Mean value properties on Sierpinski type fractals

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1. Introduction

For a domain $\Omega$ in Euclidean space, a continuous function $u$ is harmonic ($\Delta u = 0$) if and only if

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) dy = u(x)$$

if $B_r(x) \subset \Omega$ where $B_r(x)$ is the ball of radius $r$ about $x$. 
More generally, if \( u \) is not assumed harmonic but \( \Delta u \) is a continuous function, then

\[
\lim_{r \to 0} \frac{1}{r^2} \left( \frac{1}{|B_r(x)|} \right) \int_{B_r(x)} u(y) dy - u(x) = c_n \Delta u(x) \tag{1}
\]

for the appropriate dimensional constant \( c_n \).
What are the fractal analogs of these results? What are the fractal analogs of ”balls” on which we do the averaging? So if $K$ is a fractal and $x \in K$, we would like to know that there is a sequence of sets $B_k(x)$ containing $x$ with $\bigcap_k B_k(x) = \{x\}$ such that

$$\frac{1}{\mu(B_k(x))} \int_{B_k(x)} u(y) dy = u(x)$$

for every harmonic function $u$. 
We call $B_k(x)$ the \textit{k-level mean value neighborhood} of $x$.

Let $K = SG$ equipped with the standard harmonic structure and the standard self-similar measure.

If $x$ is a nonboundary vertex point, we can easily answer the question with $B_k(x)$ being the two union of the $k$ level neighboring cells of $x$.

What about the generic points?
More generally, for general $u$ not assumed harmonic but belonging to $dom\Delta$, does an analogous formula of (1) still hold?
2. Mean value neighborhoods on Sierpinski gasket

Consider any cell $F_w(SG) = C_w$ with boundary points $F_w(q_i) = p_i$.
For any point $x \in C_w$ there exist coefficients $a_i(x)$ such that

$$h(x) = \sum_i a_i(x)h(p_i)$$

for any given harmonic function $h$. 
Since constants are harmonic we must have

\[ \sum_i a_i(x) = 1, \]

and by the maximum principle all \( a_i(x) \geq 0. \)

We can compute \( \{a_i(x)\} \) for \( x \) any junction point by using the harmonic extension algorithm.
Let $W$ denote the triangle in $\mathbb{R}^3$ with boundary points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

So $\{a_i(x)\} \in W$ for any $x \in C_w$.

Of course not every point in $W$ occurs in this way.
Figure 1.
Given a set \( B \subset SG \), define
\[
M_B(f) = \frac{1}{|B|} \int_B f d\mu
\]
the *mean value* of function \( f \) on \( B \).

By linearity,
\[
M_B(h) = \sum_i a_i h(p_i)
\]
for some coefficients \( \{a_i\} \) for any harmonic function \( h \).

We also have \( \sum_i a_i = 1 \) by considering \( h \equiv 1 \).
Idea: We hope that \( \{a_i\} \in W \) if \( C_w \subset B \). If this is true, we have a map

\[ \{B\} \rightarrow W. \]

If we can show that the map is onto for some reasonable class of sets \( B \), then we can get our first \( B_0(x) \) for every \( x \in C_w \), and then by zooming in get a full sequence \( B_k(x) \).
**Observation:**

Note that $M_{C_w}(h) = \sum_i \frac{1}{3} h(p_i)$ so we hit the center point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of $W$ by taking $B = C_w$.

If we take $B$ to be the union of $C_w$ and one of its neighboring cell of the same size, then

$$M_B(h) = h(p_i)$$

for the point of intersection.
So we can get the 3 vertices $(1,0,0), (0,1,0), (0,0,1)$ of $W$ in this way.

By varying $B$ continuously between the two we can hit a curve in $W$ joining the center to any vertex.

Figure 2.
This looks like the beginning of an argument to show that the mapping is onto.

By the observation, we should require $B$ to be a subset of the union of $C_w$ and its 3 neighbors, containing $C_w$.  

**Reasonable choice of $B$:** connected, symmetric, depends only on the relative geometry of $x$ and $C_w$, independent of the size of $C_w$ and the location of $C_w$.  

Denote by $C_0, C_1, C_2$ the three neighboring cells of the same size of $C_w$, intersecting $C_w$ at $p_i$ for each $i$. Write $D_w$ the union of $C_w$ and its 3 neighbors, i.e., $D_w = C_w \cup C_0 \cup C_1 \cup C_2$.

Consider a set $B$, $C_w \subset B \subset D_w$.

$$B = C_w \cup E_0 \cup E_1 \cup E_2,$$

where $E_i = B \cap C_i$, $i = 0, 1, 2$. 
We restrict each $E_i$ to be a triangle sub-domain of $C_i$, symmetric under the reflection symmetry that fixes $p_i$, containing $p_i$ as one of its boundary points.

Figure 3.
Suppose the height of $C_w$ is $r$, then the height of $E_i$ should be $c_i r$ for $0 \leq c_i \leq 1$. Hence we can write the set 

$$B = B(c_0, c_1, c_2).$$

For example, $B(0, 0, 0) = C_w$ and $B(1, 1, 1) = D_w$.

Denote by

$$\mathcal{B}^* = \{ B(c_0, c_1, c_2) : 0 \leq c_i \leq 1 \}$$

the family of all such sets.
Let $T_w$ denote the map from $B^*$ to $\pi_W$ (the plane containing $W$) described before.

$T_w$ can be viewed as a nonlinear vector valued function from $[0,1]^3$ to $\pi_W$. For simplicity, we may write $T_w(\{c_i\}) = \{a_i\}$ for each set $B(c_0, c_1, c_2)$. 
Fact 1. The map $\mathcal{T}_w$ is independent of the particular choice of $C_w$.

Hence we can drop the subscript $w$ on $\mathcal{T}$ for simplicity.

The proof of Fact 1 benefits from the symmetric properties of $E_i$'s.
Fact 2. There exists $B \in \mathcal{B}^*$, such that $\mathcal{T}(B) \notin W$.

For example, a easy computation shows that $\mathcal{T}(0, 1, 1) = \{-\frac{1}{9}, \frac{5}{9}, \frac{5}{9}\}$.

However, if we can show that the image of the map $\mathcal{T}$ can cover the triangle $W$, things will still go well.
Theorem 1. The map $T$ from $B^*$ to $\pi_W$ fills out a region $\tilde{W}$ which contains the triangle $W$.

Sketch of the proof.

Step 1. Consider a subfamily $B_1 = \{B(0, 0, c_2) : 0 \leq c_2 \leq 1\}$ of $B^*$. If we restrict the map $T$ to $B_1$, by varying $c_2$ continuously between 0 and 1 we can hit a curve (it is a line segment which follows from the symmetry of $E_2$) in $W$ joining the center $O$ to an vertex point $N$. 
Step 2. Consider another subfamily \( \mathcal{B}_2 = \{ B(0, c, c) : 0 \leq c \leq 1 \} \) of \( \mathcal{B}^* \). If we restrict the map \( T \) to \( \mathcal{B}_2 \), by varying \( c \) continuously between 0 and 1 we can hit a curve (it is a line segment which follows from the symmetric effect of \( E_1 \) and \( E_2 \)) in \( W \) joining the center \( O \) to an point \( Q \). A directly computation shows that \( Q \) lies outside of the triangle \( W \).

Step 3. \( T(\{ B(0, c, 1) : 0 \leq c \leq 1 \}) \) is a curve located outside of the triangle \( W \), joining \( N \) to \( Q \). The intersection of this curve and \( W \) consists exactly only one point \( N \).
Step 4. Fix a number $0 \leq y \leq 1$. Consider a subfamily $C_y = \{B(0, c, y) : 0 \leq c \leq y\}$ of $\mathcal{B}^*$. If we restrict the map $T$ on $C_y$, by varying $c$ continuously between 0 and $y$ we can hit a curve $\Gamma_y$ joining the two points $T(B(0, 0, y))$ and $T(B(0, y, y))$. The first endpoint $T(B(0, 0, y))$ lies on the line segment $\overline{ON}$ and the second endpoint $T(B(0, y, y))$ lies on the line segment $\overline{OQ}$. Hence if we vary $y$ continuously between 0 and 1, we can fill out the $1/6$ region of $\widetilde{W}$.

Then by exploiting the symmetry, we get the desired result. □
Figure 4.
Let \( B \) be a subfamily of \( B^* \) defined by

\[
B = \{ B(0, c_2, c_3) : 0 \leq c_2, c_3 \leq 1 \} \\
\cup \{ B(c_1, 0, c_3) : 0 \leq c_1, c_3 \leq 1 \} \\
\cup \{ B(c_1, c_2, 0) : 0 \leq c_1, c_2 \leq 1 \},
\]

i.e., \( B \) consisting of those elements \( B \) in \( B^* \) which have the decomposition form \( B = C_w \cup E_1 \cup E_2 \) or \( B = C_w \cup E_1 \cup E_3 \), or \( B = C_w \cup E_2 \cup E_3 \).
Remark of Theorem 1. Actually, we have proved that \( \tilde{W} = T(B) \). Moreover, if we use \( B \) in stead of \( B^* \). Then the map

\[ T : B \rightarrow \tilde{W} \]

is one-to-one.

Hence for each \( x \in C_w \), there exists a unique set \( B \in \mathcal{B} \), \( C_w \subset B \subset D_w \), such that \( M_B(h) = h(x) \) for any harmonic function \( h \). We call the set \( B \) associated to this \( C_w \) a \textit{k level mean value neighborhood of} \( x \) where \( k \) is the length of \( w \).
Given a point $x \in SG \setminus V_0$, let $k_0$ be the smallest value of $k$ such that there exists a $k$ level cell $C_w$ containing $x$ but not intersecting $V_0$. $k_0$ depends on the location of $x$ in $SG$.

Then we can find a sequence of words $w^{(k)}$ of length $k$ ($k \geq k_0$) and a sequence of mean value neighborhoods $B_k(x)$ associated to $C_{w^{(k)}}$.

$\{B_k(x)\}_{k \geq k_0}$ forms a system of neighborhoods of the point $x$ satisfying $\bigcap_{k \geq k_0} B_k(x) = \{x\}$.
3. Mean value properties of functions in $\text{dom}\Delta$

Given a point $x \in SG \setminus V_0$, we want to define $c_B$ such that

$$M_B(u) - u(x) \approx c_B \Delta u(x)$$

for $u \in \text{dom}\Delta$.

Let $v$ be a function satisfying $\Delta v \equiv 1$. Define $c_B$ by

$$c_B = M_B(v) - v(x).$$
Note that the value of $c_B$ is independent of which $v$, because any two differ by a harmonic function $h$ and $M_B(h) - h(x) = 0$.

So we can choose

$$v = - \int_{SG} G(\cdot, y) d\mu(y).$$

($v$ vanishes on the boundary of $SG$, $G$ is the Green’s function).
$c_B$ depends only on the relative geometry of $B$ and $C_w$ and the size of $C_w$, not on the location of $x$ or $C_w$ in $SG$. Moreover,

**Theorem 2.** There exist two positive constants $c_0$ and $c_1$, such that

$$c_0 \frac{1}{5k} \leq c_B \leq c_1 \frac{1}{5k}$$

for any $k$ level mean value neighborhood $B$. 
Given a point $x$ and $C_w$ a $k$ level neighborhood of $x$, for any $u \in \text{dom} \Delta^2$, we can write

$$u = h^{(k)} + (\Delta u(x))v + R^{(k)}$$

on $C_w$, where $h^{(k)}$ is a harmonic function defined by

$$h^{(k)} + (\Delta u(x))v|_{\partial C_w} = u|_{\partial C_w}.$$
It is not hard to prove that

**Fact 3.** *The remainder satisfies*

\[ R^{(k)} = O\left(\left(\frac{3}{5} \cdot \frac{1}{5}\right)^k\right) \]

*on* \( C_w \) (*also on* \( B_k(x) \)).

(This looks like the Taylor expansion of \( u \) at \( x \).)
Proof of Fact 3.

It is easy to check that $\Delta_y R^{(k)}(y) = \Delta_y u(y) - \Delta_y u(x)$ and $R^{(k)}(y)$ vanishes on the boundary of $C_w$. Hence $R^{(k)}$ is given by the integral of $\Delta_y u(y) - \Delta_y u(x)$ on $C_w$ against a scaled Green’s function.

Since the scaling factor is $(\frac{1}{5})^k$, and

$$|\Delta_y u(y) - \Delta_y u(x)| \leq c(\frac{3}{5})^k$$

($\Delta u$ satisfies the Holder condition with $\gamma = \frac{3}{5}$), we get
the desired result. □

A more generalized version of Fact 3 is

**Fact 3’** Let \( u \in \text{dom} \Delta \) with \( g = \Delta u \) satisfying the following Holder condition

\[
|g(y) - g(x)| \leq c \gamma^k, \quad (0 < \gamma < 1)
\]

for all \( y \in C_w \), then the remainder satisfies

\[
R^{(k)} = O((\gamma \cdot \frac{1}{5})^k)
\]

on \( C_w \) (also on \( B_k(x) \)).
Using the Taylor expansion of $u$ at $x$ and Theorem 2, we have

\[
\frac{1}{c_{B_k}(x)} (M_{B_k}(x)(u) - u(x)) - \Delta u(x) = 1 \frac{1}{c_{B_k}(x)} (M_{B_k}(x)(R^k) - R^k(x))
\]

\[
= \frac{1}{c_{B_k}(x)} O((\gamma \cdot \frac{1}{5})^k) = O(\gamma^k).
\]
Hence we get

\[
\lim_{k \to \infty} \frac{1}{c_{B_k}(x)}(M_{B_k}(x)(u) - u(x)) = \Delta u(x),
\]

which is a fractal analogous formula of (1).
4. Generalization on a class of Sierpinski type fractals

Let $K$ be a p.c.f. self-similar fractal. We assume that $\#V_0 = 3$ and all structures possess full $D3$ symmetry.

A) We can choose all $c_{i,j} = 1$ in defining the graph energy on $\Gamma_0$, and all the resistance renormalization factors $r_i$ are the same number $r$.

B) We must have $\mu_1 = \mu_2 = \mu_3$, i.e., $\mu$ is the standard self-similar measure on $K$. 
\(SG_3, \ SG_n,\) Hexagasket, 3-dimensional SG...

For the first question on how to find the mean value neighborhoods, it is done.

However, for the second question on how to estimate the \(c_B\) constants, it is still on working.
Thank you!