Math5160 Final HW **Fall 2008**

All the homework has to be submitted by Friday December 12, any time either in my mailbox or under my office door (a submission later than Friday may result in an "incomplete" grade)**.**

During the exam week, regular office hours will be held on
Monday and Friday at 2pm–2:40pm, and on Wednesday noon–1:00pm.

If you use a theorem or a lemma, either give its name, or its number in the textbook or lecture notes (and don't forget to show that the assumptions hold).

1. (a) Show that if $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > n) < \infty$ then $\limsup_{n\to\infty} X_n/n \le 1$ almost surely.

(b) Deduce from (a) that if X_n have the same distribution and $\mathbb{E}|X_n| < \infty$ then $\lim_{n\to\infty} X_n/n = 0$ almost surely.

2. (a) If ϵ_n are iid Bernoulli random variables with $\mathbb{P}\{\epsilon_n{=}1\} = \mathbb{P}\{\epsilon_n{=}{-1}\} = \frac{1}{2}$, give necessary and sufficient conditions for non-random real coefficients a_n so that the series $\sum_{n=1}^{\infty} a_n\epsilon_n$ converges almost surely.

(b) If X_n are iid standard Gaussian random variables, which are also independent of all ϵ_n, give necessary and sufficient conditions for non-random real coefficients a_n so that the series $\sum_{n=1}^{\infty} a_n\epsilon_n X_n$ converges almost surely. Is it a Gaussian random variable?

(c) Answer the same questions if $\mathbb{P}\{\epsilon_n = 1\} = \mathbb{P}\{\epsilon_n = 0\} = \frac{1}{2}$.

3. (a) If Y_n are independent random variables with Poisson distribution and $\mathbb{E}Y_n = n^2$. Can you find real numbers a_n and b_n such that $(Y_n - a_n)/b_n$ converges to the standard normal in distribution?

(b) Answer the same question if $\mathbb{E}Y_n = 2^n$.

4. (a) If N is a standard Poisson random variable, which is independent of an iid sequence of standard normal random variables X_n, find the mean, variance, and the characteristic function of the random variable $Y = \sum_{n=1}^{N} X_n$.

(b)* Let N_k be independent random variables with Poisson distribution and $\mathbb{E}N_k = k$, which are independent of an iid sequence of standard normal random variables X_n, and $Y_k = \sum_{n=1}^{N_k} X_n$. Can you find real numbers a_k and b_k such that $(Y_k - a_k)/b_k$ converges to the standard normal in distribution?

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5. Show that if X_n is a martingale with $\sup_n \mathbb{E}[|X_n|(1+\log^+|X_n|)] < \infty$, where $\log^+ x = \max(0, \log x)$, then X_n is uniformly integrable. Does it imply $\mathbb{E}[\sup_n |X_n|] < \infty$, and why?

6. Let X_t be Brownian motion, $t \in [0,1]$. Let $f, g \in C[0,1]$ be nonrandom.

(a) Show that the random variables $Y_1 = \int_0^1 f(t)X_t dt$ and $Y_2 = \int_0^1 g(s)X_s ds$ are jointly Gaussian random variables, and find their covariance in terms of f and g. The integrals here are the usual Riemann integrals.

(b) Define a stochastic integral

$$\int_0^1 f(t)dX_t = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} f\left(\tfrac{k}{2^n}\right)\left(X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}\right)$$

and show that the limit exists in L^2. Show that $Y_1 = \int_0^1 f(t)dX_t$ and $Y_2 = \int_0^1 g(s)X_s ds$ are jointly Gaussian random variables, and find their covariance.

7.* Let Y_t be a process, not necessarily Brownian motion, for which it is true that

$$\mathbb{P}\left(\sup_{r\le s\le t} |Y_s - Y_r| > \lambda\right) \le c_1|t-s|^3/\lambda^3.$$

Show that Y_t has a modification with a.s. continuous paths (say, $\frac{1}{8}$-Hölder).

Thanks for you work!