The abelian sandpile model on fractals

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September 12, 2011

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- When too many grains accumulate at a given vertex it *topples*.
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- What can we say about the avalanches and stable configurations and how do these behave in the limit as the graph grows to infinity?

The model I

Let $G = (V, E)$ be a directed graph. A *sandpile* on G is given by a height function $h : V \to \mathbb{Z}^*$.

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Grains of sand are added by the grain addition operators,

$$A_n h = h + \delta_{X_n}, \tag{1}$$

where X_n is some random walk on V.

Fix h_0. Then $h_n = A_n \cdots A_1 h_0$ is a sequence of sandpiles on G.

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Each vertex v has a threshold γ_v, usually taken to be $\mathrm{outdeg}(v)$, and if $h(v) > \gamma_v$ for some v we say the sandpile is *unstable* and that v is an unstable vertex.

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Let $\Delta = D - A$, where $D = \mathrm{diag}(\mathrm{outdeg}(v_i))$ and A is the adjacency matrix of X. Toppling v corresponds to $h' = h - \Delta_{(v,\cdot)}$.

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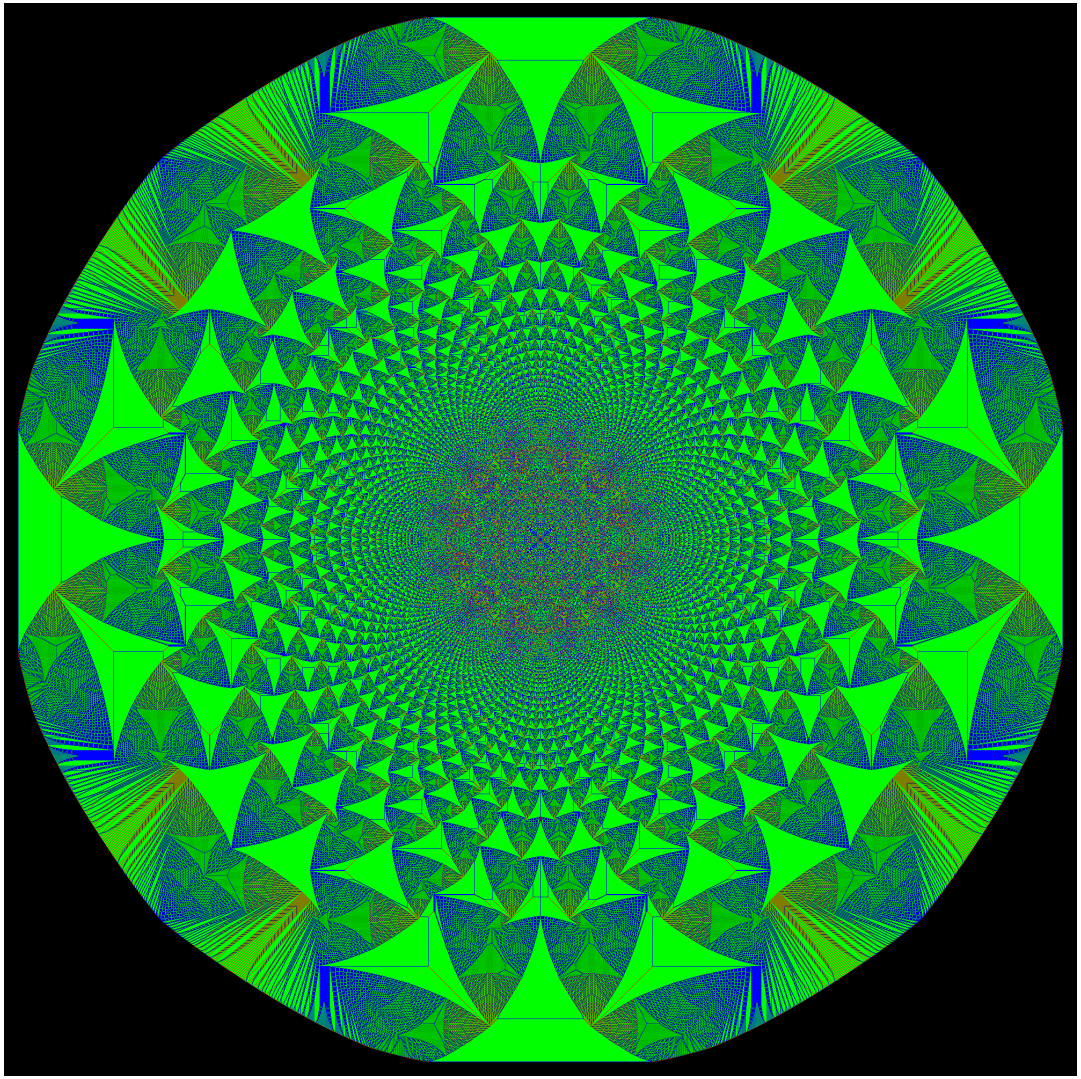
Lemma (Dhar)

If G has a global sink then

- *every sandpile stabilizes,*
- *the order of topplings for an unstable sandpile does not affect the stabilized sandpile, and*
- *the order in which grains are added does not affect the final stabilized sandpile.*



A huge sandpile (courtesy Seth Terashima)

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$$\mathcal{S}(G) = \mathbb{Z}^n / \Delta' \mathbb{Z}^n, \tag{2}$$

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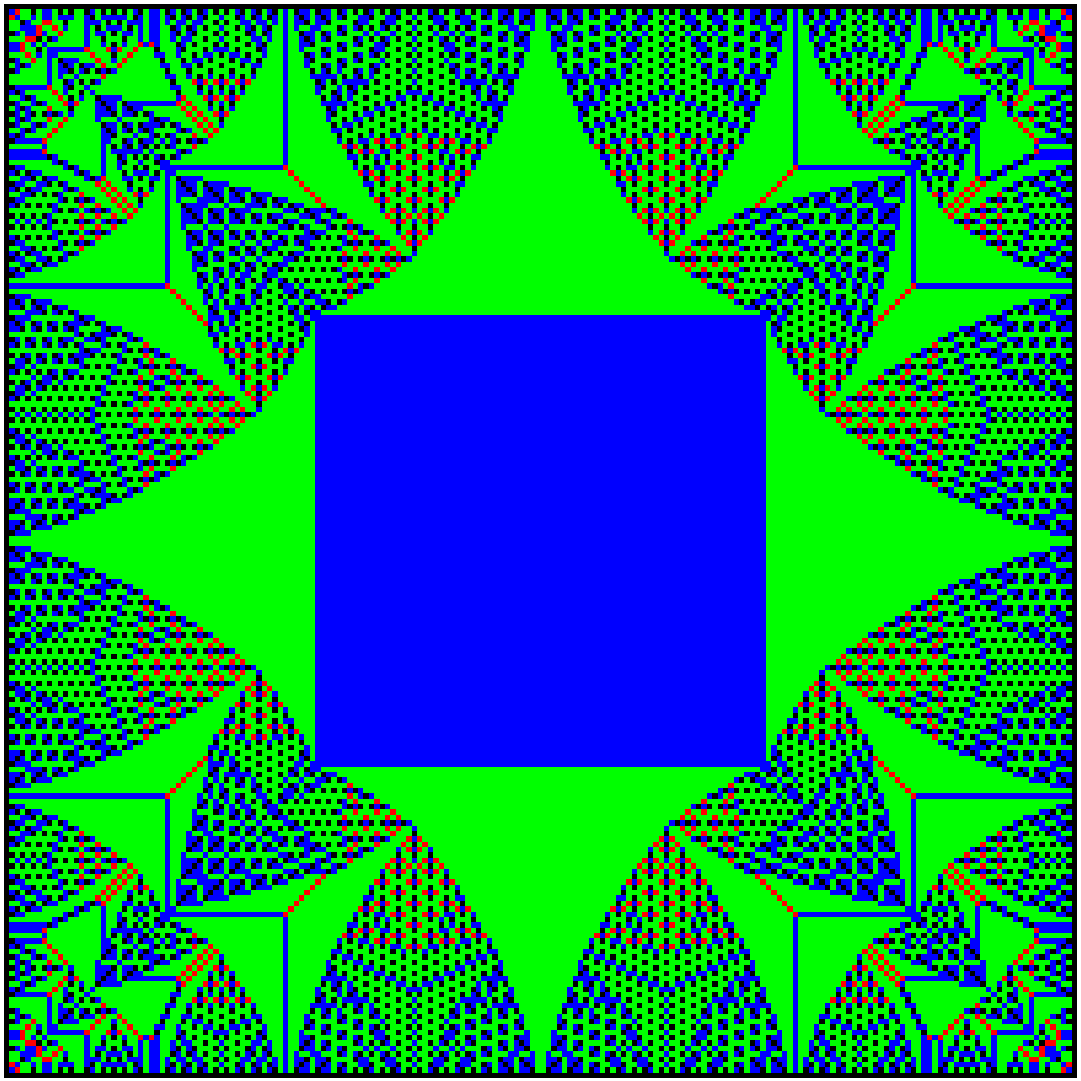
Theorem (Matrix-Tree Theorem)

Let G be a digraph and choose a vertex v whose incoming edges we prune to make it a sink. The order of the corresponding sandpile group is $\det(\Delta')$, *which is the number of oriented spanning trees in G rooted at v.*



The identity sandpile on a 198 by 198 grid (courtesy Mike Creutz).

The Infinite Volume Limit

For an infinite graph G, a sandpile h is said to be recurrent if for all finite $V \subset G_V$, $h|_V$ is recurrent for the induced subgraph of G.

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Theorem (Jarai, Redig)

For $\mathbb{Z}^d$, $d \geq 3$, there exist infinite volume addition operators which leave the infinite volume limit μ invariant, and there exists a Markov process for which μ is the stationary measure.

Criticality

Let Y be a random variable associated with some property of an avalanche (number of sites in an avalanche, total number of topplings). We say the ASM on a sequence of graphs $G_n \nearrow G$ is *critical* if there exists a *critical exponent* $\delta_Y > 0$ such that

$$\lim_{n\to\infty} P_{\mu_n}(Y = y) \sim y^{-\delta_Y}, \tag{3}$$

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- No criticality for $\mathbb{Z}$.
- Criticality for $\mathbb{Z}^d$, $d > 1$.
- Regular trees are critical.
- Non-rigorous arguments also exist for the Sierpinski gasket with critical exponents depending on walk and fractal dimensions [Daerden, Vanderzande].

Let $G_n \nearrow G$. For each G_n pick a root uniformly at random. This provides a sequence of measures, ν_n, on the space of finite rooted graphs. The limit $\nu = \lim_{n\to\infty} \nu_n$ is called the *random weak limit* of the sequence ν_n.

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Theorem (D'Angeli, Donno, Matter, Nagnibeda)

For $\Gamma < Aut(T)$, let G_n be the Schreier graph of the action of Γ on the nth level of the tree, and let ν_n be the uniform measure on the vertices of of G_n. The random weak limit of this sequence of graphs is the set of Schreier graphs G_ξ, $\xi \in \partial T$ of the action of G on ∂T with the uniform measure on ∂T.

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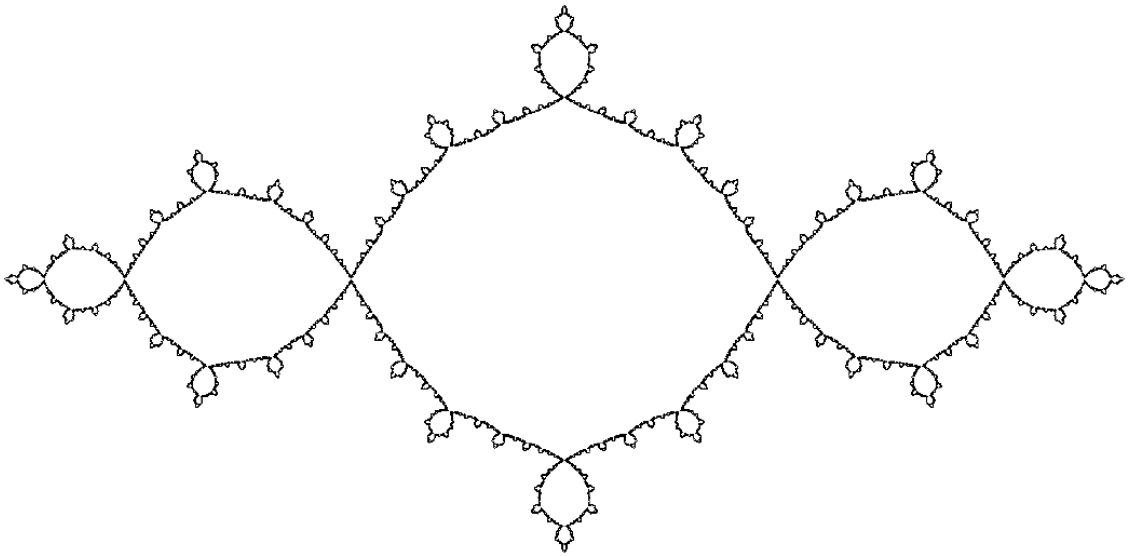
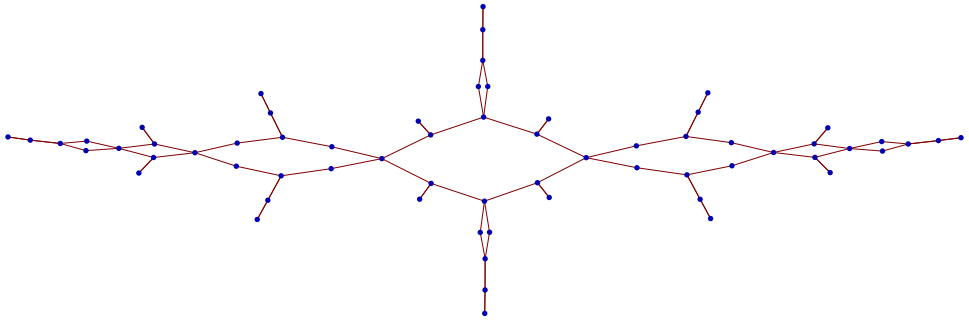
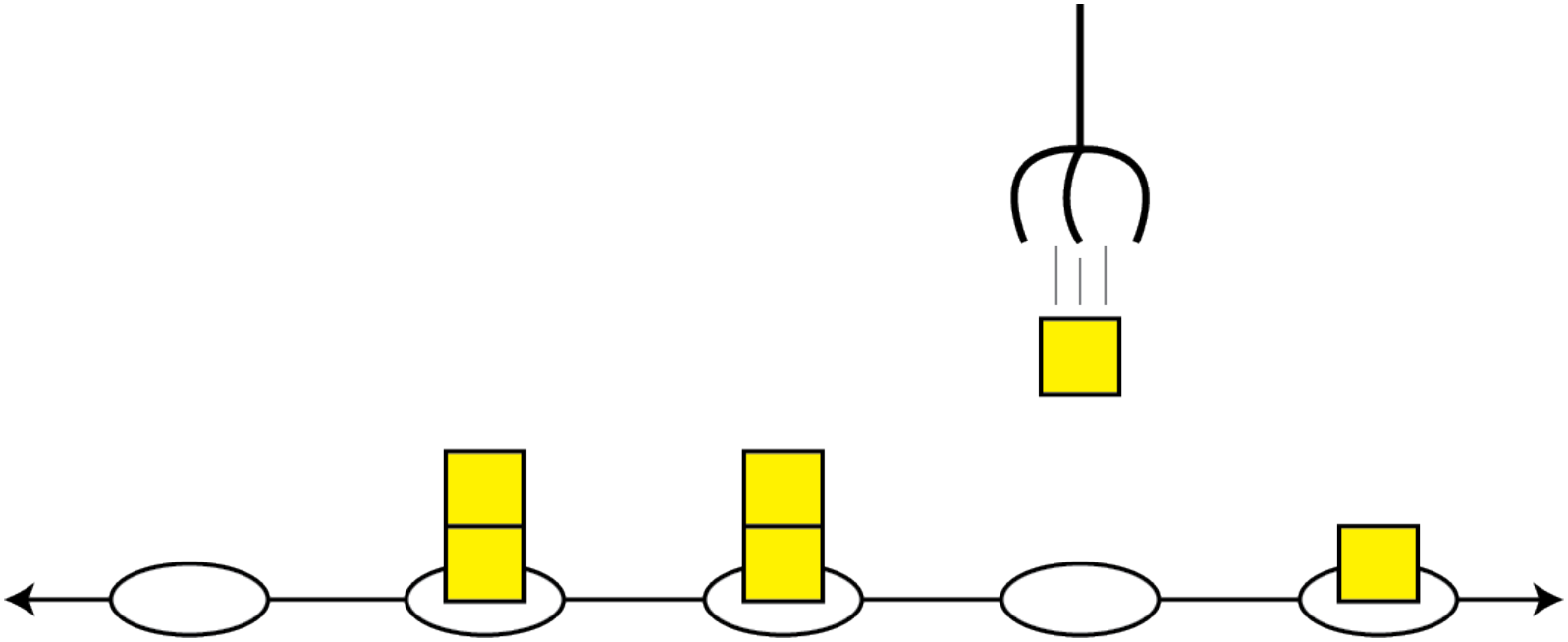
The ASM on a sequence of Schreier graphs of the Basilica group is a.s. critical in the random weak limit





Sandman groups



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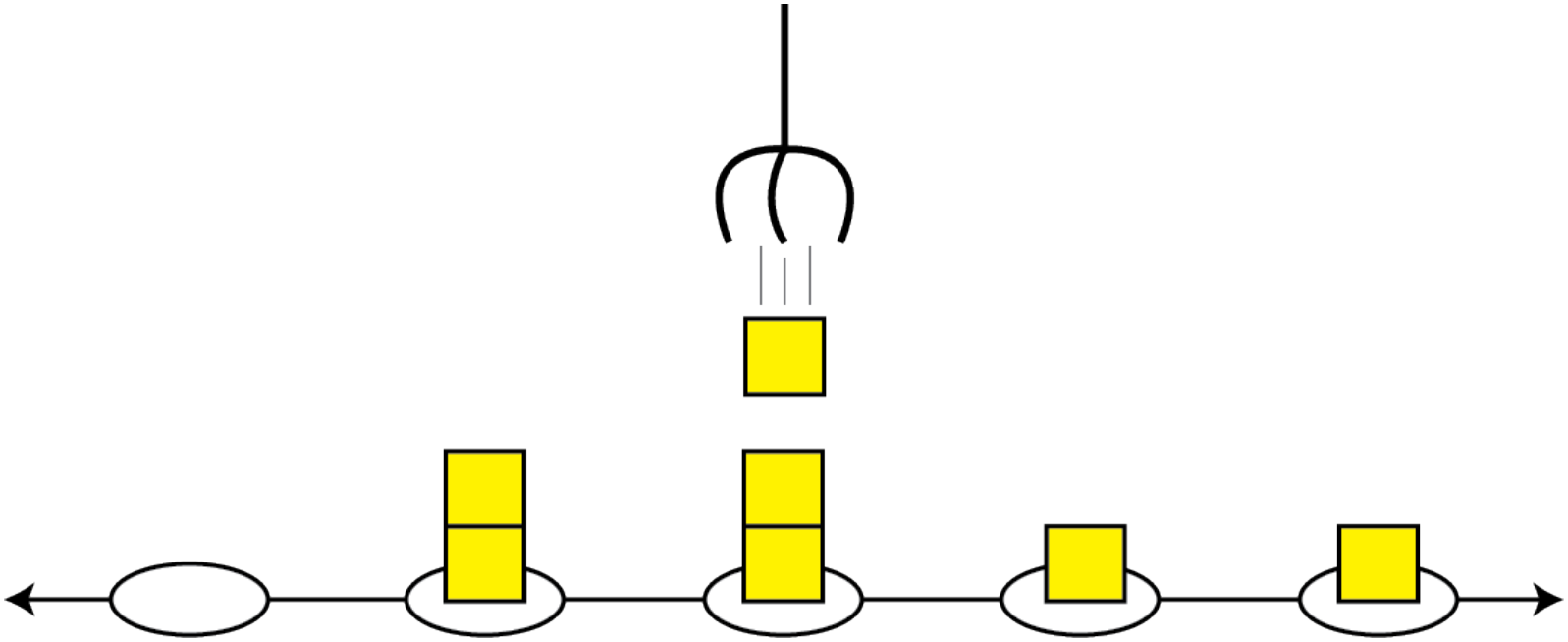

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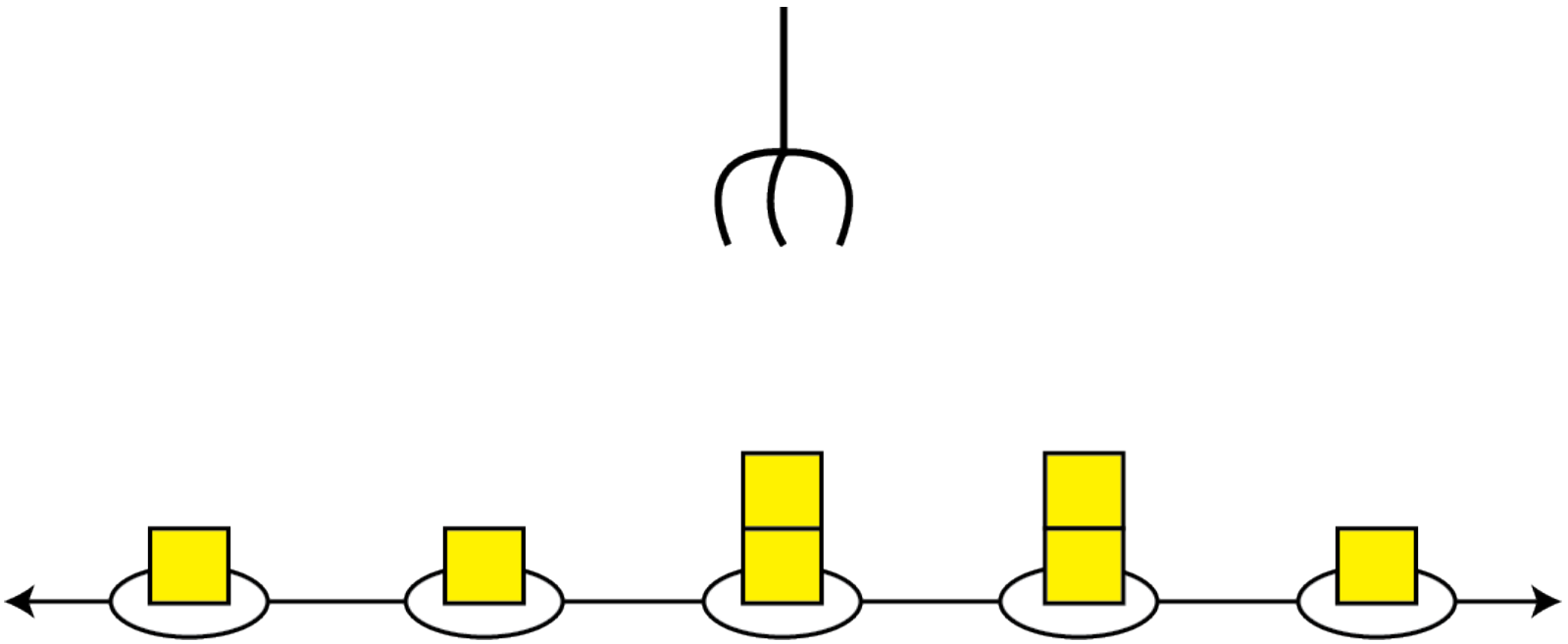

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Permutational wreath products

Let H and K be groups and suppose K acts on a set X. The *permutational wreath product* $H \wr_X K$ is the semidirect of $\sum_X A$ by G. Group elements are (f, k) where $f : X \to H$ with finite support and $k \in K$.

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Fixing a basepoint $o \in X$ lets you talk about the X location, $o \cdot h^{-1}$, of an element (f, h).

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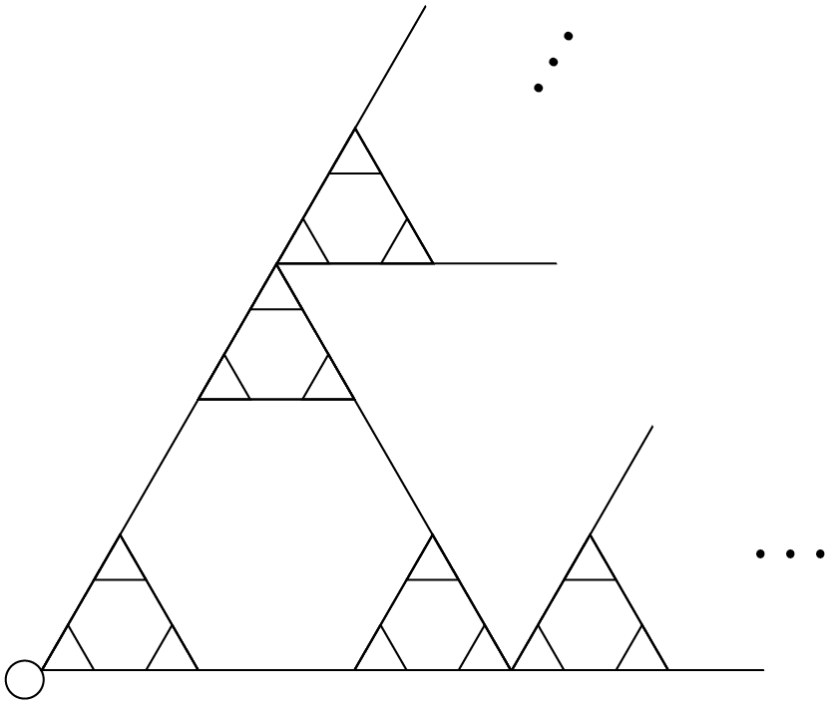
- $a^t a^{-n}$ gives $\mathrm{BS}(1, n)$,
- $a^{t^4} a^{-2t^3} a^{t^2} a^{-2t} a$.

The Sandman for the Sierpinski Gasket

Let $\mathcal{H}$ be the Hanoi towers groups and let X be the Schreier graph of the quotient $\mathcal{H}/\mathrm{Stab}(0)$. We denote the standard generators of $\mathcal{H}$ as r, s, t. The toppling relation $a^r a^s a^t a^{-3}$ corresponds to the classical ASM on X.



An application

Theorem (T.)

If τ is "strongly" dissipative there exist simple random walks, X_n, on the Cayley graph of $\mathcal{S}(X, G, \tau)$ which satisfy $\mathbb{E}|X_n| \simeq n^{1/2}$ and a law of iterated logarithm.

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This result depends on the number of sites a random walk visits on $\mathbb{Z}$. For $\mathbb{Z}^2$ or $\mathbb{Z}^3$ the number of sites visited is much larger, but fractals can provide intermediate behavior.

Some Questions

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- What can we say about the Martin or Poisson boundary of sandman groups and how are these related to recurrent configurations in the infinite volume limit?
- What other algebraic or geometric information can be gleaned about a sandman group from the nature of the embedded ASM?