The abelian sandpile model on fractals

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- Grains of sand are placed according to some process on a finite directed graph.
- When too many grains accumulate at a given vertex it *topples*.
- This causes an *avalanche* of topplings until the sandpile stabilizes (assuming it ever stabilizes).
- What can we say about the avalanches and stable configurations and how do these behave in the limit as the graph grows to infinity?
Let $G = (V, E)$ be a directed graph. A sandpile on $G$ is given by a height function $h : V \to \mathbb{Z}^*$. 
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Grains of sand are added by the grain addition operators,  

$$A_n h = h + \delta_{X_n},$$  

where $X_n$ is some random walk on $V$.  

Fix $h_0$. Then $h_n = A_n \cdots A_1 h_0$ is a sequence of sandpiles on $G$.  

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Each vertex $v$ has a threshold $\gamma_v$, usually taken to be $\text{outdeg}(v)$, and if $h(v) > \gamma_v$ for some $v$ we say the sandpile is \textit{unstable} and that $v$ is an unstable vertex.
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A sandpile is stabilized by toppling unstable vertices, one at a time, until no vertices are unstable. This results in a stable sandpile $h^\circ$. We assume sandpiles stabilize before adding more grains.
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Let $\Delta = D - A$, where $D = \text{diag}(\text{outdeg}(v_i))$ and $A$ is the adjacency matrix of $X$. Toppling $v$ corresponds to $h' = h - \Delta_{(v,\cdot)}$. 

The model II
A *sink* is a vertex with out-degree zero, and a sink $s$ is said to be *global* if there exists a path from every vertex to $s$. 

*Lemma (Dhar)* If $G$ has a global sink then every sandpile stabilizes, the order of topplings for an unstable sandpile does not affect the stabilized sandpile, and the order in which grains are added does not affect the final stabilized sandpile.
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A huge sandpile (courtesy Seth Terashima)
The sandpile group

There are reduced sandpiles which can be obtained from any other sandpile by adding some grains and stabilizing. These are called recurrent sandpiles, and they form an abelian group under addition.
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This group, called the sandpile group of a graph $G$. If $G$ has $n$ non-sink vertices,

$$S(G) = \mathbb{Z}^n / \Delta' \mathbb{Z}^n,$$

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Theorem (Matrix-Tree Theorem)

Let $G$ be a digraph and choose a vertex $v$ whose incoming edges we prune to make it a sink. The order of the corresponding sandpile group is $\det(\Delta')$, which is the number of oriented spanning trees in $G$ rooted at $v$. 

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The identity sandpile on a 198 by 198 grid (courtesy Mike Creutz).
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- Do the uniform measures, $\mu_V$, on recurrent sandpiles on $V$ converge weakly to a measure $\mu$ called the *infinite volume limit*? Is this limit concentrated on recurrent configurations of $G$?
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- Is $\mu$ stationary for some Markov process?
- Is grain addition well behaved in the infinite volume limit and is the model still abelian with respect to toppling order?

**Theorem (Jarai, Redig)**

For $\mathbb{Z}^d$, $d \geq 3$, there exist infinite volume addition operators which leave the infinite volume limit $\mu$ invariant, and there exists a Markov process for which $\mu$ is the stationary measure.

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*For $\mathbb{Z}^d$, $d \geq 3$, there exist infinite volume addition operators which leave the infinite volume limit $\mu$ invariant, and there exists a Markov process for which $\mu$ is the stationary measure.*
Let $Y$ be a random variable associated with some property of an avalanche (number of sites in an avalanche, total number of topplings). We say the ASM on a sequence of graphs $G_n \xrightarrow{\mu} G$ is *critical* if there exists a *critical exponent* $\delta_Y > 0$ such that

$$\lim_{n \to \infty} P_{\mu_n} (Y = y) \sim y^{-\delta_Y},$$

where $\mu_n$ is the uniform measure on recurrent sandpiles on $G_n$. 

No criticality for $Z$. Criticality for $Z_d$, $d > 1$. Regular trees are critical. Non-rigorous arguments also exist for the Sierpinski gasket with critical exponents depending on walk and fractal dimensions [Daerden, Vanderzande].
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- Regular trees are critical.
- Non-rigorous arguments also exist for the Sierpinski gasket with critical exponents depending on walk and fractal dimensions [Daerden, Vanderzande].
Let $G_n \nearrow G$. For each $G_n$ pick a root uniformly at random. This provides a sequence of measures, $\nu_n$, on the space of finite rooted graphs. The limit $\nu = \lim_{n \to \infty} \nu_n$ is called the *random weak limit* of the sequence $\nu_n$. 

Theorem (D’Angeli, Donno, Matter, Nagnibeda) For $\Gamma < \text{Aut}(T)$, let $G_n$ be the Schreier graph of the action of $\Gamma$ on the $n$th level of the tree, and let $\nu_n$ be the uniform measure on the vertices of $G_n$. The random weak limit of this sequence of graphs is the set of Schreier graphs $G_\xi$, $\xi \in \partial T$ of the action of $G$ on $\partial T$ with the uniform measure on $\partial T$.

Theorem (Matter, Nagnibeda) The ASM on a sequence of Schreier graphs of the Basilica group is a.s. critical in the random weak limit.
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Let $G_n \xrightarrow{\sim} G$. For each $G_n$ pick a root uniformly at random. This provides a sequence of measures, $\nu_n$, on the space of finite rooted graphs. The limit $\nu = \lim_{n \to \infty} \nu_n$ is called the random weak limit of the sequence $\nu_n$.

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Sandman groups

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The Heisenberg group.
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• $\mathbb{Z} \wr \mathbb{Z} / \langle a^t a^{-2t} a \rangle$ is isomorphic to?  The Heisenberg group.
Let $H$ and $K$ be groups and suppose $K$ acts on a set $X$. The \textit{permutational wreath product} $H \wr_X K$ is the semidirect of $\sum_X A$ by $G$. Group elements are $(f, k)$ where $f : X \to H$ with finite support and $k \in K$. 
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Multiplication is defined by

$$(f_1, h_1)(f_2, h_2) = (f_1(h_1 \cdot f_2), h_1 h_2).$$
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Multiplication is defined by

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Fixing a basepoint $o \in X$ lets you talk about the $X$ location, $o \cdot h^{-1}$, of an element $(f, h)$. 

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Some example toppling relations for $\mathbb{Z} \wr \mathbb{Z}$: $a^2 - 3a$ gives Sol, $a^2 - n$ gives $\text{BS}(1,n)$, $a^4 - 2a^3a^2 - 2a^3a - 2a^3$.

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Some example toppling relations for $\mathbb{Z} \wr \mathbb{Z}$

- $a^{t^2} a^{-3t} a$ gives Sol,
- $a^t a^{-n}$ gives BS$(1, n)$,
- $a^{t^4} a^{-2t^3} a^{t^2} a^{-2t} a$. 
Let $\mathcal{H}$ be the Hanoi towers groups and let $X$ be the Schreier graph of the quotient $\mathcal{H}/\text{Stab}(0)$. We denote the standard generators of $\mathcal{H}$ as $r, s, t$. The toppling relation $a^r a^s a^t a^{-3}$ corresponds to the classical ASM on $X$. 

\[
\begin{array}{c}
\includegraphics[width=0.7\textwidth]{sierpinski_gasket_diagram}
\end{array}
\]
Theorem (T.)

If \( \tau \) is “strongly” dissipative there exist simple random walks, \( X_n \), on the Cayley graph of \( S(X, G, \tau) \) which satisfy \( \mathbb{E}|X_n| \asymp n^{1/2} \) and a law of iterated logarithm.

Idea.
Given a sandpile \( h \) show that the number of grains in \( h \) is determined by \( |\text{supp}(h)| \) and \( \log(1 + \max|h|) \). This provides an upper bound on the length of the random walk. This result depends on the number of sites a random walk visits on \( \mathbb{Z} \). For \( \mathbb{Z}^2 \) or \( \mathbb{Z}^3 \) the number of sites visited is much larger, but fractals can provide intermediate behavior.
An application

Theorem (T.)

If $\tau$ is “strongly” dissipative there exist simple random walks, $X_n$, on the Cayley graph of $S(X,G,\tau)$ which satisfy $\mathbb{E}|X_n| \simeq n^{1/2}$ and a law of iterated logarithm.

Idea. Given a sandpile $h$ show that the number of grains in $h^0$ is determined by $|\text{supp}(h)|$ and $\log^{1+\epsilon}(\max|h|)$. This provides an upper bound on the length of the random walk.
Theorem (T.)

If $\tau$ is “strongly” dissipative there exist simple random walks, $X_n$, on the Cayley graph of $\mathcal{S}(X, G, \tau)$ which satisfy $\mathbb{E}|X_n| \sim n^{1/2}$ and a law of iterated logarithm.

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This result depends on the number of sites a random walk visits on $\mathbb{Z}$. For $\mathbb{Z}^2$ or $\mathbb{Z}^3$ the number of sites visited is much larger, but fractals can provide intermediate behavior.
Some Questions

- What does a sequence of stable sandpiles obtained from a random walk on a sandman group look like?
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- What can we say about the Martin or Poisson boundary of sandman groups and how are these related to recurrent configurations in the infinite volume limit?
- What other algebraic or geometric information can be gleaned about a sandman group from the nature of the embedded ASM?