Diffusions and spectral analysis on fractals: an overview

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Mathematics - XXI century. PDMI 70th anniversary.
September 14, 2010

Plan:

Introduction and initial motivation

Main classes of fractals considered

Selected results: existence, uniqueness, heat kernel estimates

Selected results: spectral analysis

Open problems and further directions

Selected references

Overview: Diffusions and spectral analysis on fractals are about 25 years old

Analysis and Probability on fractals is a large and diverse area of mathematics, rapidly expanding in new directions. However, the subject is not easy: even the "simplest" case of the "standard" Sierpinski triangle is very difficult. The most basic reason for the difficulty is that we can not differentiate Hölder continuous functions (which can be done much easier for Lipschitz functions).

Three books:

- M. T. Barlow, *Diffusions on fractals.* Lectures on Probability Theory and Statistics (Saint-Flour, 1995), Lecture Notes in Math., **1690**, Springer, 1998.
- J. Kigami, *Analysis on fractals.* Cambridge Tracts in Mathematics **143**, Cambridge University Press, 2001.
- R. S. Strichartz, *Differential equations on fractals: a tutorial.* Princeton University Press, 2006.

Initial motivation

- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984)
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Main early results

Sheldon Goldstein, *Random walks and diffusions on fractals.* Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., 8, Springer

Summary: we investigate the asymptotic motion of a random walker, which at time $\mathbf{n}$ is at $\mathbf{X(n)}$, on certain 'fractal lattices'. For the 'Sierpiński lattice' in dimension $\mathbf{d}$ we show that, as $\mathbf{l} \to \infty$, the process $\mathbf{Y_l(t) \equiv X([(d+3)^l t])/2^l}$ converges in distribution (so that, in particular, $|\mathbf{X(n)}| \sim \mathbf{n}^{\gamma}$, where $\gamma = \mathbf{(\ln 2)/\ln(d+3))}$ to a diffusion on the Sierpin'ski gasket, a Cantor set of Lebesgue measure zero. The analysis is based on a simple 'renormalization group' type argument, involving self-similarity and 'decimation invariance'.

Shigeo Kusuoka, *A diffusion process on a fractal.* Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 1987.

- M.T. Barlow, E.A. Perkins, *Brownian motion on the Sierpinski gasket.* (1988)
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- J. Béllissard, *Renormalization group analysis and quasicrystals,* Ideas and methods in quantum and statistical physics (Oslo, 1988) Cambridge Univ. Press, 1992.
- M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket.* (1992)
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Main classes of fractals considered

- $[0, 1]$
- Sierpiński gasket
- nested fractals
- p.c.f. self-similar sets, possibly with various symmetries
- finitely ramified self-similar sets, possibly with various symmetries
- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)
- Dirichlet metric measure spaces with heat kernel estimates (DMMS+HKE)

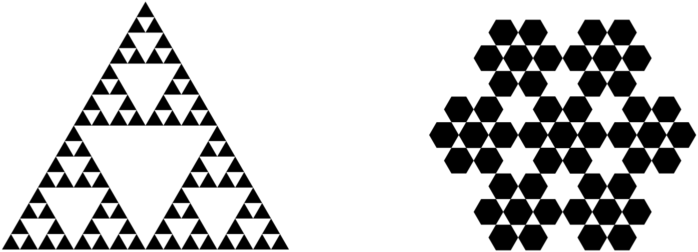


Figure: Sierpiński gasket and Lindstrøm snowfalke (nested fractals), p.c.f., finitely ramified)

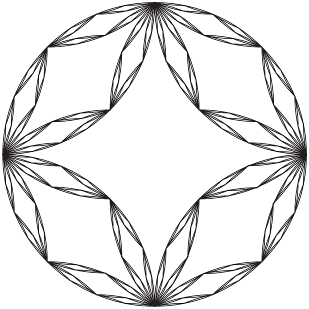


Figure: Diamond fractals, non-p.c.f., but finitely ramified

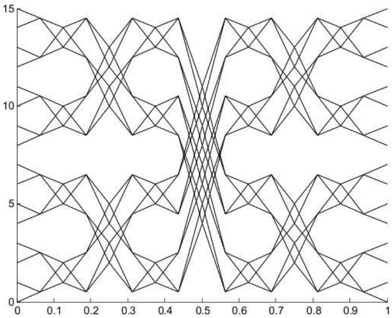


Figure: Laakso Spaces (Ben Steinhurst), infinitely ramified

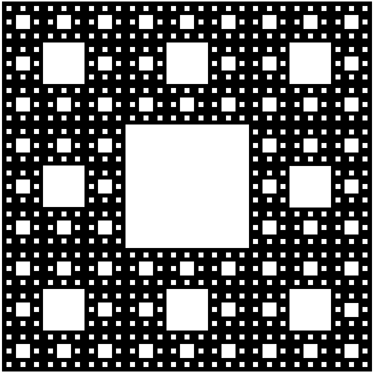


Figure: Sierpiński carpet, infinitely ramified

Selected results: existence, uniqueness, heat kernel estimates

Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

$$\text{distance} \sim \sqrt{\text{time}}$$

"Einstein space–time relation for Brownian motion"

Gaussian transition density :

$$\mathbf{p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)}$$

De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;
Li-Yau (1986) type estimates on a geodesically complete Riemannian manifold with **Ricci** $\geqslant \mathbf{0}$:

$$\mathbf{p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)}$$

$$\mathbf{distance \sim \sqrt{time}}$$

Gaussian:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

Li-Yau Gaussian-type:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

Sub-Gaussian:

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

$$\text{distance} \sim (\text{time})^{\frac{1}{d_w}}$$

Brownian motion on $\mathbb{R}^{\mathbf{d}}$: $\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| = \mathbf{ct^{1/2}}$.

Anomalous diffusion: $\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| = \mathbf{o(t^{1/2})}$, or (in regular enough situations),

$$\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| \approx \mathbf{t^{1/d_w}}$$

with $\mathbf{d_w} > \mathbf{2}$.

Here $\mathbf{d_w}$ is the so-called **walk dimension** (should be called **“walk index”** perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$\text{distance} \sim (\text{time})^{\frac{1}{d_w}}$$

$$\begin{aligned} d_H &= \text{Hausdorff dimension} \\ d_w &= \text{“walk dimension”} \\ \frac{2d_H}{d_w} = d_S &= \text{“spectral dimension”} \end{aligned}$$

First example: **Sierpiński gasket**; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

Theorem. (Barlow, Bass, Kumagai, T. (1989–2010)) On any fractal in the class of generalized Sierpiński carpets there exists a unique, up to a scalar multiple, local regular Dirichlet form that is invariant under the local isometries. Therefore there there is a unique corresponding symmetric Markov process and a unique Laplacian. Moreover, the Markov process is Feller and its transition density satisfies sub-Gaussian heat kernel estimates.

If it is not a cube in $\mathbb{R}^{\mathbf{n}}$, then

- $\mathbf{d_S < d_H,\ d_w > 2}$
- the energy measure and the Hausdorff measure are mutually singular;
- the domain of the Laplacian is not an algebra;
- if $\mathbf{d(x, y)}$ is the shortest path metric, then $\mathbf{d(x,)}$ is not in the domain of the Dirichlet form.

Theorem. (Barlow, Bass, Kumagai (2006)) Under natural assumptions on the metric space with a regular symmetric Dirichlet form, the sub-Gaussian **heat kernel estimates are stable under rough isometries**, *i.e. under maps that preserve distance and energy up to scalar factors.*

- The classical diffusion process was first studied by Einstein, and later a mathematical theory was developed by Wiener, Kolmogorov, Levy et al. One of the basic principle is that displacement in a small time is proportional to the square root of time. This law is related to the properties of the Gaussian transition density and the heat equation.
- On fractals diffusions have to obey scaling laws what are different from the classical Gaussian diffusion, but are of so called sub-Gaussian type. In some situations the diffusion, and therefore the correspondent Laplace operator, is uniquely determined by the geometry of the space.
- As a consequence, there are uniquely defined spectral and walk dimensions, which are related by so called Einstein relation and determine the behavior of the natural diffusion processes by (these dimensions are different from the well known Hausdorff dimension, which describes the distribution of the mass in a fractal).

$$\mathbf{2d_f/d_s = d_f + \tilde{\rho} = d_w}$$

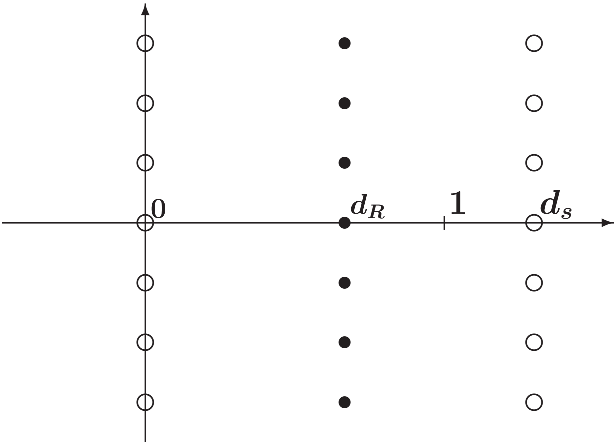
Selected results: spectral analysis

Theorem. (Derfel, Grabner, Vogl, T. (2007–2008)) For a large class of **finitely ramified symmetric fractals**, which includes the Sierpiński gaskets, but does not include the Sierpiński carpets, the spectral zeta function

$$\zeta(s) = \sum \lambda_j^{s/2}$$

has a meromorphic continuation from the half-pain $\mathbf{Re(s)} > \mathbf{d_S}$ to $\mathbb{C}$. Moreover, all the poles and residues are computable from the geometric data of the fractal. Here λ_j are the eigenvalues if the unique symmetric Laplacian.

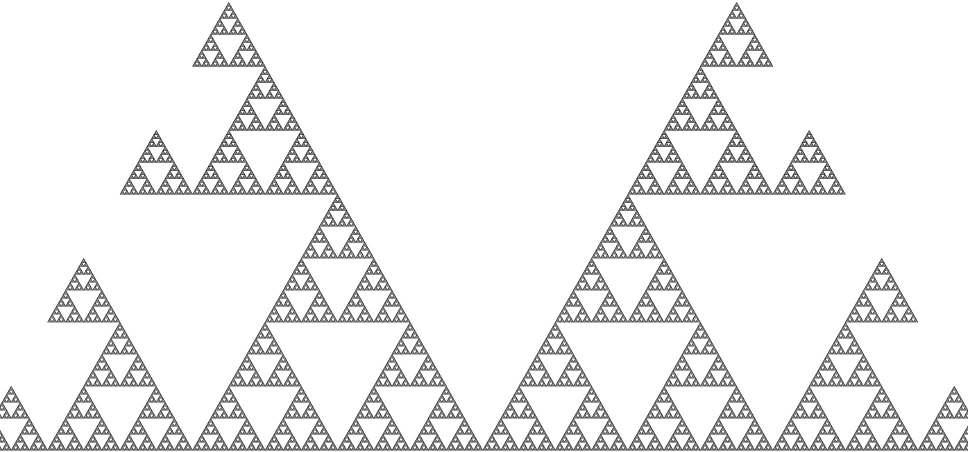
- Example: $\zeta(s)$ is the Riemann zeta function up to a trivial factor in the case when our fractal is $[0, 1]$.
- In more complicated situations, such as the Sierpiński gasket, there are infinitely many non-real poles, which can be called complex spectral dimensions, and are related to oscillations in the spectrum.



$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

Poles (white circles) of the spectral zeta function of the Sierpiński gasket.



A part of an infinite Sierpiński gasket.

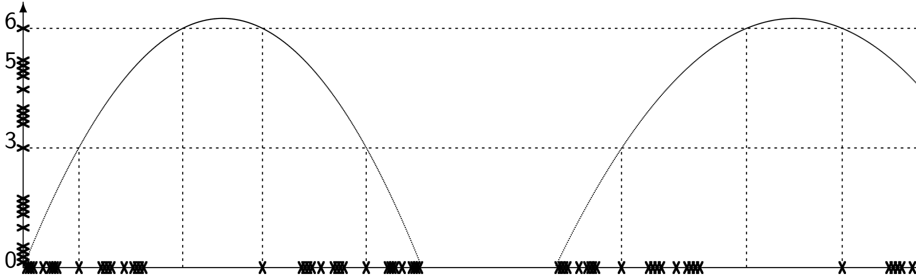
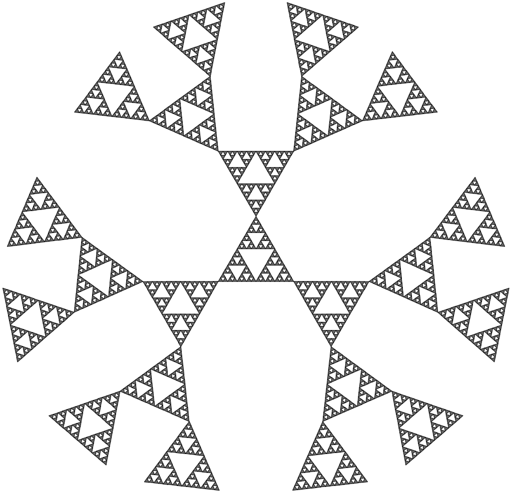
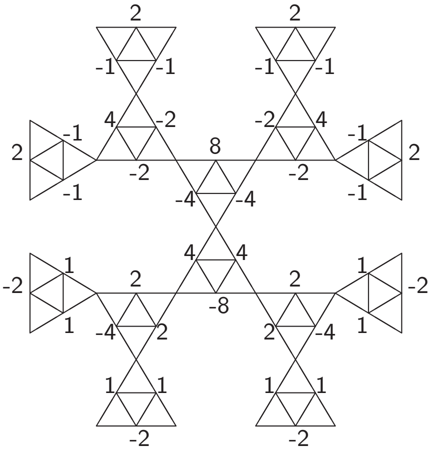


Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

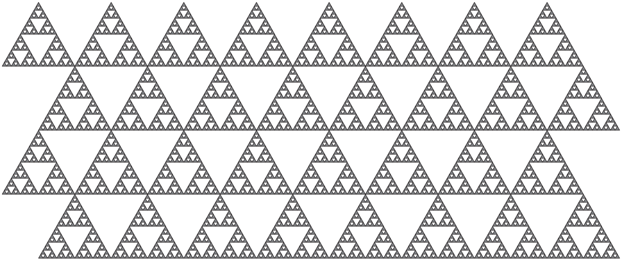
Theorem. (T. 1998, Quint 2009) On the Barlow-Perkins infinite Sierpiński fractafold the spectrum of the Laplacian consists of a **dense set of eigenvalues** $\mathfrak{R}^{-1}(\Sigma_0)$ **of infinite multiplicity** and a **singularly continuous component of spectral multiplicity one supported on** $\mathfrak{R}^{-1}(\mathcal{J}_R)$.

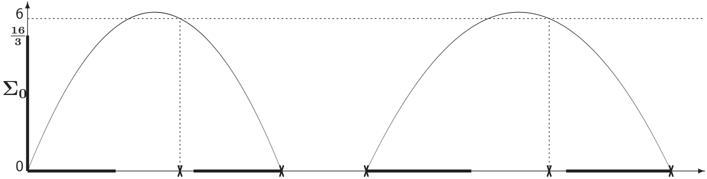


The Tree Fractafold.



An eigenfunction on the Tree Fractafold.





Theorem. (Strichartz, T. 2010) The Laplacian on the periodic triangular lattice finitely ramified Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum. The **absolutely continuous spectrum is $\mathfrak{R}^{-1}[0, \frac{16}{3}]$.** The **pure point spectrum** consists of two infinite series of eigenvalues of infinite multiplicity. The spectral resolution is given in the main theorem.

Open problems

- Existence of self-similar diffusions on finitely ramified fractals (algebraic geometry?)
- on limit sets of self-similar groups (algebra and probability?)
- on any self-similar fractal (noncommutative analysis?)
- Spectral analysis: finitely ramified fractals but with few symmetries, infinitely ramified fractals, such as Julia sets. In particular, does the Laplacian on the Sierpiński carpet has spectral gaps? Meromorphic spectral zeta function?
- Distributions (generalized functions) on DMMS+HKE?
- Derivatives on fractals (even in the simplest case of the Sierpiński gasket are not well defined).
- Differential geometry of fractals?
- PDEs involving derivatives, such as the Navier-Stokes equation.

Further directions

- Mathematical physics, in particular, more general diffusion processes than in Einstein theory, behavior of fractals in magnetic field, Feynman integrals and field theories in general spaces.
- Fractal behavior of processes in algebra and geometry and probabilistic approach to stability under Hölder continuous transformations.
- Computational tools for natural sciences, such as geophysics, chemistry, biology etc.

More on motivations and connections to other areas: Cheeger, Heinonen, Koskela, Shanmugalingam, Tyson

- J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces,* Geom. Funct. Anal. **9** (1999)
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- J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*. J. Anal. Math. 85 (2001)
  In this paper the authors give a definition for the class of Sobolev functions from a metric measure space into a Banach space. They characterize Sobolev classes and study the absolute continuity in measure of Sobolev mappings in the "borderline case". Specifically, the authors prove that the validity of a Poincaré inequality for mappings of a metric space is independent of the target Banach space; they obtain embedding theorems and *Lipschitz approximation* of Sobolev functions; they also prove that pseudomonotone Sobolev mappings in the "borderline case" are absolutely continuous in measure, which is a generalization of the existing results by Y. G. Reshetnyak [Sibirsk. Mat. Zh. 28 (1987)] and by J. Malý and O. Martio [J. Reine Angew. Math. 458 (1995)]. The authors show that quasisymmetric homeomorphisms belong to a Sobolev space of borderline degree. The work in this paper was partially motivated by questions in the theory of quasiconformal mappings in metric spaces.

More on possible connections to other areas

- Works of Barhtoldi, Grigorchuk, Nekrashevich, Kaimanovich, Virag on self-similar groups.

  Possible relation to formal languages and Noam Chomsky hierarchy.

  Epstein, David B. A.; Cannon, James W.; Holt, Derek F.; Levy, Silvio V. F.; Paterson, Michael S.; Thurston, William P. Word processing in groups. Jones and Bartlett Publishers, Boston, MA, 1992.

- Works on random structures, and on various random networks in computer science.

- Relation to infinite dimensional analysis, probability, differential geometry.

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- R. S. Strichartz and M. Usher, *Splines on fractals*. Math. Proc. Cambridge Philos. Soc. (2000)
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