

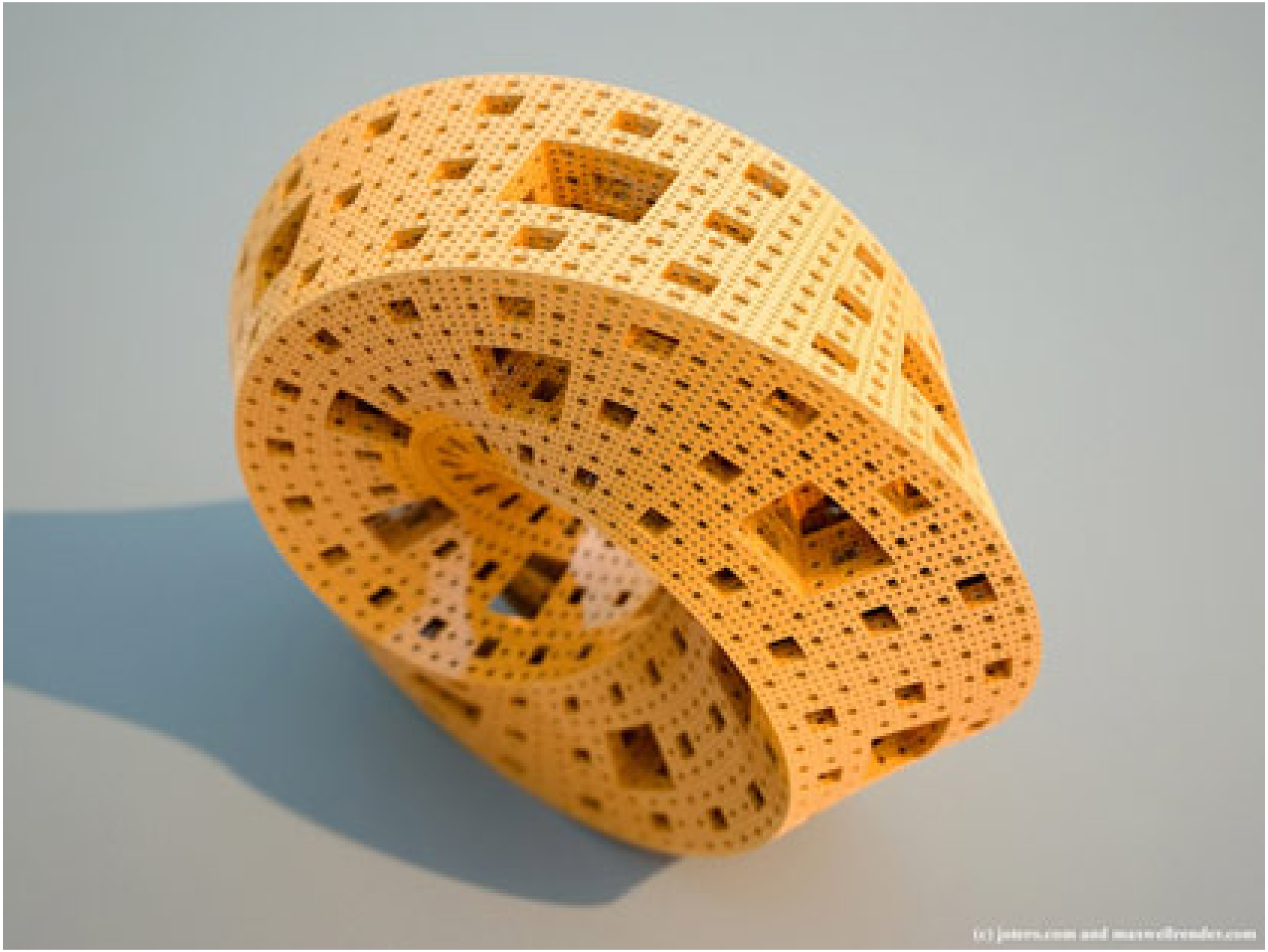
Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data

Martin Reuter and Frank Saueressig

a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered



Q: can we do **intrinsic differential** geometry and analysis ?



Early results on diffusions on fractals

Sheldon Goldstein, *Random walks and diffusions on fractals.* Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., 8, Springer

Summary: we investigate the asymptotic motion of a random walker, which at time $\mathbf{n}$ is at $\mathbf{X(n)}$, on certain 'fractal lattices'. For the 'Sierpiński lattice' in dimension $\mathbf{d}$ we show that, as $\mathbf{L} \to \infty$, the process $\mathbf{Y_L(t) \equiv X([(d+3)^L t])/2^L}$ converges in distribution to a diffusion on the Sierpin'ski gasket, a Cantor set of Lebesgue measure zero. The analysis is based on a simple 'renormalization group' type argument, involving self-similarity and 'decimation invariance'. In particular,

$$|\mathbf{X(n)}| \sim \mathbf{n}^{\gamma},$$

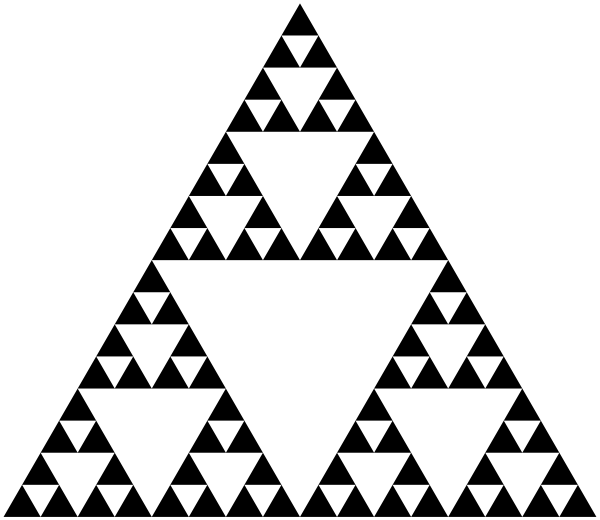
where $\gamma = \mathbf{(\ln 2)/\ln(d+3))} \leqslant \mathbf{2}$.

Shigeo Kusuoka, *A diffusion process on a fractal.* Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 1987.

- M.T. Barlow, E.A. Perkins, *Brownian motion on the Sierpinski gasket.* (1988)
- M. T. Barlow, R. F. Bass, *The construction of Brownian motion on the Sierpiński carpet.* Ann. Inst. Poincaré Probab. Statist. (1989)
- S. Kusuoka, *Dirichlet forms on fractals and products of random matrices.* (1989)
- T. Lindstrøm, *Brownian motion on nested fractals.* Mem. Amer. Math. Soc. **420**, 1989.
- J. Kigami, *A harmonic calculus on the Sierpiński spaces.* (1989)
- J. Béllissard, *Renormalization group analysis and quasicrystals,* Ideas and methods in quantum and statistical physics (Oslo, 1988) Cambridge Univ. Press, 1992.
- M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket.* (1992)
- J. Kigami, *Harmonic calculus on p.c.f. self–similar sets.* Trans. Amer. Math. Soc. **335** (1993)
- J. Kigami and M. L. Lapidus, *Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals.* Comm. Math. Phys. **158** (1993)

Main classes of fractals considered

- $\mathbf{[0,1]}$
- Sierpiński gasket
- nested fractals
- p.c.f. self-similar sets, possibly with various symmetries
- finitely ramified self-similar sets, possibly with various symmetries
- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)
- metric measure Dirichlet spaces, possibly with heat kernel estimates (MMD+HKE)



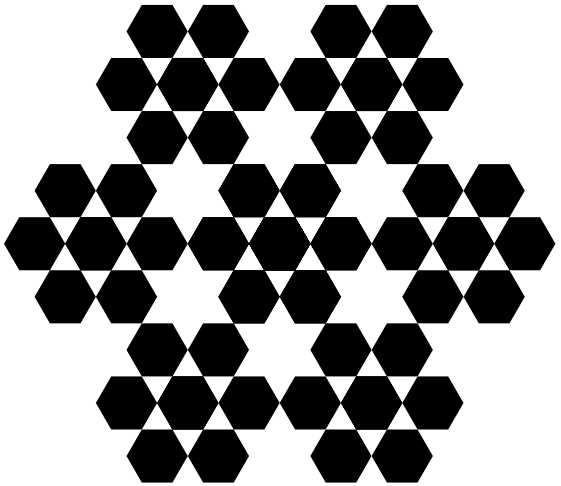


Figure: Sierpiński gasket and Lindstrøm snowflake (nested fractals), p.c.f., finitely ramified)

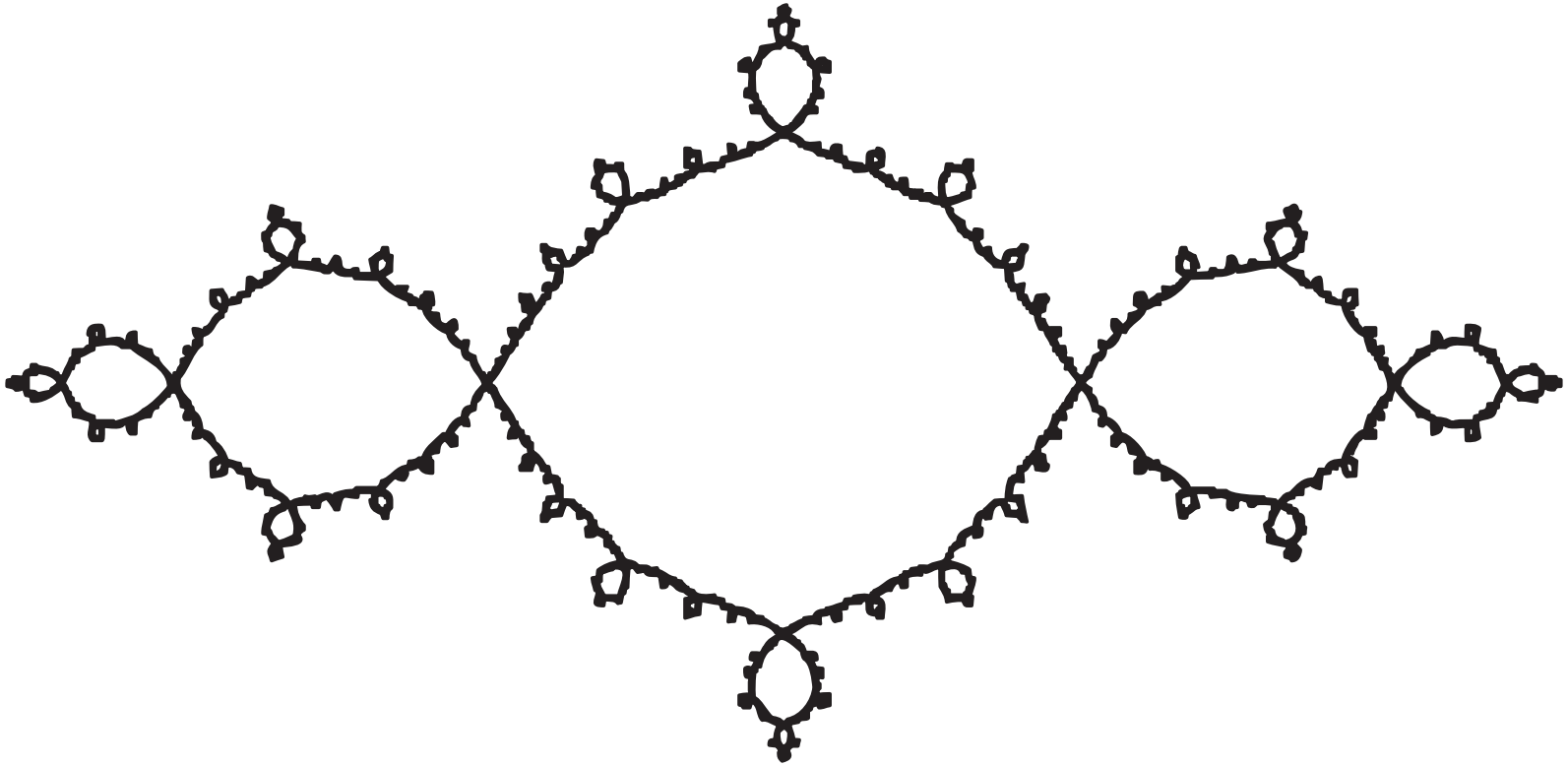


Figure: The basilica Julia set, the Julia set of $\mathbf{z^2 - 1}$ and the limit set of the basilica group of exponential growth (Grigorchuk, Żuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda et al., Rogers-T.).

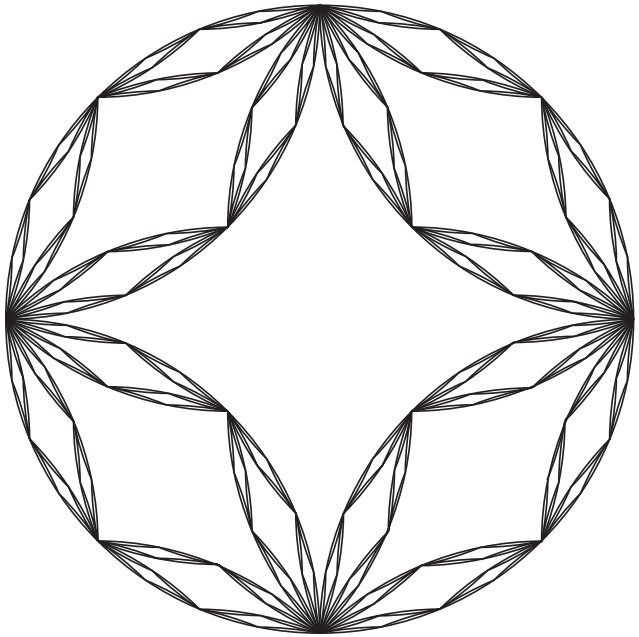


Figure: Diamond fractals, non-p.c.f., but finitely ramified

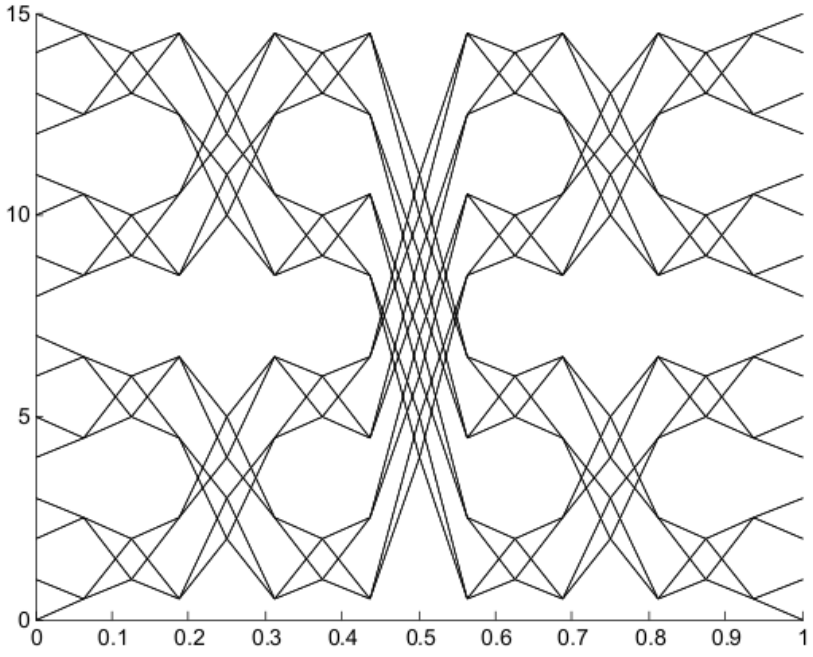


Figure: Laakso Spaces (Ben Steinhurst), infinitely ramified

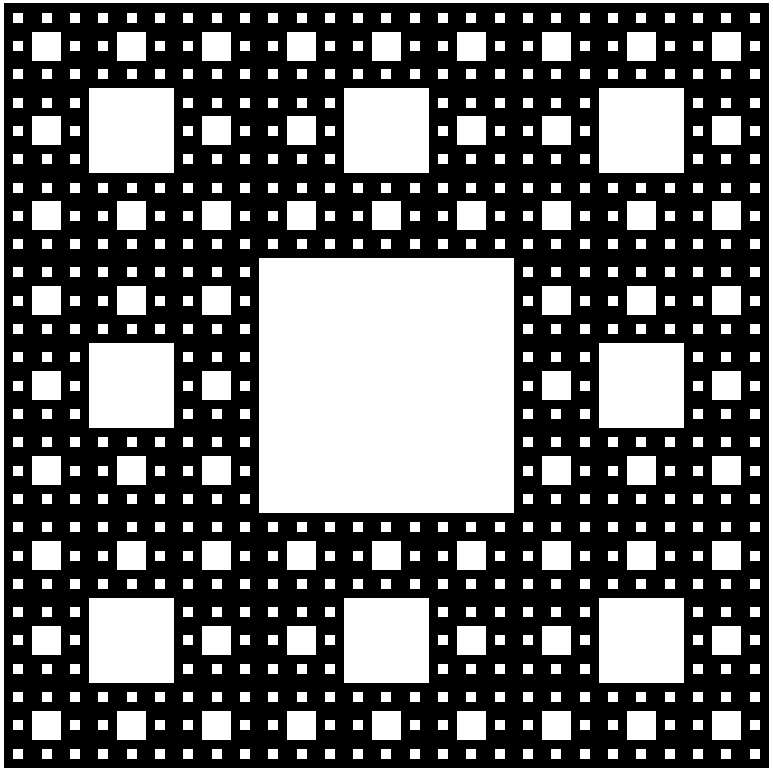


Figure: Sierpiński carpet, infinitely ramified

Existence, uniqueness, heat kernel estimates

Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

$$\text{distance} \sim \sqrt{\text{time}}$$

"Einstein space–time relation for Brownian motion"

De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;

Li-Yau (1986) type estimates on a geodesically complete Riemannian manifold with **Ricci** $\geqslant$ **0**:

$$\mathbf{p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)}$$

$$\mathbf{distance \sim \sqrt{time}}$$

Brownian motion on $\mathbb{R}^{\mathbf{d}}$: $\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| = \mathbf{ct^{1/2}}$.

Anomalous diffusion: $\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| = \mathbf{o(t^{1/2})}$, or (in regular enough situations),

$$\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| \approx \mathbf{t^{1/d_w}}$$

with $\mathbf{d_w} > \mathbf{2}$.

Here $\mathbf{d_w}$ is the so-called **walk dimension** (should be called **"walk index"** perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$\text{distance} \sim (\text{time})^{\frac{1}{d_w}}$$

$$\begin{aligned} d_H &= \text{Hausdorff dimension} \\ \frac{1}{\gamma} = d_w &= \text{“walk dimension” } (\gamma\text{=diffusion index}) \\ \frac{2d_H}{d_w} = d_S &= \text{“spectral dimension” (diffusion dimension)} \end{aligned}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

Theorem (Barlow, Bass, Kumagai (2006)).

Under natural assumptions on the MMD (geodesic Metric Measure space with a regular symmetric conservative Dirichlet form), the sub-Gaussian **heat kernel estimates are stable under rough isometries**, *i.e. under maps that preserve distance and energy up to scalar factors.*

Gromov-Hausdorff + energy

Theorem. (Barlow, Bass, Kumagai, T. (1989–2010).) On any fractal in the class of generalized Sierpiński carpets (includes cubes in $\mathbb{R}^{\mathbf{d}}$) there exists a unique, up to a scalar multiple, local regular Dirichlet form that is invariant under the local isometries.

Therefore there there is a unique corresponding symmetric Markov process and a unique Laplacian. Moreover, the Markov process is Feller and its transition density satisfies sub-Gaussian heat kernel estimates.

Remark: intrinsic uniqueness is proved by Steinhurst for the Laakso spaces (to appear in Potential Analysis) and for non-self-similar Sierpinski carpets (work in progress)

Main difficulties for the Sierpinski carpet:

If it is not a cube in $\mathbb{R}^n$, then

- $d_S < d_H$, $d_w > 2$
- the energy measure and the Hausdorff measure are mutually singular;
- the domain of the Laplacian is not an algebra;
- if $d(x, y)$ is the (Euclidean-induced) shortest path metric, then $d(x, \cdot)$ is not in the domain of the Dirichlet form (not of finite energy) and so methods of Differential geometry seem to be not applicable;**
- Lipschitz functions are not of finite energy;**
- in fact, we can not compute any non-constant functions of finite energy;
- Fourier and complex analysis methods seem to be not applicable.

** see recent papers by Koskela and Zhou and by Hinz, Kelleher, T

Theorem. (Grigor'yan and Telcs, also [BBK])

On a MMD space the following are equivalent

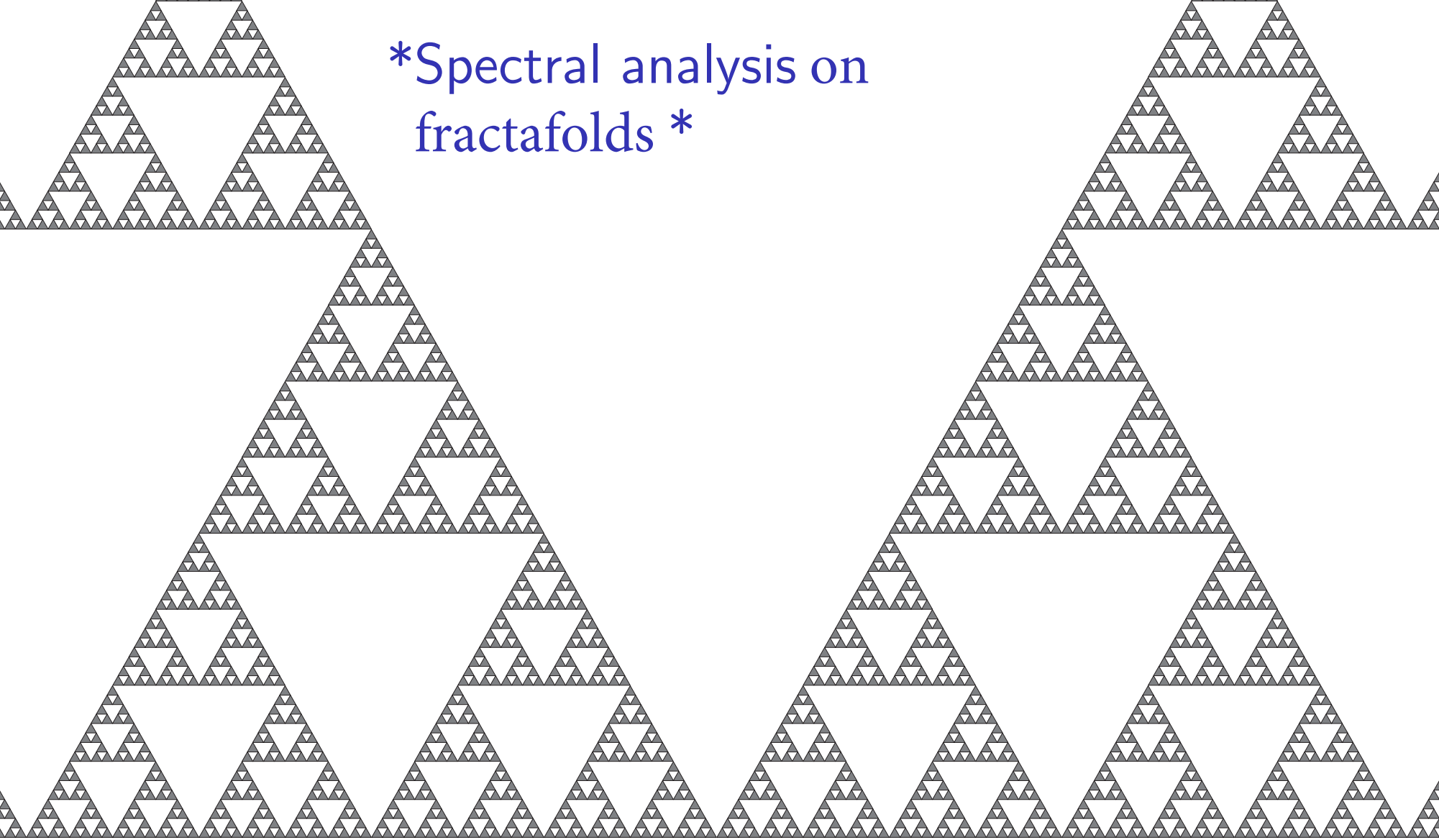
- **(VD)**, **(EHI)** and **(RES)**
- **(VD)**, **(EHI)** and **(ETE)**
- **(PHI)**
- **(HKE)**

and the constants in each implication are effective.

Abbreviations: Metric Measure Dirichlet spaces, Volume Doubling, Elliptic Harnack Inequality, Exit Time Estimates, Parabolic Harnack Inequality, Heat Kernel Estimates.

Remark: recent improvements in Grigor'yan and Hu, Heat kernels and Green functions on metric measure spaces, to appear in Canad. J. Math.

*Spectral analysis on fractafolds *



A part of an infinite Sierpiński gasket.

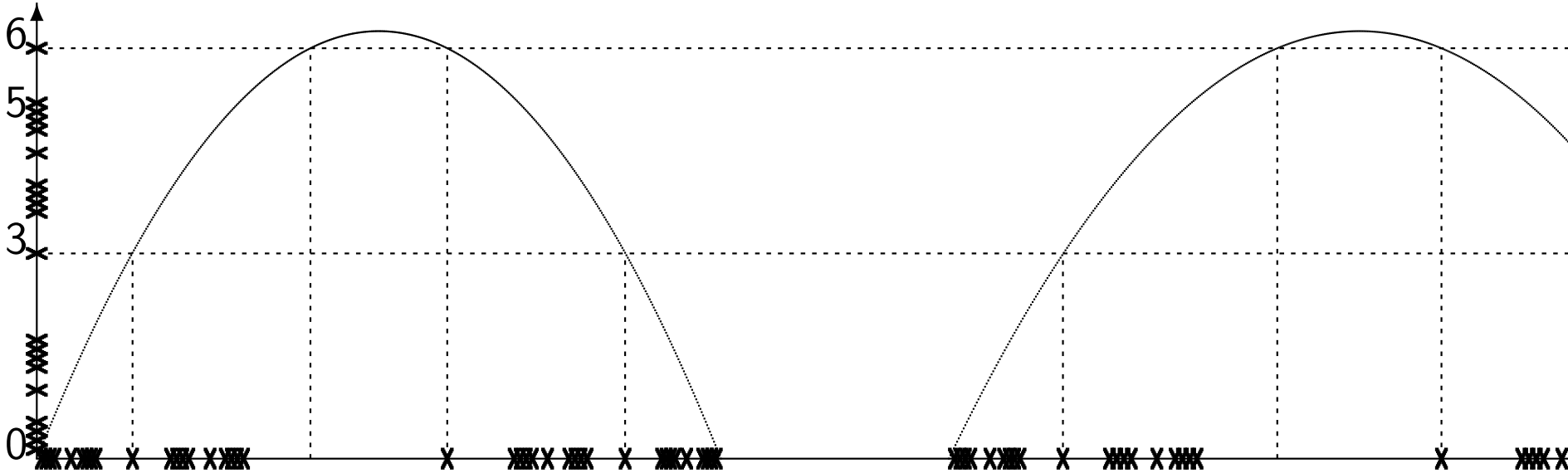
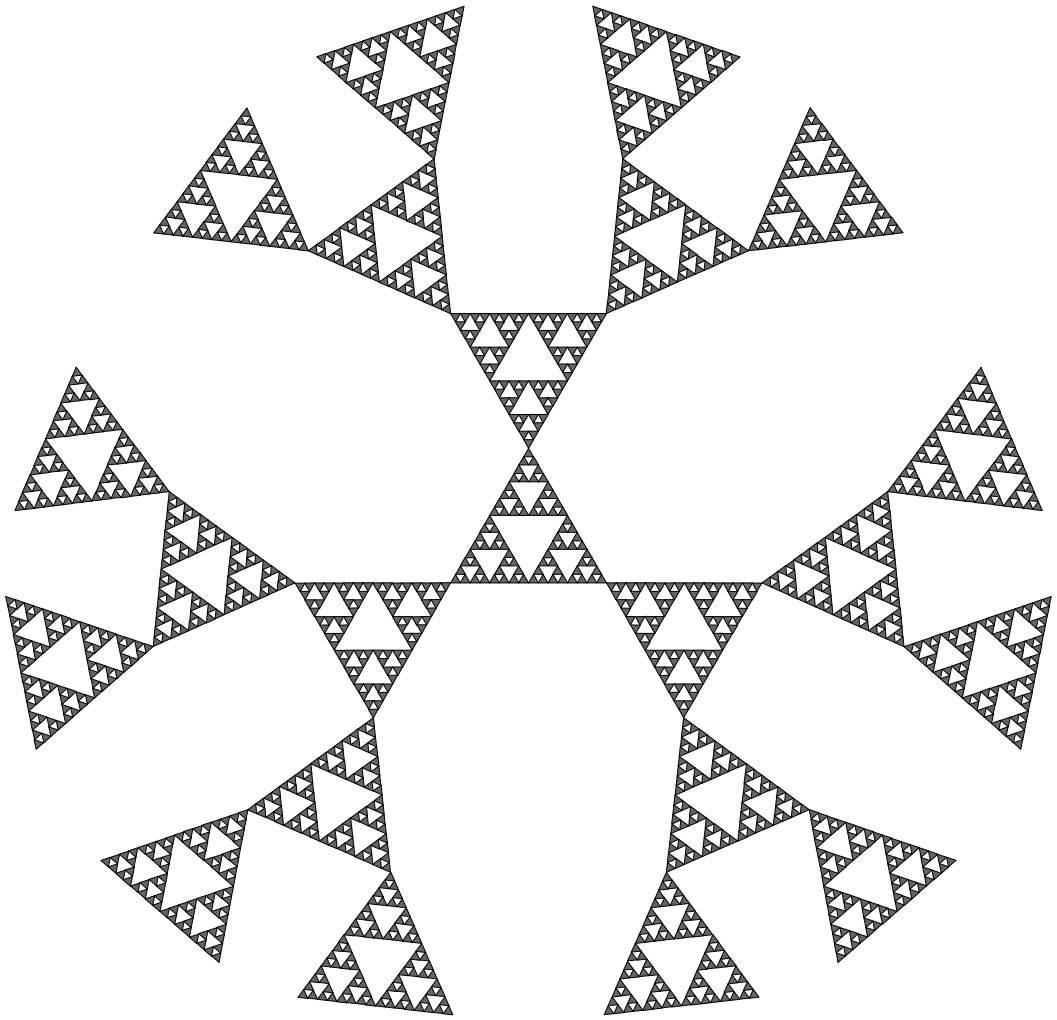
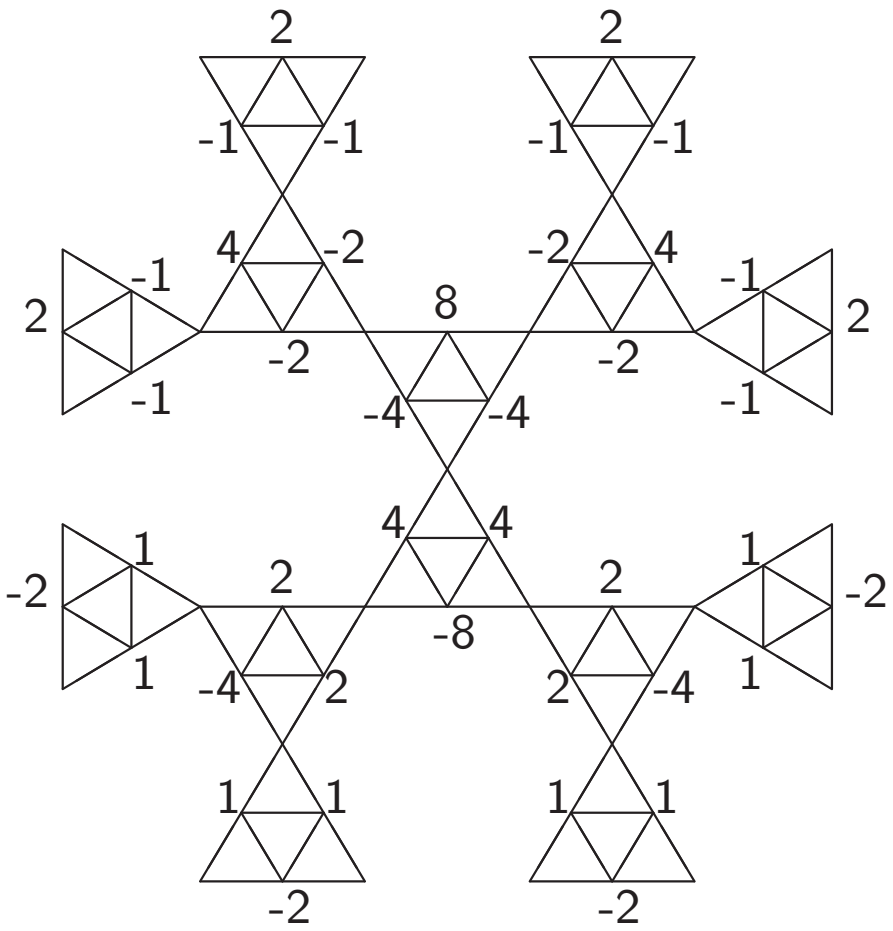


Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

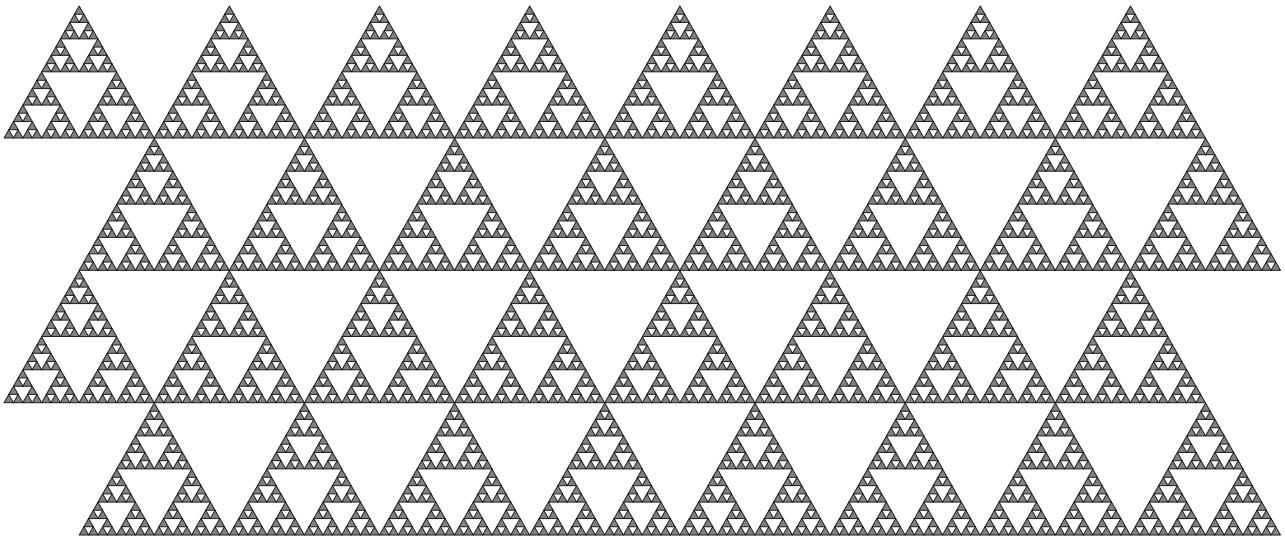
Theorem. (Béllissard 1988, T. 1998, Quint 2009)
On the infinite Sierpiński gasket the spectrum of the Laplacian consists of a **dense set of eigenvalues** $\mathfrak{R}^{-1}(\Sigma_0)$ **of infinite multiplicity** and a **singularly continuous component of spectral multiplicity one supported on** $\mathfrak{R}^{-1}(\mathcal{J}_{\mathsf{R}})$.

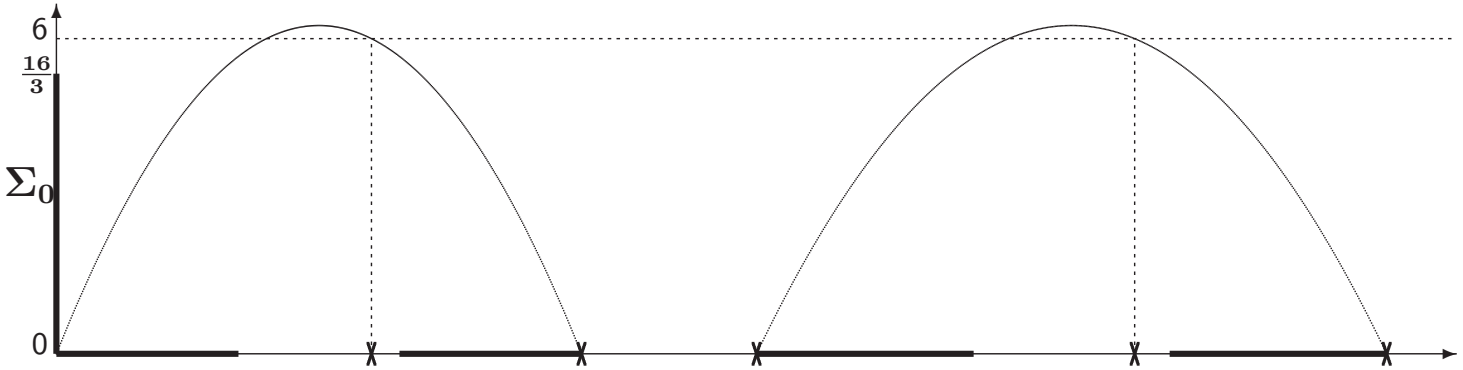


The Tree Fractafold.



An eigenfunction on the Tree Fractafold.





Theorem. (Strichartz, T. 2010) The Laplacian on the periodic triangular lattice finitely ramified Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum. The **absolutely continuous spectrum is $\mathfrak{R}^{-1}[0, \frac{16}{3}]$.** The **pure point spectrum** consists of two infinite series of eigenvalues of infinite multiplicity. The spectral resolution is given in the main theorem.

Recent results on spectral analysis and applications

G. Derfel, P. J. Grabner, and F. Vogl: Laplace Operators on Fractals and Related Functional Equations, J. Phys. A, 45(46), (2012), 463001

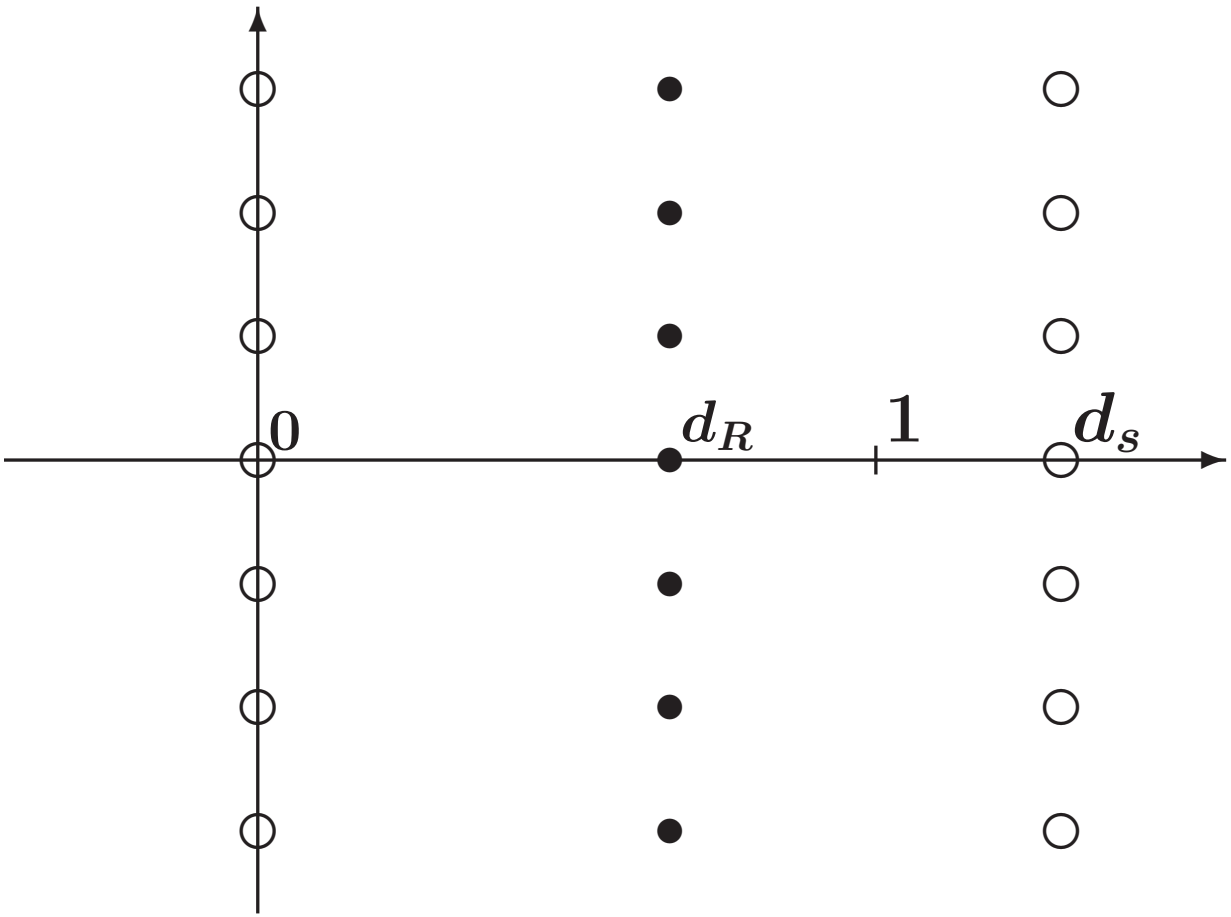
N. Kajino, *Spectral asymptotics for Laplacians on self-similar sets*. J. Funct. Anal. 258 (2010)
N. Kajino, T, *Spectral gap sequence and on-diagonal oscillation of heat kernels*, work in progress

Joe Chen, R. Strichartz, *Spectral asymptotics and heat kernels on three-dimensional fractal sponges*

J. F.-C. Chan, S.-M. Ngai, T, *One-dimensional wave equations defined by fractal Laplacians*

U. Freiberg, L. Rogers, T, *Eigenvalue convergence of second order operators on the real line*

B. Steinhurst, T, *Spectral Analysis and Dirichlet Forms on Barlow-Evans Fractals* arXiv:1204.5207



$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

Poles (white circles) of the spectral zeta function of the Sierpiński gasket.

See work of Grabner et al on relation to complex analysis and of Steinhurst et al on applications to Casimir energy.

Remark: what are dimensions of the Sierpiński gasket?

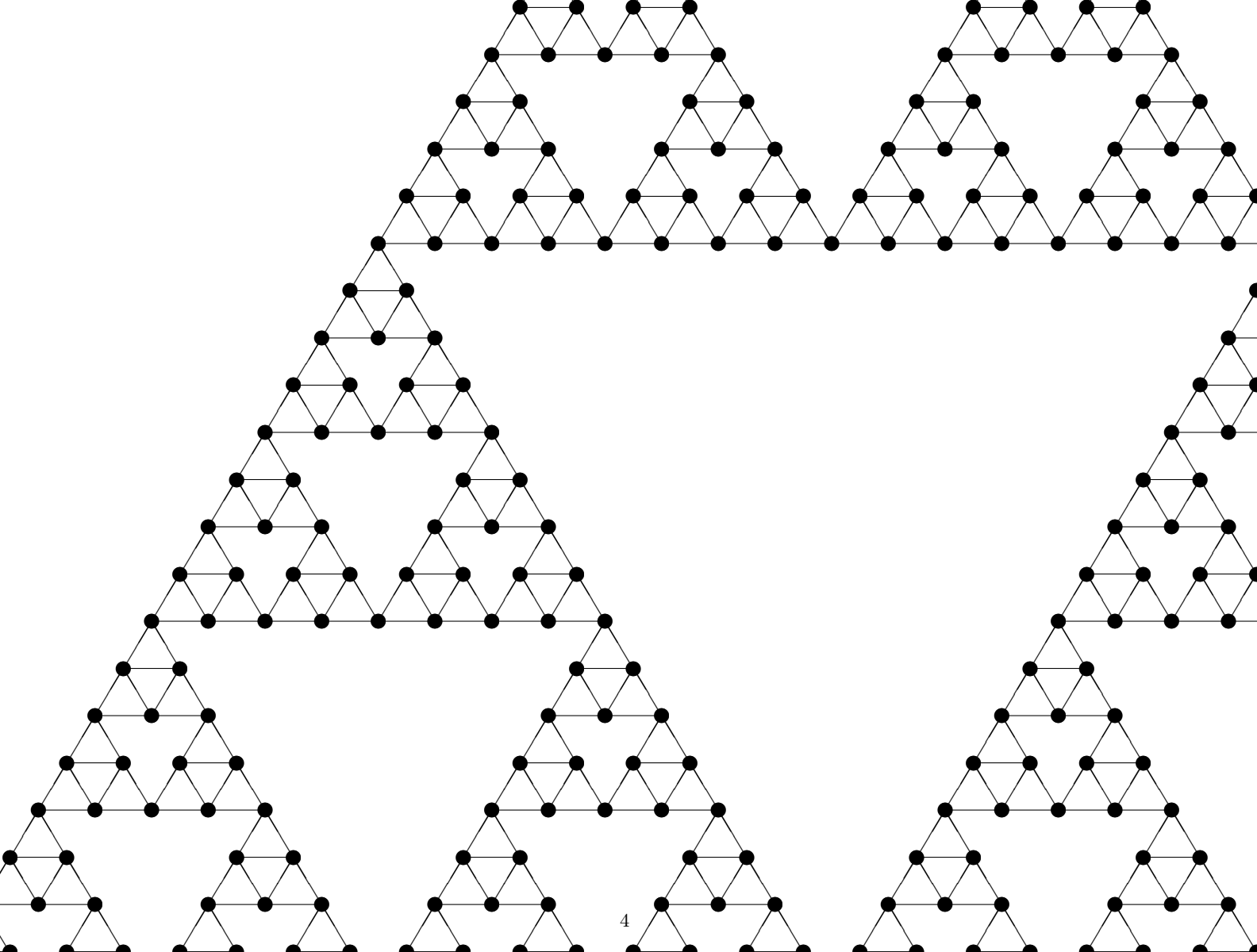
- $\frac{\log 3}{\log \frac{5}{3}} \approx \mathbf{2.15} =$ Hausdorff dimension in effective resistance metric
- $\mathbf{2} =$ geometric, linear dimension
- $\frac{\log 3}{\log 2} \approx \mathbf{1.58} =$ usual Hausdorff (Minkowsky, box, self-similarity) dimension in Euclidean coordinates (geodesic metric)
- $\frac{2 \log 3}{\log 5} \approx \mathbf{1.37} =$ usual spectral dimension
- ... ... =
  there are several Lyapunov exponent type dimensions related to harmonic functions and harmonic coordinates (Kajino, Ionescu-Rogers-T)
- $\mathbf{1} =$ topological dimension, martingale dimension
- $\frac{2 \log 2}{\log 5} \approx \mathbf{0.86} =$ polynomial spectral co-dimension (Grabner)?

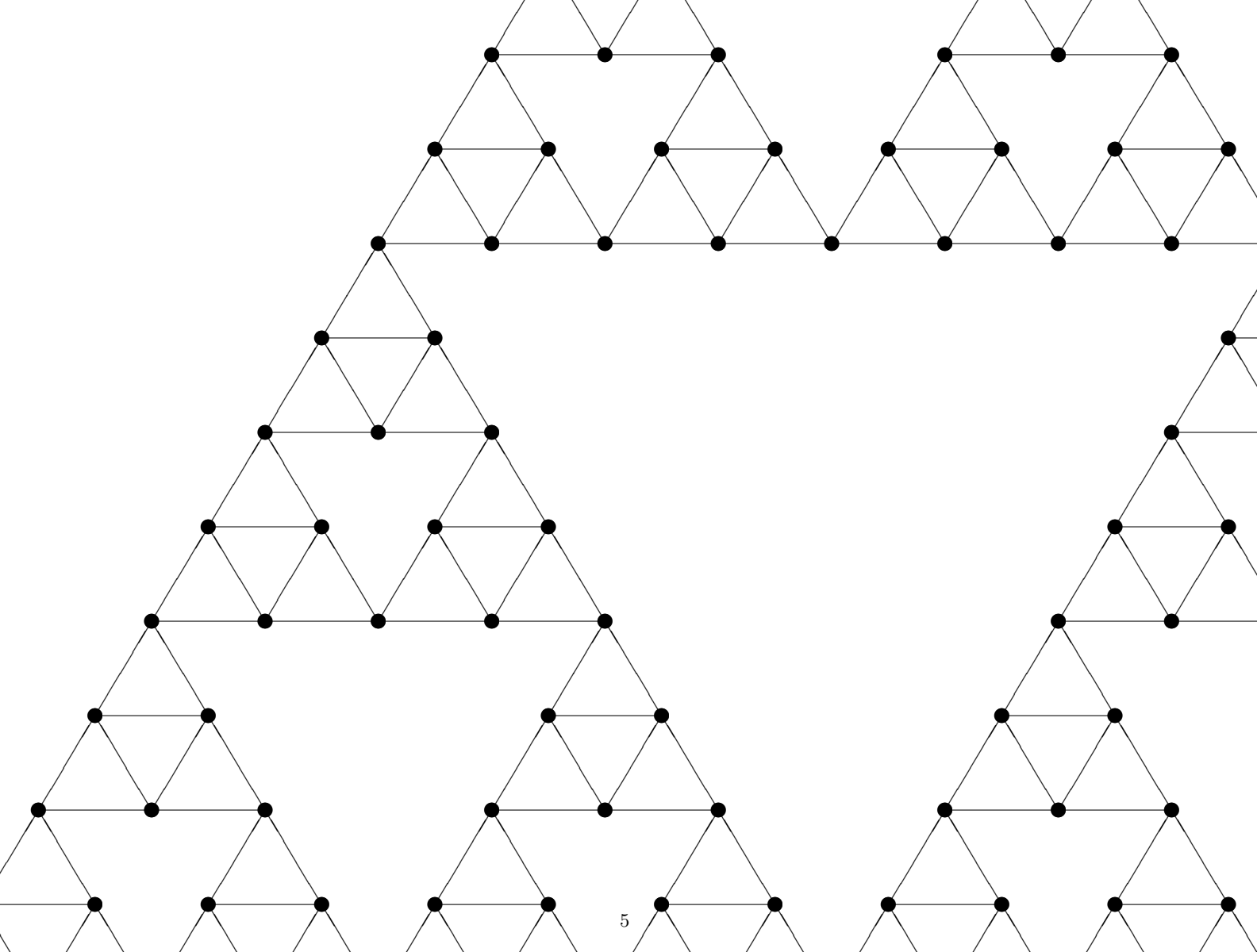
*conclusion

perhaps
analysis and probability on fractals
analysis on metric measure spaces
sub-Riemannian geometry
converge

***

Thank you for your attention :-)





Lecture 1
Laplacians on self-similar graphs and relation to self-similar groups

R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983), L13–L22.

R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984).

S. Alexander, *Some properties of the spectrum of the Sierpiński gasket in a magnetic field.* Phys. Rev. B **29** (1984).

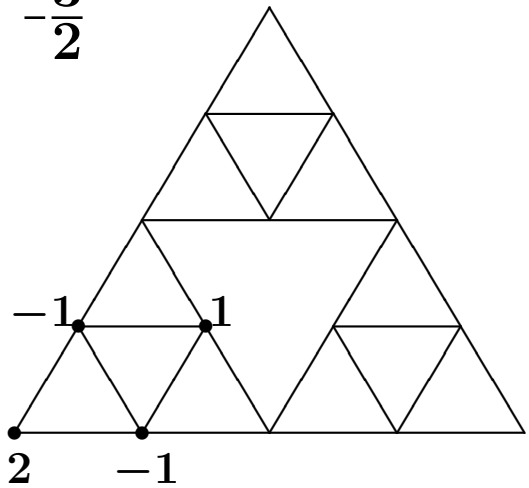
E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984).

Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983–1984).

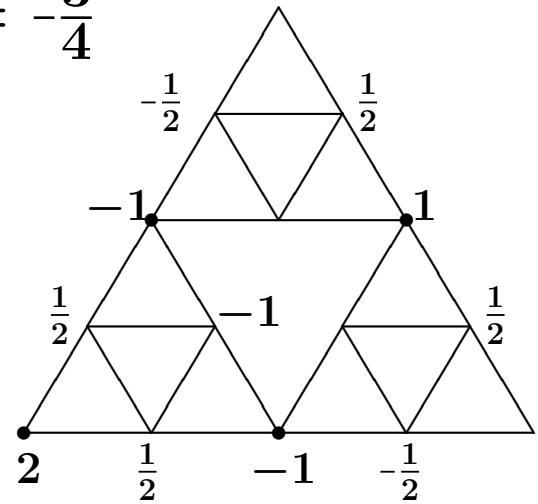
R. B. Stinchcombe, *Fractals, phase transitions and criticality.* Fractals in the natural sciences. Proc. Roy. Soc. London Ser. A **423** (1989), 17–33.

J. Béllissard, *Renormalization group analysis and quasicrystals,* Ideas and methods in quantum and statistical physics (Oslo, 1988). Cambridge Univ. Press, 1992.

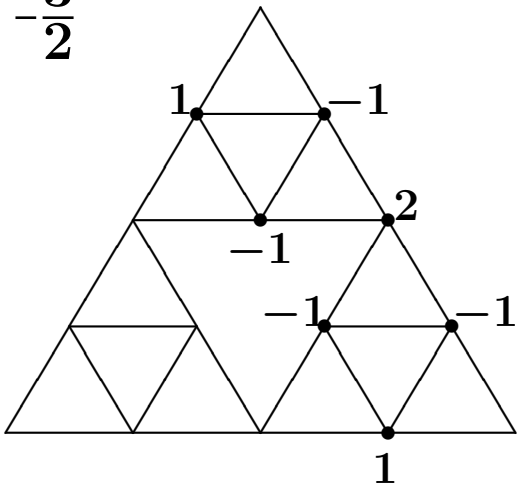
$z = -\frac{3}{2}$



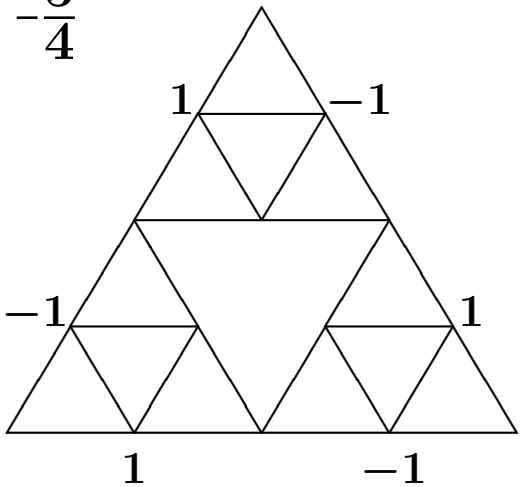
$z = -\frac{3}{4}$



$z = -\frac{3}{2}$



$z = -\frac{5}{4}$



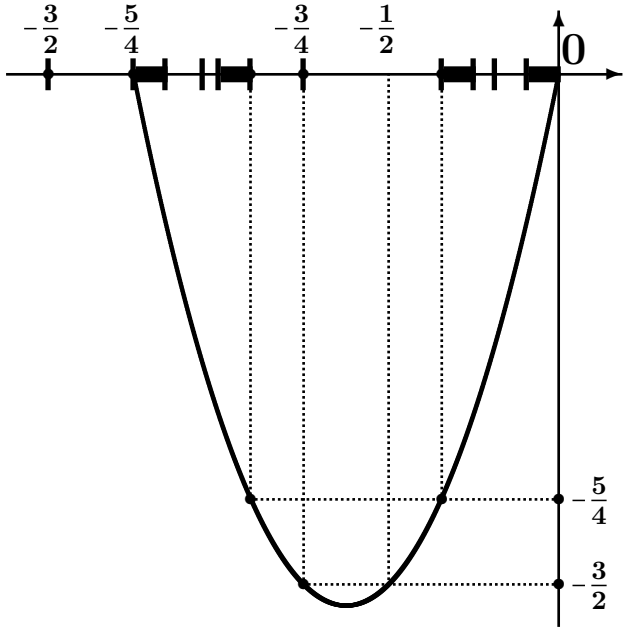
Let $\boldsymbol{\Delta}$ be the probabilistic Laplacian (generator of a simple random walk) on the **Sierpiński lattice**. If $z \neq -\frac{3}{2}, -\frac{5}{4}, -\frac{1}{2}$, and $R(z) = z(4z+5)$, then

$$R(z) \in \sigma(\Delta) \Longleftrightarrow z \in \sigma(\Delta)$$

$$\sigma(\Delta) = \mathfrak{J}_R \bigcup \mathfrak{D}$$

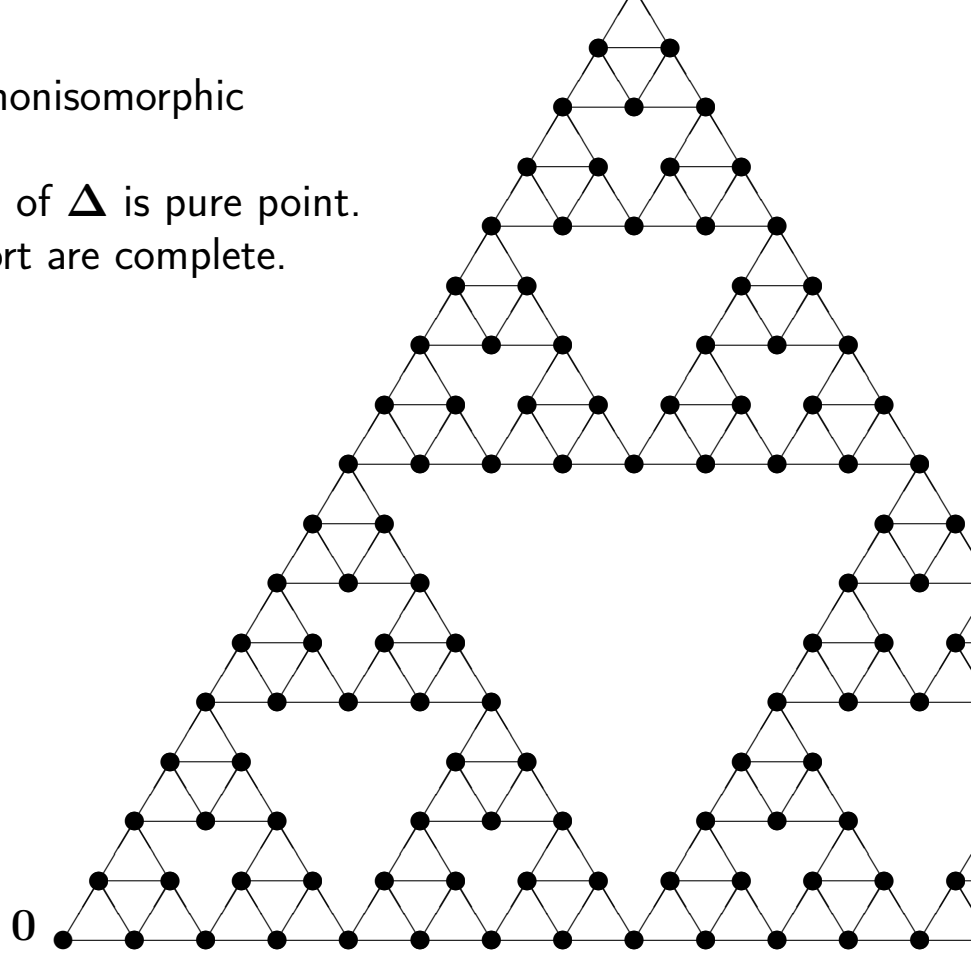
where $\mathfrak{D} \stackrel{\text{def}}{=} \{-\frac{3}{2}\} \bigcup \left(\bigcup_{m=0}^{\infty} R^{-m}\{-\frac{3}{4}\} \right)$

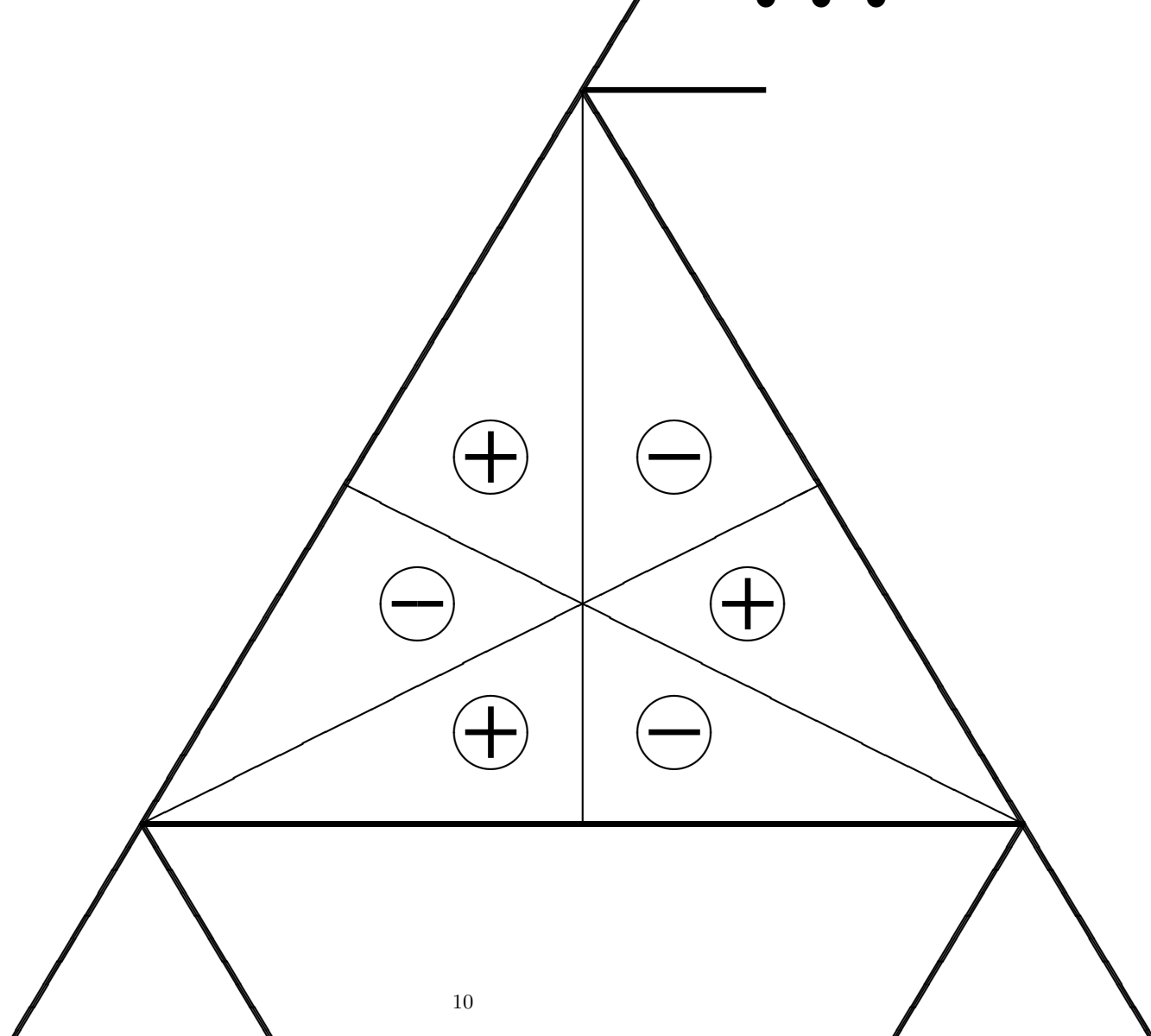
and $\mathfrak{J}_R$ is the Julia set of $R(z)$.



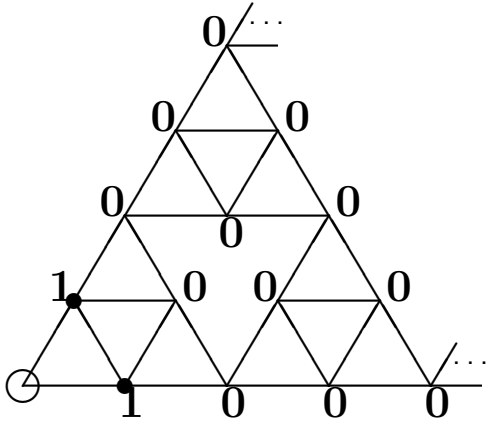
There are uncountably many nonisomorphic Sierpiński lattices.

Theorem (T). The spectrum of $\boldsymbol{\Delta}$ is pure point. Eigenfunctions with finite support are complete.





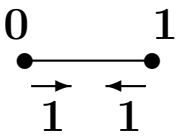
Let $\Delta^{(0)}$ be the Laplacian with zero (Dirichlet) boundary condition at ∂L. Then the compactly supported eigenfunctions of $\Delta^{(0)}$ are **not** complete (eigenvalues in $\mathcal{E}$ is not the whole spectrum).

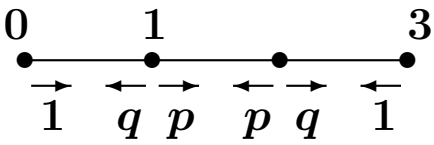


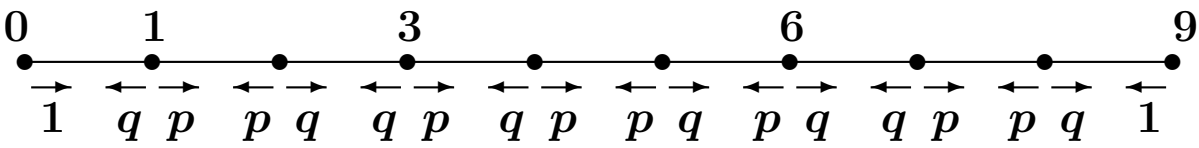
Let $\partial L^{(0)}$ be the set of two points adjacent to ∂L and $\omega_\Delta^{(0)}$ be the spectral measure of $\Delta^{(0)}$ associated with $\mathbb{1}_{\partial L^{(0)}}$. Then $\mathrm{supp}(\omega_\Delta^{(0)}) = \mathcal{J}_R$ has Lebesgue measure zero and

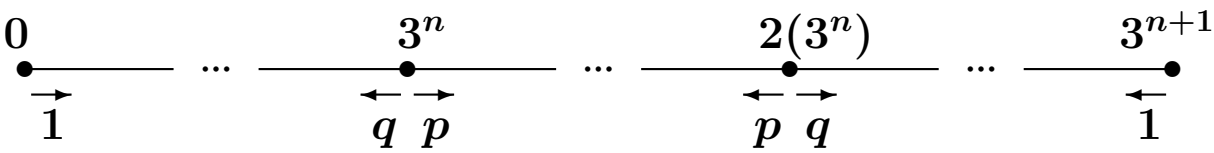
$$\frac{d(\omega_\Delta^{(0)} \circ R_{1,2})}{d\omega_\Delta^{(0)}}(z) = \frac{(8z+5)(2z+3)}{(2z+1)(4z+5)}$$

Fix $p, q > 0$, $p + q = 1$, and define probabilistic Laplacians Δ_n on the segments $[0, 3^n]$ of $\mathbb{Z}_+$ inductively as a generator of the random walks:



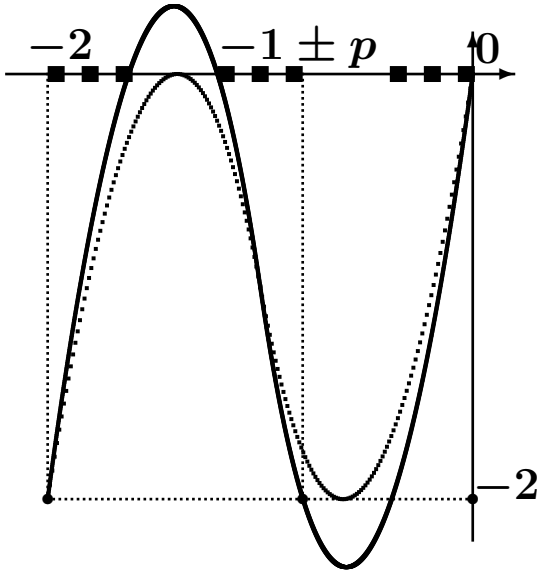






and let $\Delta = \lim_{n \to \infty} \Delta_n$ be the corresponding probabilistic Laplacian on $\mathbb{Z}_+$.

If $z \neq -1 \pm p$ and $R(z) = z(z^2+3z+2+pq)/pq$, then $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$

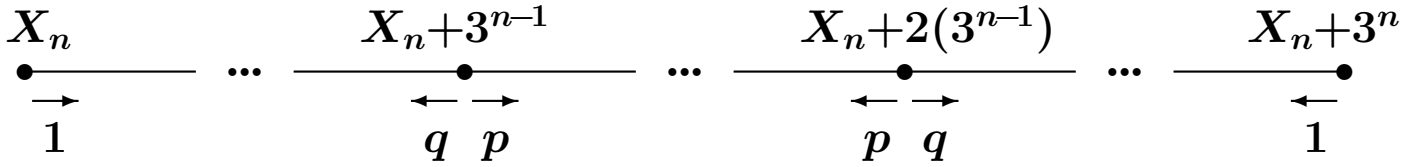


Theorem (Joe Chen + T, almost completed). $\sigma(\Delta) = \mathcal{J}_R$, the Julia set of $R(z)$.
If $p=q$, then $\sigma(\Delta)=[-2, 0]$, spectrum is a.c.
If $p \neq q$, then $\sigma(\Delta)$ is a Cantor set of Lebesgue measure zero, spectrum is singularly continuous.

There are uncountably many "random" self-similar Laplacians $\boldsymbol{\Delta}$ on $\mathbb{Z}$:
For a sequence $\mathcal{K} = \{k_j\}_{j=1}^{\infty}$, $k_j \in \{0, 1, 2\}$, let

$$X_n = -\sum_{j=1}^{n} k_j 3^j$$

and $\boldsymbol{\Delta}_n$ is a probabilistic Laplacian on $[X_n, X_n + 3^n]$:

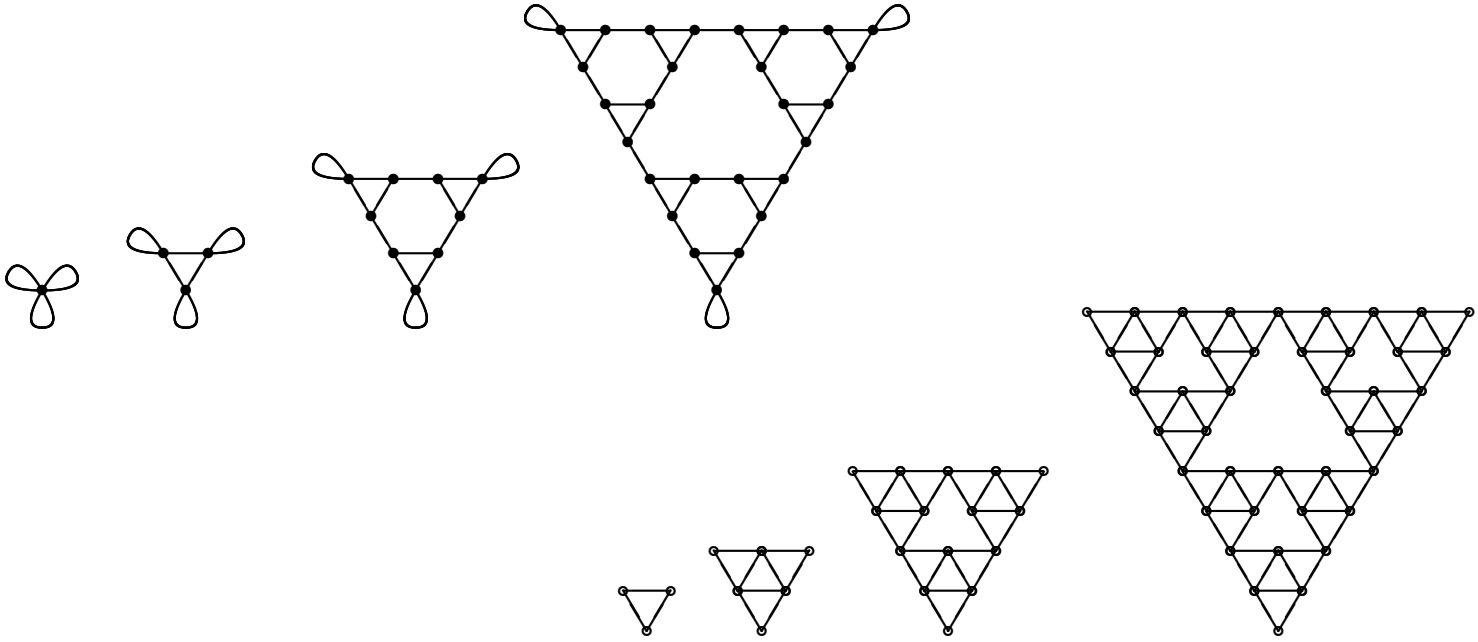


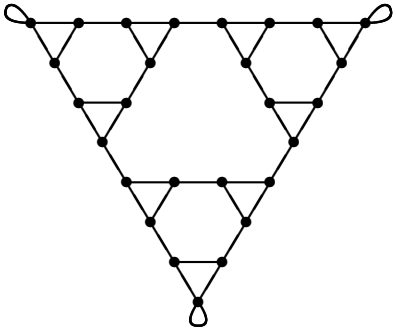
In the previous example $k_j = 0$ for all j.

Theorem.

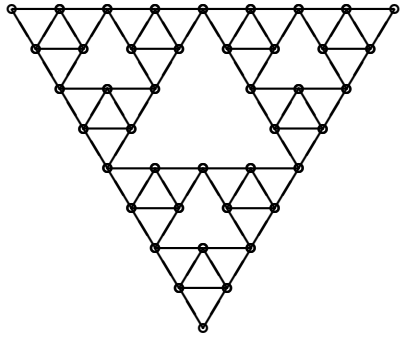
For any sequence $\mathcal{K}$ we have $\sigma(\boldsymbol{\Delta}) = \mathcal{J}_R$. The same is true for the Dirichlet Laplacian on $\mathbb{Z}_+$ (when $k_j \equiv 0$).

R. Grigorchuk and Z. Sunik, *Asymptotic aspects of Schreier graphs and Hanoi Towers groups,* preprint.





Sierpiński 3-graph
(Hanoi Towers-3 group)



Sierpiński 4-graph
(standard)

These three polynomials are conjugate:

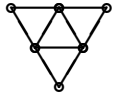
Sierpiński 3-graph (Hanoi Towers-3 group): $f(x) = x^2 - x - 3$
$f(3) = 3$, $f'(3) = 5$

Sierpiński 4-graph, "adjacency matrix" Laplacian: $P(\lambda) = 5\lambda - \lambda^2$
$P(0) = 0$, $P'(0) = 5$

Sierpiński 4-graph, probabilistic Laplacian: $R(z) = 4z^2 + 5z$
$R(0) = 0$, $R'(0) = 5$

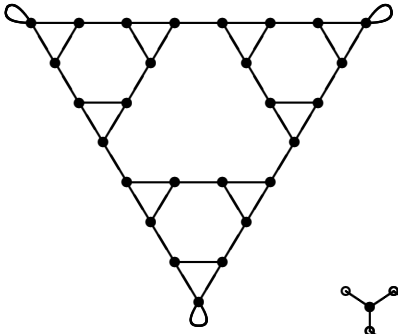
Theorem. Eigenvalues and eigenfunctions on the Sierpiński 3-graphs and Sierpiński 4-graphs are in one-to-one correspondence, with the exception of the eigenvalue $z = -\frac{3}{2}$ for the 4-graphs.

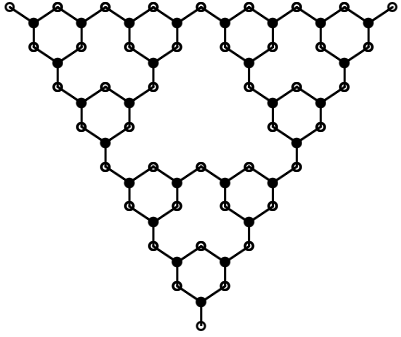
$$\xleftarrow[4z^2+5z]{}$$

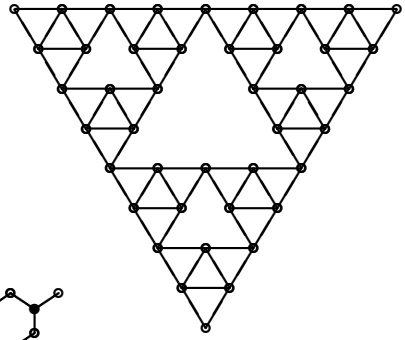


$$\xleftarrow[\frac{4}{3}z^2+\frac{8}{3}z]{}$$

$$\xleftarrow[2z^2+4z]{}$$

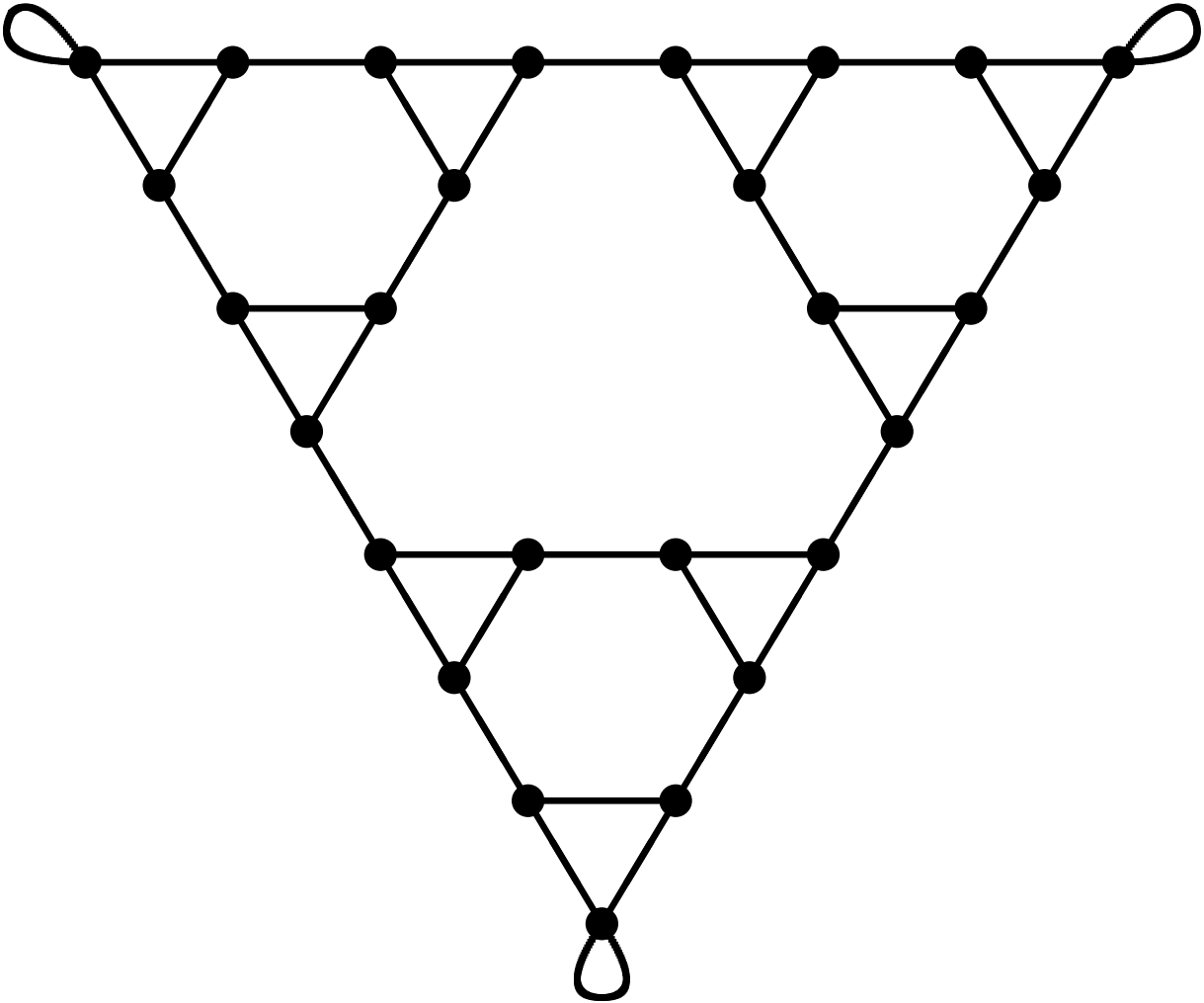


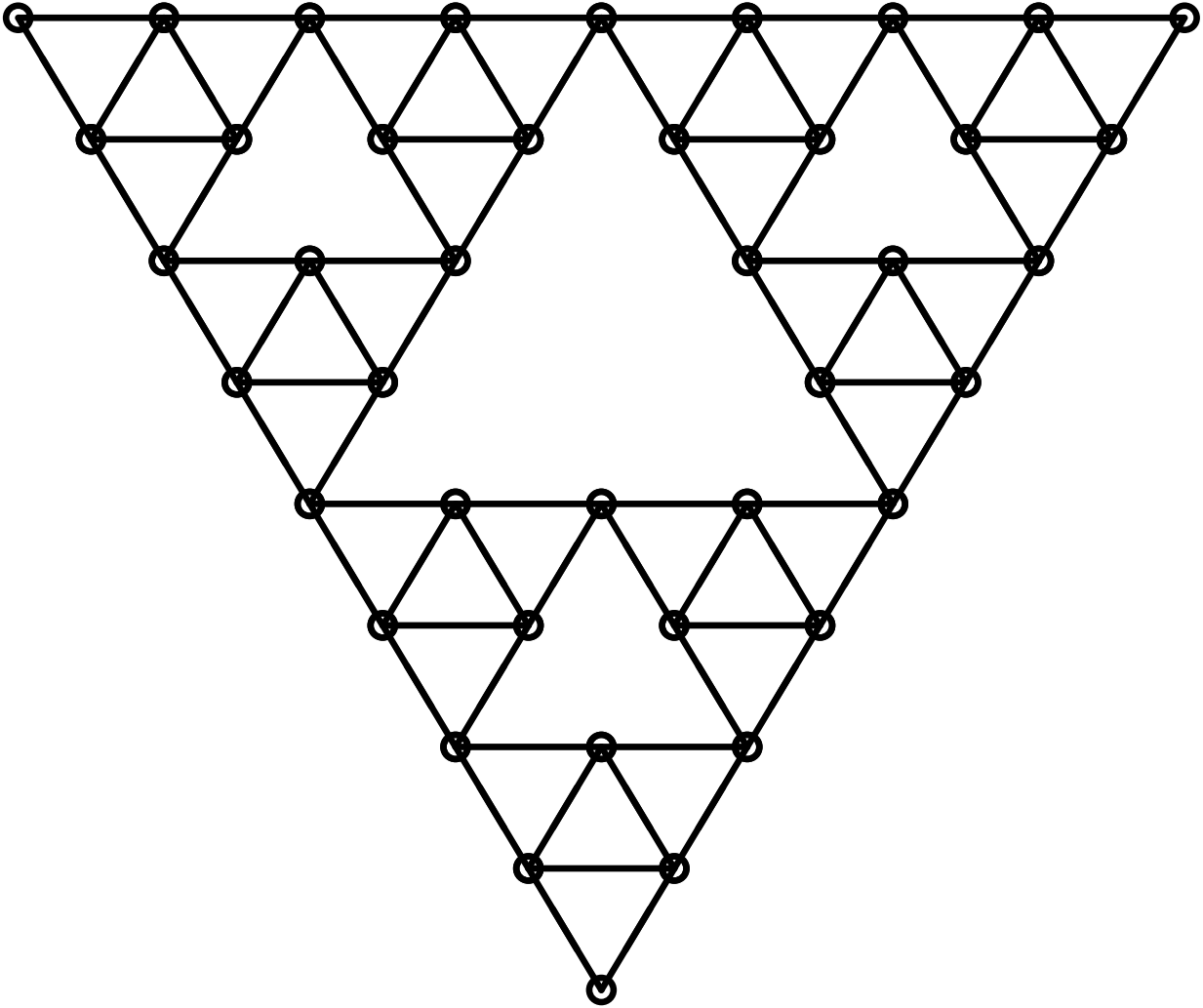


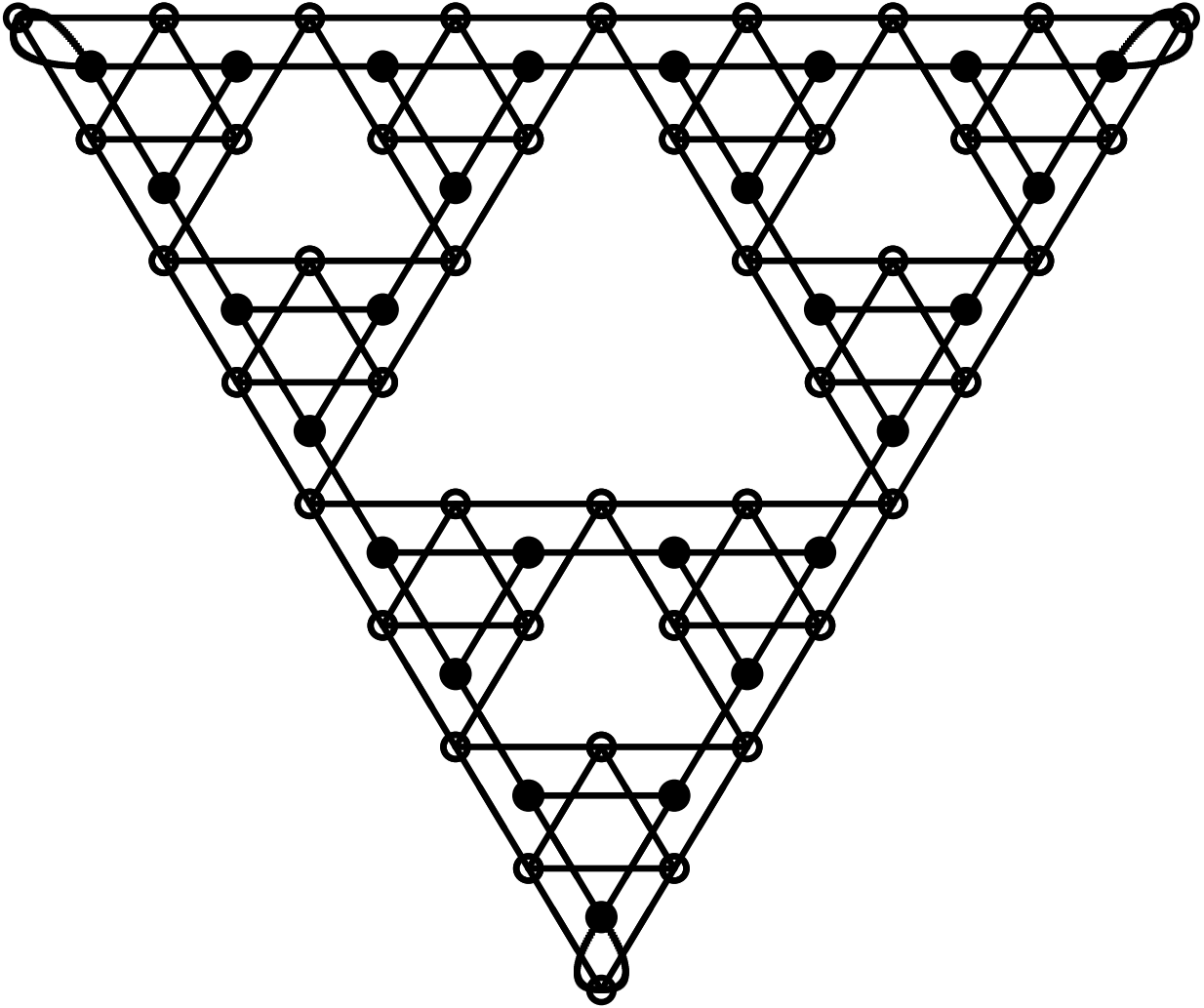


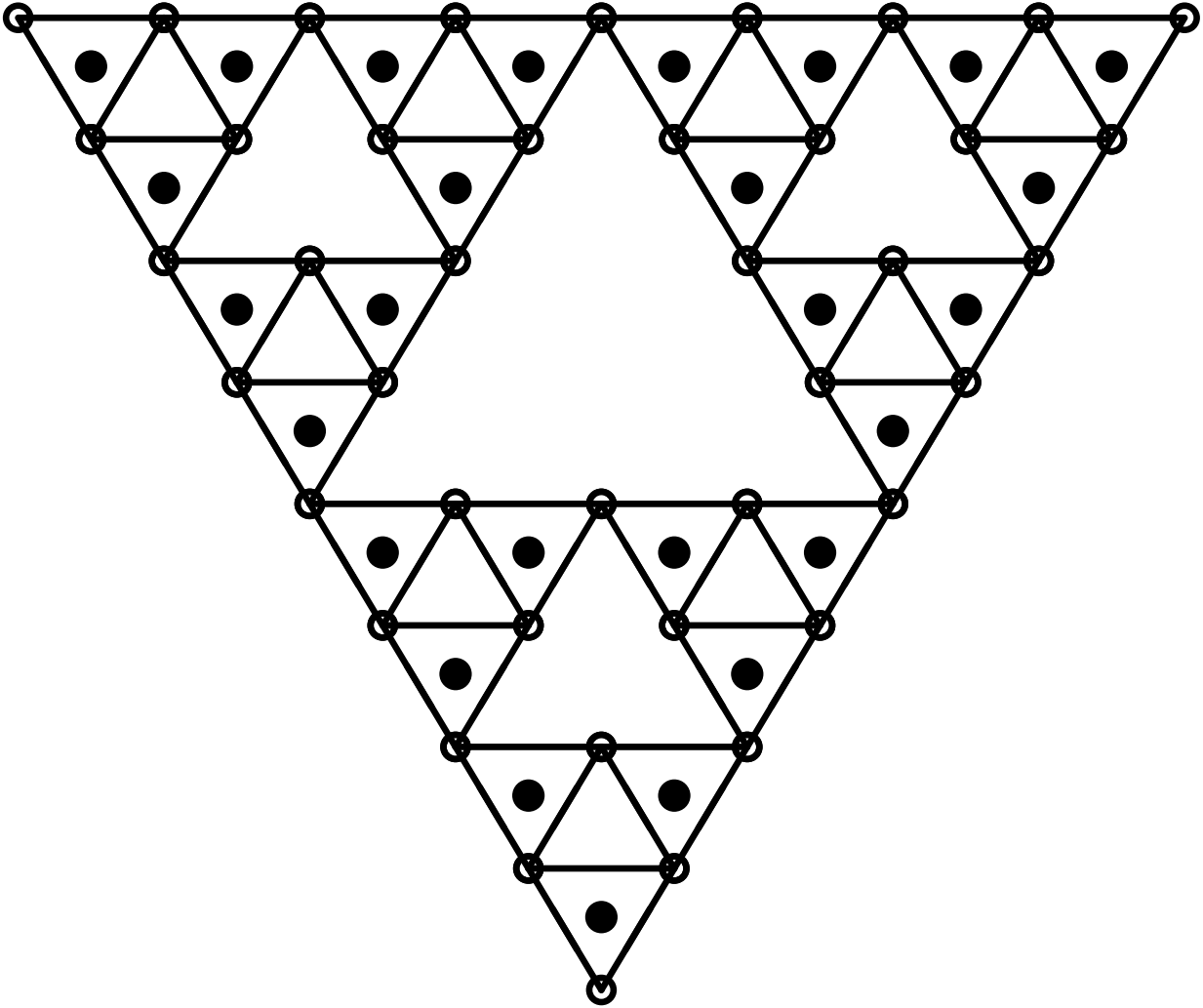
Sierpiński 3-graph
(Hanoi Towers-3 group)
$\boldsymbol{R(z) = 2z^2 + 4z}$

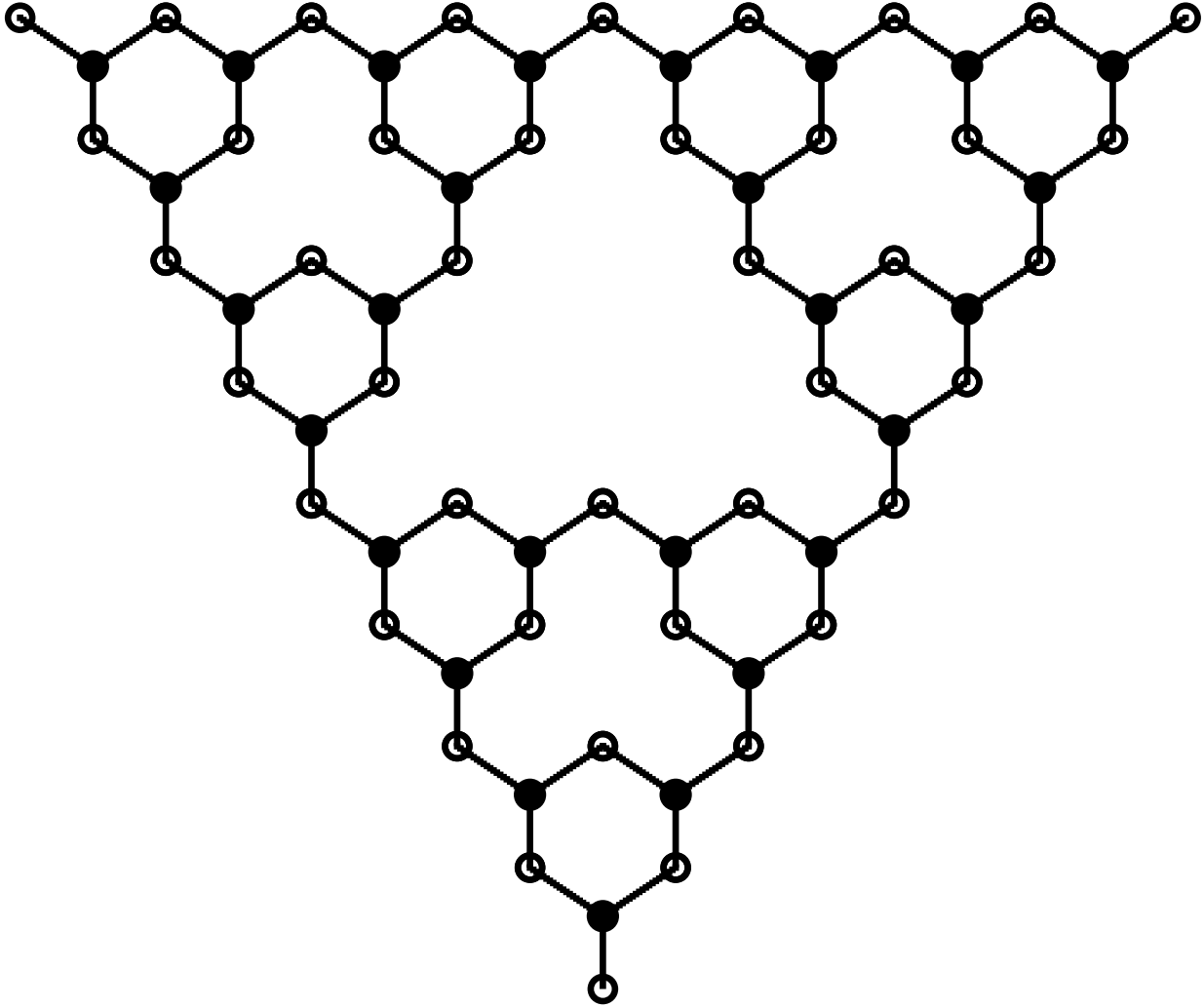
Sierpiński 4-graph
(standard)
$\boldsymbol{R(z) = \frac{4}{3}z^2 + \frac{8}{3}z}$

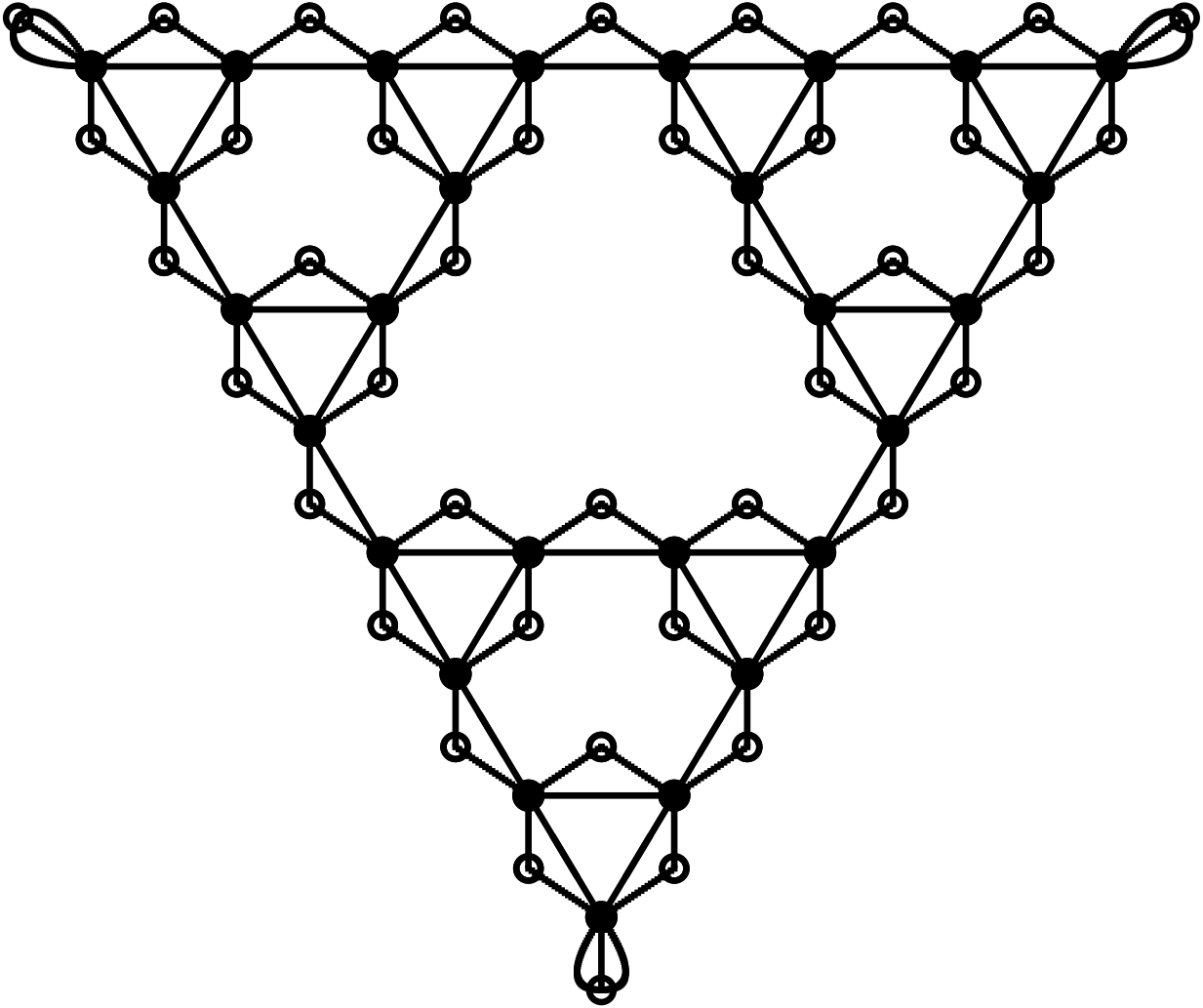


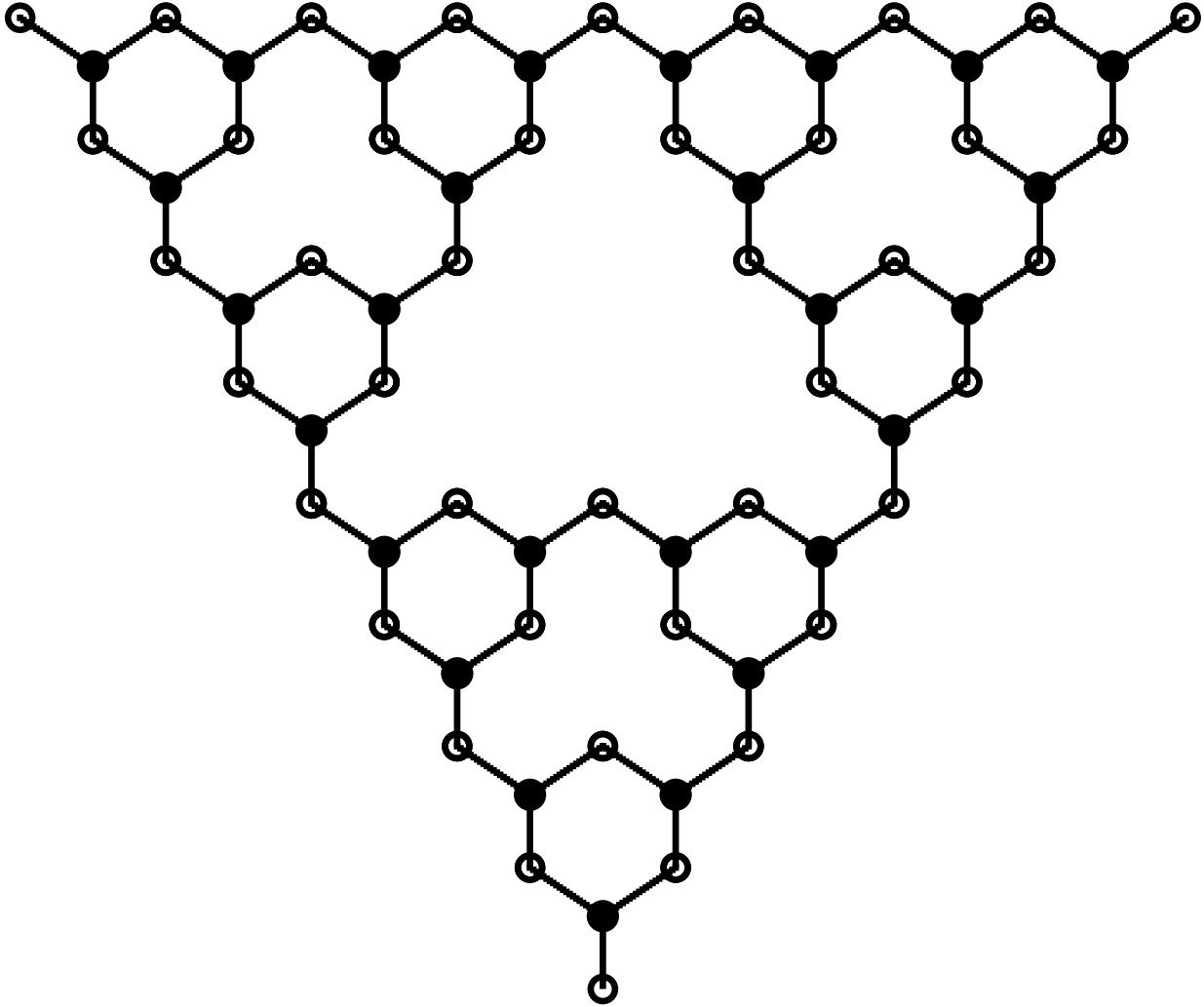


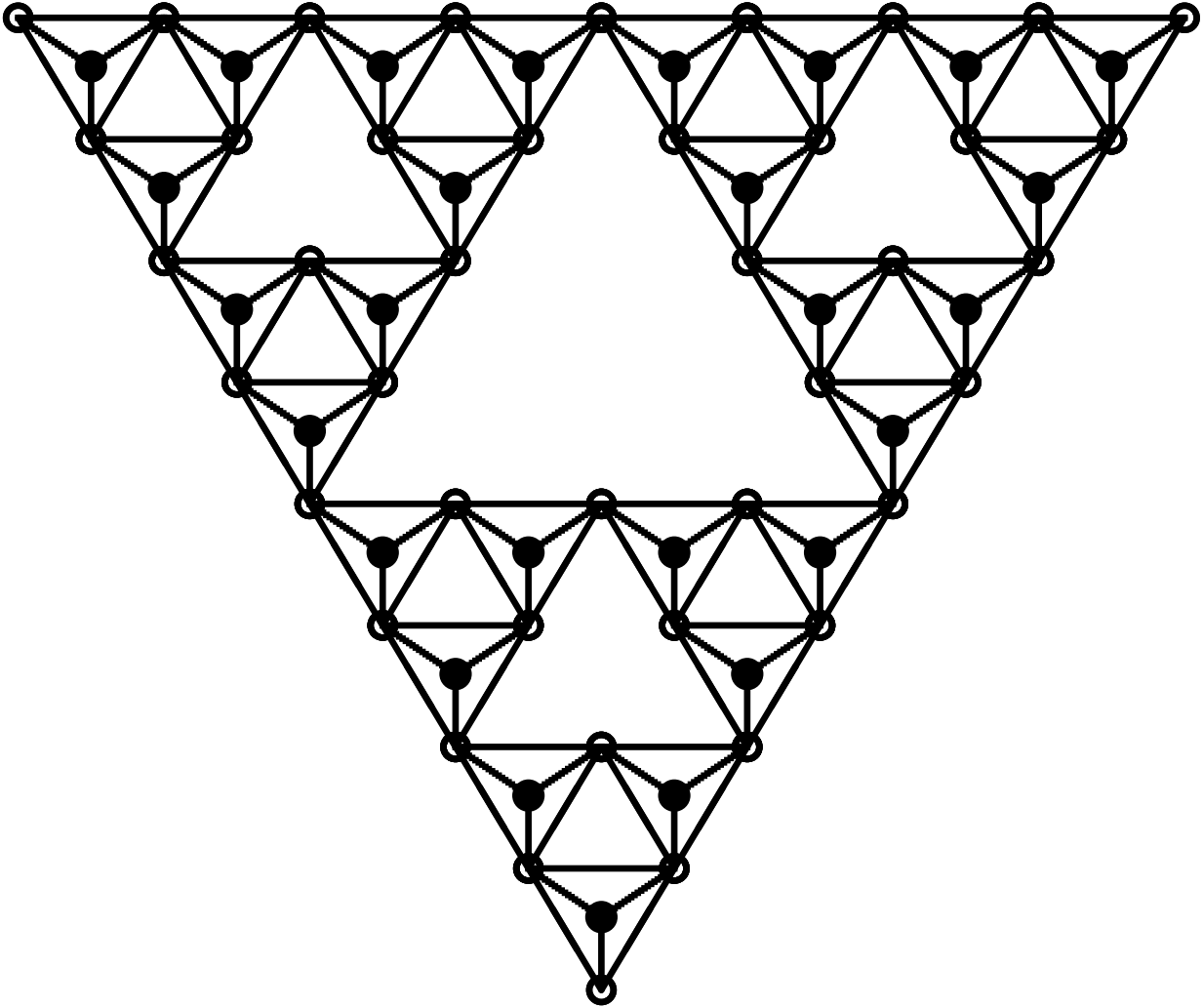












Let $\mathcal{H}$ and $\mathcal{H}_0$ be Hilbert spaces, and $U : \mathcal{H}_0 \longrightarrow \mathcal{H}$ be an isometry.

Definition. We call an operator H **spectrally similar** to an operator H_0 with functions φ_0 and φ_1 if

$$U^*(H - z)^{-1}U = (\varphi_0(z)H_0 - \varphi_1(z))^{-1}$$

In particular, if $\varphi_0(z) \neq 0$ and $R(z) = \varphi_1(z)/\varphi_0(z)$, then

$$U^*(H - z)^{-1}U = \frac{1}{\varphi_0(z)}(H - R(z))^{-1}.$$

If $H = \begin{pmatrix} S & \bar{X} \\ X & Q \end{pmatrix}$ then

$$S - zI_0 - \bar{X}(Q - zI_1)^{-1}X = \varphi_0(z)H_0 - \varphi_1(z)I_0$$

Theorem (Malozemov, Teplyaev). If Δ is the graph Laplacian on a self-similar symmetric infinite graph, then

$$\mathcal{J}_R \subseteq \sigma(\Delta_\infty) \subseteq \mathcal{J}_R \cup \mathcal{D}_\infty$$

where $\mathcal{D}_\infty$ is a discrete set and $\mathcal{J}_R$ is the Julia set of the rational function R.

Abbreviated list of references

[1] S. Alexander, *Some properties of the spectrum of the Sierpiński gasket in a magnetic field.* Phys. Rev. B **29** (1984), 5504-5508.

[2] Bartholdi, Laurent ; W. Woess, *Spectral computations on lamplighter groups and Diestel-Leader graphs.* J. Fourier Anal. Appl. **11** (2005), 175–202.

[3] J. Béllissard, *Renormalization group analysis and quasicrystals,* Ideas and methods in quantum and statistical physics (Oslo, 1988), 118–148. Cambridge Univ. Press, Cambridge, 1992.

[4] S. Brofferio, W. Woess, *Green kernel estimates and the full Martin boundary for random walks on lamplighter groups and Diestel-Leader graphs.* Ann. Inst. H. Poincare' Probab. Statist. **41** (2005), 1101–1123.

[5] S. Brofferio, W. Woess, *Positive harmonic functions for semi-isotropic random walks on trees, lamplighter groups, and DL-graphs.* Potential Anal. **24** (2006), 245–265.

[6] E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984), 3110–3123.

[7] Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983), 1267–1278; **17** (1984), 435–444 and 1277–1289.

[8] S. Goldstein, *Random walks and diffusions on fractals,* in "Percolation Theory and Ergodic Theory of Infinite Particle Systems" (H.Kesten, ed.), 121–129, IMA Math. Appl., Vol. 8, Springer, New York, 1987.

[9] P. Grabner, *Functional iterations and stopping times for Brownian motion on the Sierpiński gasket.* Mathematika **44** (1997), 374–400.

[10] R. Grigorchuk and Z. Sunik, *Asymptotic aspects of Schreier graphs and Hanoi Towers groups,* preprint.

[11] V. A. Kaĭmanovich and A. M. Vershik, *Random walks on discrete groups: boundary and entropy.* Ann. Probab., **11** (1983), 457–490.

[12] B. Krön, *Green functions on self-similar graphs and bounds for the spectrum of the Laplacian.* Ann. Inst. Fourier (Grenoble) **52** (2002), 1875–1900.

[13] B. Krön, *Growth of self-similar graphs.* J. Graph Theory **45** (2004), 224–239.

[14] B. Krön and E. Teufl, *Asymptotics of the transition probabilities of the simple random walk on self-similar graph,* Trans. Amer. Math. Soc., **356** (2003) 393–414.

[15] T. Lindstrøm, *Brownian motion on nested fractals.* Mem. Amer. Math. Soc. **420**, 1989.

[16] L. Malozemov and A. Teplyaev, *Pure point spectrum of the Laplacians on fractal graphs.* J. Funct. Anal. **129** (1995), 390–405.

[17] L. Malozemov and A. Teplyaev, *Self-similarity, operators and dynamics.* Math. Phys. Anal. Geom. **6** (2003), 201–218.

[18] B. B. Mandelbrot, *Hasards et tourbillons: quatre contes à clef.* Annales des Mines, November 1975, 61–66.

[19] B. B. Mandelbrot, *Les Objets Fractals. Forme, Hasard et Dimension.* Nouvelle Bibliothèque Scientifique. Flammarion, Editeur, Paris, 1975.

[20] B. B. Mandelbrot, *Fractals: form, chance, and dimension.* Translated from the French. Revised edition. W. H. Freeman and Co., San Francisco, Calif., 1977.

[21] B. B. Mandelbrot, *The Fractal Geometry of Nature.* W. H. Freeman and Co., San Francisco, Calif., 1982.

[22] G. Musser, *Wireless communications,* Scientific American, July 1999.

[23] C. Puente-Baliarda, J. Romeu., Ra. Pous and A. Cardama, *On the behavior of the Sierpinski multiband fractal antenna.* IEEE Trans. Antennas and Propagation **46** (1998), 517–524.

[24] R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984), 191–206.

[25] R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983), L13–L22.

[26] C. Sabot, *Pure point spectrum for the Laplacian on unbounded nested fractals.* J. Funct. Anal. **173** (2000), 497–524.

[27] C. Sabot, *Integrated density of states of self-similar Sturm-Liouville operators and holomorphic dynamics in higher dimension.* Ann. Inst. H. Poincaré Probab. Statist. **37** (2001), 275–311.

[28] L. Rogers, R. Strichartz and A. Teplyaev, *Smooth bumps, a Borel theorem and partitions of unity on p.c.f. fractals* preprint.

[29] C. Sabot, *Existence and uniqueness of diffusions on finitely ramified self-similar fractals.* Ann. Sci. École Norm. Sup. (4) **30** (1997), 605–673.

[30] C. Sabot, *Electrical networks, symplectic reductions, and application to the renormalization map of self-similar lattices.* J. Physique Letters **44** (1983), L13–L22. Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot, Part 1. Proceedings of Symposia in Pure Mathematics **72**, Amer. Math. Soc., (2004), 155–205.

[31] T. Shima, *On eigenvalue problems for Laplacians on p.c.f. self-similar sets,* Japan J. Indust. Appl. Math. **13** (1996), 1–23.

[32] R. B. Stinchcombe, *Fractals, phase transitions and criticality.* Fractals in the natural sciences. Proc. Roy. Soc. London Ser. A **423** (1989), 17–33.

[33] R. S. Strichartz, *The Laplacian on the Sierpinski gasket via the method of averages.* Pacific J. Math. **201** (2001), 241–256.

[34] A. Teplyaev, *Spectral Analysis on Infinite Sierpiński Gaskets,* J. Funct. Anal., **159** (1998), 537-567.

[35] E. Teufl, *The average displacement of the simple random walk on the Sierpiński graph.* Combin. Probab. Comput. **12** (2003), 203–222.

[36] W. Woess, *A note on the norms of transition operators on lamplighter graphs and groups.* Internat. J. Algebra Comput. **15** (2005), 1261–1272.

[37] W. Woess, *Lamplighters, Diestel-Leader graphs, random walks, and harmonic functions.* Combin. Probab. Comput. **14** (2005), 415–433.

[38] W. Woess, *Random walks on infinite graphs and groups.* Cambridge Tracts in Mathematics, **138**. Cambridge University Press, Cambridge, 2000.

Lecture 2
Laplacians on self-similar fractals and spectral zeta functions

Three **contractions** $F_1, F_2, F_3 : \mathbb{R}^1 \to \mathbb{R}^1$, $F_j(x) = \frac{1}{3}(x+p_j)$, with fixed points $p_j = 0, \frac{1}{2}, 1$. The interval $I=[0,1]$ is a unique compact set such that

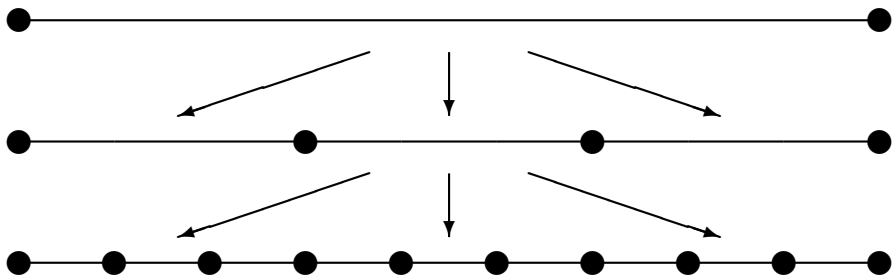
$$I = \bigcup_{j=1,2,3} F_j(I)$$

The ***boundary*** of I is $\partial I = V_0 = \{0,1\}$ and the ***discrete approximations*** to I are $V_n = \bigcup_{j=1,2,3} F_j(V_{n-1}) = \left\{\frac{k}{3^n}\right\}_{k=0}^{3^n}$

$V_0 = \partial I$:

V_1 :

V_2 :



Definition. The ***discrete Dirichlet (energy) form*** on V_n is

$$\mathcal{E}_n(f) = \sum_{\substack{x,y \in V_n \\ y \sim x}} (f(y) - f(x))^2$$

and the ***Dirichlet (energy) form*** on I is $\mathcal{E}(f) = \lim_{n \to \infty} 3^n \mathcal{E}_n(f) = \int_0^1 |f'(x)|^2 dx$

Definition. A function h is ***harmonic*** if it minimizes the energy given the boundary values.

Proposition. $3\mathcal{E}_{n+1}(f) \geqslant \mathcal{E}_n(f)$ and $3\mathcal{E}_{n+1}(h) = \mathcal{E}_n(h) = 3^{-n}\mathcal{E}(h)$ for a harmonic h.

Proposition. The Dirichlet (energy) form on I is ***self-similar*** in the sense that

$$\mathcal{E}(f) = 3 \sum_{j = 1, 2, 3} \mathcal{E}(f \circ F_j)$$

Definition. The *discrete Laplacians* on V_n are

$$\Delta_n f(x) = \frac{1}{2} \sum_{\substack{y \in V_n \\ y \sim x}} f(y) - f(x), \quad x \in V_n \setminus V_0$$

and the Laplacian on I is $\Delta f(x) = \lim_{n \to \infty} 9^n \Delta_n f(x) = f''(x)$

Gauss–Green (integration by parts) formula:

$$\mathcal{E}(f) = -\int_0^1 f \Delta f dx + f f' \Big|_0^1$$

Spectral asymptotics: Let $\rho(\lambda)$ be the *eigenvalue counting function* of the Dirichlet or Neumann Laplacian Δ:

$$\rho(\lambda) = \#\{j : \lambda_j < \lambda\}.$$

Then

$$\lim_{\lambda \to \infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} = \frac{1}{\pi}$$

where $d_s = 1$ is the spectral dimension.

Definition. The ***spectral zeta function*** is $\zeta_\Delta(s) = \sum_{\lambda_j \neq 0} (-\lambda_j)^{-s/2}$
Its poles are the ***complex spectral dimensions.***

Let $R(z)$ be a polynomial of degree N such that its Julia set $\mathcal{J}_R \subset (-\infty, 0]$, $R(0) = 0$ and $c = R'(0) > 1$.

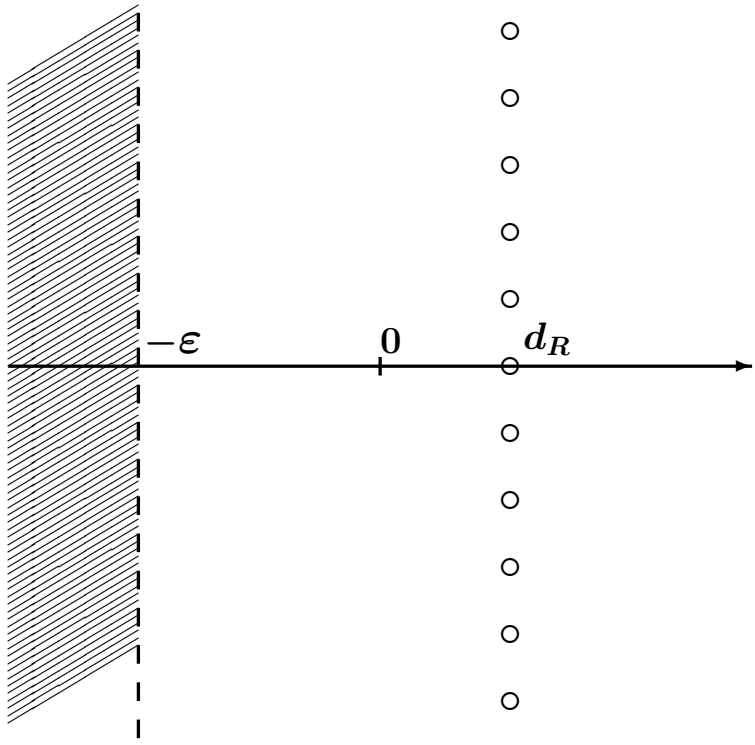
Definition. The ***zeta function of*** $R(z)$ for Re$(s) > d_R = \frac{2 \log N}{\log c}$ is

$$\zeta_R^{z_0}(s) = \lim_{n \to \infty} \sum_{z \in R^{-n}\{z_0\}} (-c^n z)^{-s/2} = \sum \lambda_j^{-s/2}$$

Theorem. $\zeta_R^{z_0}(s) = \dfrac{f_1(s)}{1 - N c^{-s/2}} + f_2^{z_0}(s)$, where $f_1(s)$ and $f_2^{z_0}(s)$ are analytic for Re$(s) > 0$. If $\mathcal{J}_R$ is totally disconnected, then this meromorphic continuation extends to Re$(s) > -\varepsilon$, where $\varepsilon > 0$.

In the case of polynomials this theorem has been improved by Grabner et al.

$$d_R \in \text{the poles of } \zeta_R^{z_0} \subseteq \left\{ \frac{2\log N + 4in\pi}{\log c} : n \in \mathbb{Z} \right\}$$



Theorem. $\zeta_\Delta(s) = \zeta_R^0(s)$ where $R(z) = z(4z^2+12z+9)$.

The Riemann zeta function $\zeta(s)$ satisfies $\zeta(s) = \pi^s \zeta_R^0(s)$ The only complex spectral dimension is the pole at $s = 1$.

<u>A sketch of the proof:</u> If $z \neq -\frac{1}{2}, -\frac{3}{2}$, then

$$R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$$

and so $\zeta_\Delta(s) = \zeta_R^0(s)$ since the eigenvalues λ_j of Δ are limits of the eigenvalues of $9^n \Delta_n$.

Also $\lambda_j = -\pi^2 j^2$ and so

$$\zeta_\Delta(s) = \sum_{j=1}^{\infty} \left(\pi^2 j^2\right)^{-s/2} = \pi^{-s}\zeta(s)$$

where $\zeta(s)$ is the Riemann zeta function. *Q.E.D.*

$$\zeta(s) = \pi^s \lim_{n\to\infty} \sum_{\substack{z \in R^{-n}\{0\} \\ z \neq 0}} \left(-9^n z\right)^{-s/2}$$

Definition. Δ_μ is μ–*Laplacian* if

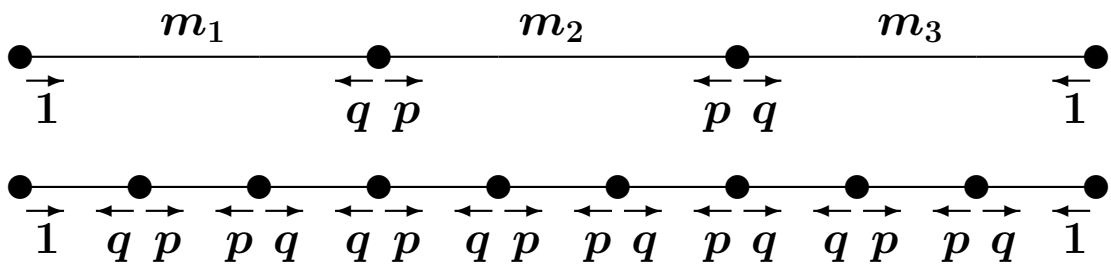
$$\mathcal{E}(f) = \int_0^1 |f'(x)|^2 dx = -\int_0^1 f\Delta_\mu f\, d\mu + f f'\Big|_0^1.$$

Definition. A probability measure μ is *self-similar* with weights m_1, m_2, m_3 if $\mu = \sum_{j=1,2,3} m_j \mu \circ F_j$.

Proposition. $\Delta_\mu f(x) = \frac{f''}{\mu} = \lim_{n\to\infty} \left(1+\frac{2}{pq}\right)^n \Delta_n f(x)$.

$$\Delta_n f\left(\tfrac{k}{3^n}\right) = \begin{cases} p f\left(\frac{k-1}{3^n}\right) + q f\left(\frac{k+1}{3^n}\right) - f\left(\frac{k}{3^n}\right) \\ q f\left(\frac{k-1}{3^n}\right) + p f\left(\frac{k+1}{3^n}\right) - f\left(\frac{k}{3^n}\right) \end{cases}$$

where $m_1 = m_3$, $p = \frac{m_2}{m_1+m_2}$, $q = \frac{m_1}{m_1+m_2}$, and



Spectral asymptotics: If $\rho(\lambda)$ is the eigenvalue counting function of the Dirichlet or Neumann Laplacian Δ_μ, then

$$0 < \liminf_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} \leqslant \limsup_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty$$

where the spectral dimension is

$$d_s = \frac{\log 9}{\log(1+\frac{2}{pq})} \leqslant 1.$$

All the inequalities are strict if and only if $p \neq q$.

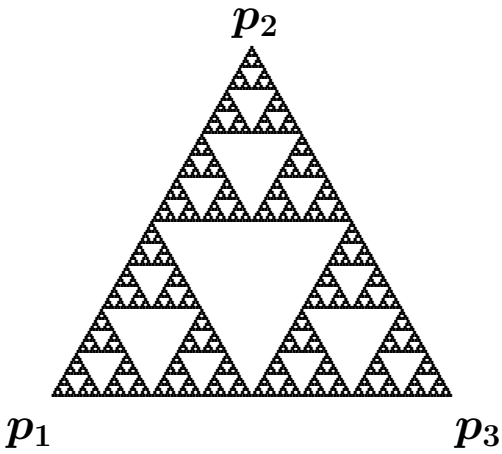
Proposition. $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$

where $z \neq -1 \pm p$ and $R(z) = z(z^2+3z+2+pq)/pq$.

Note that $R'(0) = 1 + \frac{2}{pq}$, and $d_s = d_R$.

Theorem. $\zeta_{\Delta_\mu}(s) = \zeta_R^0(s)$

Three **contractions** $F_1, F_2, F_3 : \mathbb{R}^2 \to \mathbb{R}^2$,
$F_j(x) = \frac{1}{2}(x+p_j)$, with fixed points p_1, p_2, p_3.



The **Sierpiński gasket** is a unique compact set S such that

$$S = \bigcup_{j=1,2,3} F_j(S)$$

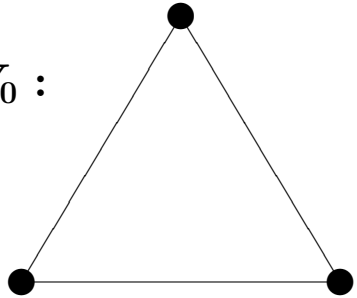
Definition. The ***boundary*** of S is

$$\partial S = V_0 = \{p_1, p_2, p_3\}$$

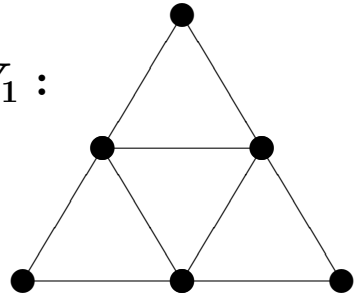
and ***discrete approximations*** to S are

$$V_n = \bigcup_{j=1,2,3} F_j(V_{n-1})$$

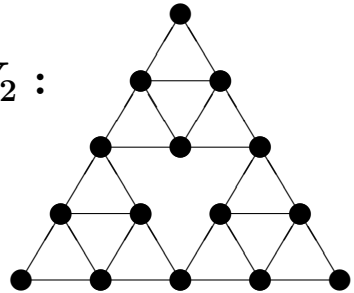
V_0 :



V_1 :



V_2 :



Definition. The ***discrete Dirichlet (energy) form*** on V_n is

$$\mathcal{E}_n(f) = \sum_{\substack{x,y\in V_n \\ y\sim x}} (f(y)-f(x))^2$$

and the ***Dirichlet (energy) form*** on S is

$$\mathcal{E}(f) = \lim_{n\to\infty} \left(\tfrac{5}{3}\right)^n \mathcal{E}_n(f)$$

Definition. A function h is ***harmonic*** if it minimizes the energy given the boundary values.

Proposition. $\frac{5}{3}\mathcal{E}_{n+1}(f) \geqslant \mathcal{E}_n(f)$
$\frac{5}{3}\mathcal{E}_{n+1}(h)=\mathcal{E}_n(h)=\left(\frac{5}{3}\right)^{-n}\mathcal{E}(h)$ for a harmonic h.

Theorem (Kigami). $\mathcal{E}$ is a local regular Dirichlet form on S which is self-similar in the sense that

$$\mathcal{E}(f) = \tfrac{5}{3} \sum_{j\,=\,1,2,3} \mathcal{E}(f\circ F_j)$$

Definition. The *discrete Laplacians* on V_n are

$$\Delta_n f(x) = \frac{1}{4} \sum_{\substack{y \in V_n \\ y \sim x}} f(y) - f(x), \quad x \in V_n \setminus V_0$$

and the Laplacian on S is

$$\Delta_\mu f(x) = \lim_{n \to \infty} 5^n \Delta_n f(x)$$

if this limit exists and $\Delta_\mu f$ is continuous.

Gauss–Green (integration by parts) formula:

$$\mathcal{E}(f) = -\int_S f \Delta_\mu f \, d\mu + \sum_{p \in \partial S} f(p) \partial_n f(p)$$

where μ is the normalized Hausdorff measure, which is self-similar with weights $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$:

$$\mu = \frac{1}{3} \sum_{j = 1, 2, 3} \mu \circ F_j.$$

Spectral asymptotics: If $\rho(\lambda)$ is the eigenvalue counting function of the Dirichlet or Neumann Laplacian Δ_μ, then

$$0 < \liminf_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \limsup_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty$$

where the spectral dimension is

$$1 < d_s = \tfrac{\log 9}{\log 5} < 2.$$

Proposition. $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$ where $z \neq -\frac{1}{2}, -\frac{3}{4}, -\frac{5}{4}$ and $R(z) = z(5+4z)$.

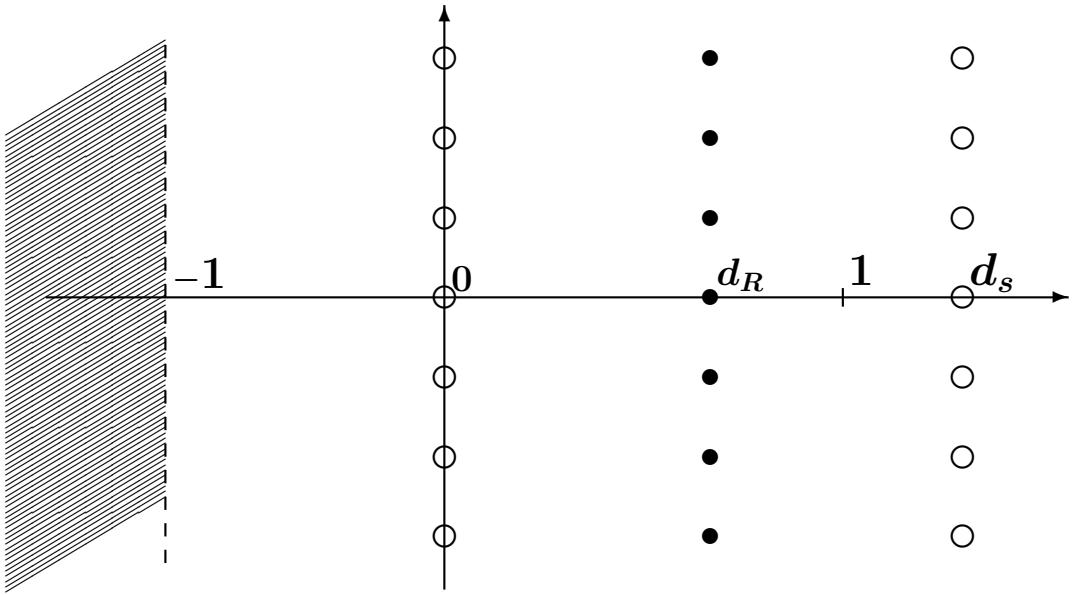
Theorem (Fukushima, Shima). Every eigenvalue of Δ_μ has a form

$$\lambda = 5^m \lim_{n\to\infty} 5^n R^{-n}(z_0)$$

where $R^{-n}(z_0)$ is a preimage of $z_0 = -\frac{3}{4}, -\frac{5}{4}$ under the n-th iteration power of the polynomial $R(z)$. The multiplicity of such an eigenvalue is $C_1 3^m + C_2$.

Theorem. Zeta function of the Laplacian on the Sierpiński gasket is

$$\zeta_{\Delta_\mu}(s) \;=\; \frac{1}{2}\,\zeta_R^{-\frac{3}{4}}(s)\left(\frac{1}{5^{s/2}-3}+\frac{3}{5^{s/2}-1}\right) \;+\; \frac{1}{2}\,\zeta_R^{-\frac{5}{4}}(s)\left(\frac{3\cdot 5^{-s/2}}{5^{s/2}-3}-\frac{5^{-s/2}}{5^{s/2}-1}\right)$$



$$d_s \;=\; \frac{\log 9}{\log 5}$$

$$d_R \;=\; \frac{\log 4}{\log 5}$$

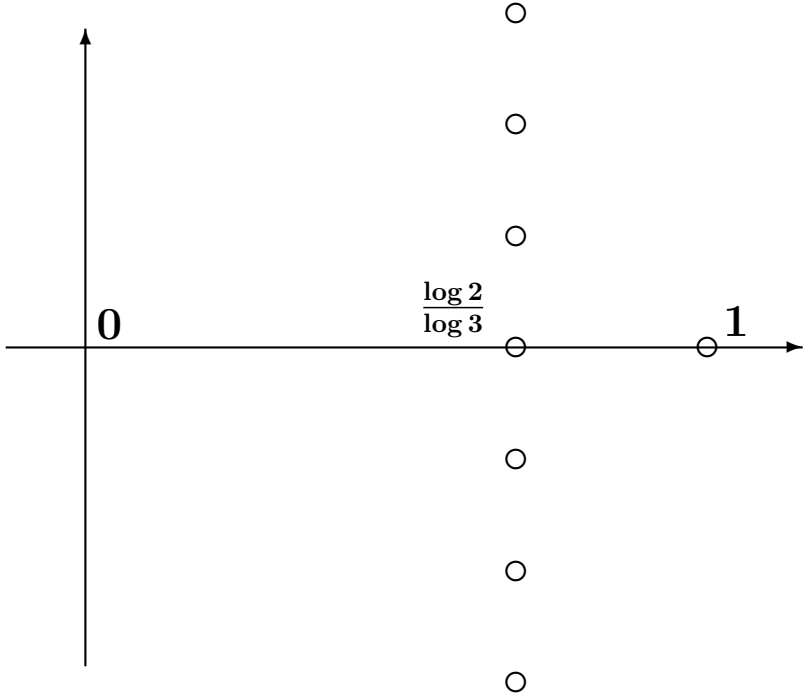
Definition. If $\mathcal{L}$ is a fractal string, that is, a disjoint collection of intervals of lengths l_j, then its *geometric zeta function* is $\zeta_{\mathcal{L}}(s) = \sum l_j^s$.

Theorem (Lapidus). If $A = -\frac{d^2}{dx^2}$ is a Neumann or Dirichlet Laplacian on $\mathcal{L}$, then $\zeta_A(s) = \pi^{-s}\zeta(s)\zeta_{\mathcal{L}}(s)$.

<u>**Example: Cantor self-similar fractal string.**</u>

If $\mathcal{L}$ is the complement of the middle third Cantor set in $[0, 1]$, then the complex spectral dimensions are 1 and $\{\frac{\log 2 + 2in\pi}{\log 3} : n \in \mathbb{Z}\}$,

$$\zeta_{\mathcal{L}}(s) = \frac{1}{1 - 2 \cdot 3^{-s}}, \quad \zeta_A(s) = \zeta(s)\frac{\pi^{-s}}{1 - 2 \cdot 3^{-s}}$$


0
$\frac{\log 2}{\log 3}$
1

Definition. A post critically finite (p.c.f.) self-similar set F is a compact connected metric space with a finite boundary $\partial F \subset F$ and contractive injections $\psi_i : F \to F$ such that

$$F = \Psi(F) = \bigcup_{i=1}^{k} \psi_i(F)$$

and

$$\psi_v(F) \bigcap \psi_w(F) \subseteq \psi_v(\partial F) \bigcap \psi_w(\partial F),$$

for any two different words v and w of the same length. Here for a finite word $w \in \{1, \ldots, k\}^m$ we define $\psi_w = \psi_{w_1} \circ \ldots \circ \psi_{w_m}$.

We assume that ∂F is a minimal such subset of F. We call $\psi_w(F)$ an m-cell. ***The p.c.f. assumption is that every boundary point is contained in a single 1-cell.***

Theorem (Kigami, Lapidus). The spectral dimension of the Laplacian Δ_μ is the unique solution of the equation

$$\sum_{i=1}^{k} (r_i \mu_i)^{d_s/2} = 1$$

Conjecture. On every p.c.f. fractal F there exists a local regular Dirichlet form $\mathcal{E}$ which gives positive capacity to the boundary points and is self-similar in the sense that

$$\mathcal{E}(f) = \sum_{i=1}^{k} \rho_i \mathcal{E}(f \circ \psi_i)$$

for a set of positive refinement weights $\rho = \{\rho_i\}_{i=1}^{k}$.

Definition. The group G of acts on a finitely ramified fractal F if each $g \in G$ is a homeomorphism of F such that $g(V_n) = V_n$ for all $n \geqslant 0$.

Proposition. Suppose a group G of acts on a self-similar finitely ramified fractal F and G restricted to V_0 is the whole permutation group of V_0. Then there exists a unique, up to a constant, G-invariant self-similar resistance form $\mathcal{E}$ with equal energy renormalization weights ρ_i and

$$\mathcal{E}_0(f, f) = \sum_{x,y \in V_0} \big(f(x) - f(y)\big)^2.$$

Moreover, for any G-invariant self-similar measure μ the Laplacian Δ_μ has the spectral self-similarity property (a.k.a. spectral decimation).