

# ONE-DIMENSIONAL WAVE EQUATIONS DEFINED BY FRACTAL LAPLACIANS

JOHN FUN-CHOI CHAN, SZE-MAN NGAI, AND ALEXANDER TEPLYAEV

ABSTRACT. We study one-dimensional wave equations defined by a class of fractal Laplacians. These Laplacians are defined by fractal measures generated by iterated function systems with overlaps, such as the well-known infinite Bernoulli convolution associated with the golden ratio and the 3-fold convolution of the Cantor measure. The iterated function systems defining these measures do not satisfy the post-critically finite condition or the open set condition. By using second-order self-similar identities introduced by Strichartz et al., we discretize the equations and use the finite element and central difference methods to obtain numerical approximations to the weak solutions. We prove that the numerical solutions converge to the weak solution, and obtain estimates for the rate of convergence.

## CONTENTS

1. Introduction	1
2. Preliminaries	5
3. The finite element method	11
4. Fractal measures defined by iterated function systems	15
4.1. Weighted Bernoulli-type measure	15
4.2. Infinite Bernoulli convolution associated with the golden ratio	16
4.3. 3-fold convolution of the Cantor measure	17
5. Convergence of numerical approximations	19
References	22

## 1. INTRODUCTION

In this paper we study approximations to the solution of the wave equation defined by a one-dimensional fractal measure. Such fractals have recently attracted considerable attention because of their relation to classical analysis on one hand, and having many unusual properties on the other hand. In such situations classical approximation methods have to be modified to produce accurate results, see [3, 28, 39, 27, and references therein]. In this paper we investigate the solution of the wave equation theoretically, and also provide numerical examples.

---

*Date:* January 24, 2014.

*2010 Mathematics Subject Classification.* Primary: 28A80, 34L16, 65L60, 65L20; Secondary: 34L15, 65L15.

*Key words and phrases.* fractal, iterated function system, self-similar measure, Laplacian, wave equation, finite element method.

The first two authors were supported in part by a Faculty Research Grant from Georgia Southern University. The second author is also supported in part by an HKRGC grant and National Natural Science Foundation of China grant 11271122. The third author is supported in part by NSF grant DMS-0505622.

Our long term goal is to combine ideas of Strichartz, including the celebrated Strichartz estimates, with some recent results, such as [18] and [36], in a comprehensive study of wave equations on fractals and fractafolds. However currently there are few mathematical tools developed to study wave equations on fractals, despite the fact that the existence of large gaps in the spectrum on many fractals, together with heat kernel estimates, implies that Fourier series on these fractals can have better convergence than in the classical case (and, as was noted by Strichartz in [38], “... is the first kind of example which improves on the corresponding results in smooth analysis”). Among the most recent results, the infinite wave propagation speed was recently proved on some *post-critically finite (p.c.f.)* (see [22]) fractals in the preprint [26] by Yin-Tat Lee. This question was open, even in the most standard case of the Sierpiński gasket, since 1999, see [4, 37]. The proof in [26] relies partially on the Kigami’s p.c.f. assumptions (see [22] and references therein), and more substantially on certain heat kernel estimates. In general, the heat kernel estimates on fractals is a difficult and extensively studied subject, with most relevant recent results and references contained in [16, 17, 23, 24]. It is not clear at present if the heat kernel estimates assumed in [26] can be verified for fractal measures that we consider, but some preliminary results can be found in [42]. The most intuitive idea, essentially due to Strichartz, is that there is no reason why the wave propagation speed should be finite on fractals, because of the difference in time and Laplacian scalings. In our paper we do not discuss the wave propagation speed directly, but rather develop approximating tools that may help in this study.

Let  $\mu$  be a continuous positive finite Borel measure on  $\mathbb{R}$  with  $\text{supp}(\mu) \subseteq [a, b]$ . Let  $L^2_\mu[a, b] := \{f : [a, b] \rightarrow \mathbb{R} : \int |f|^2 d\mu < \infty\}$ ; if  $\mu$  is Lebesgue measure, we simply denote the space by  $L^2[a, b]$ . It is well known (see e.g., [1, 19]) that  $\mu$  defines a Dirichlet Laplacian  $\Delta_\mu$  on  $L^2_\mu[a, b] = \{f : [a, b] \rightarrow \mathbb{R} : \int |f|^2 d\mu < \infty\}$ , described as follows. Let  $H^1(a, b)$  be the Sobolev space of all functions in  $L^2[a, b]$  whose weak derivatives belong to  $L^2[a, b]$ , with the inner product

$$(u, v)_{H^1(a,b)} := \int_a^b uv \, dx + \int_a^b u'v' \, dx.$$

Let  $H_0^1(a, b)$  be the completion of  $C_c^\infty(a, b)$  in  $H^1(a, b)$ .  $H_0^1(a, b)$  and  $H^1(a, b)$  are dense subspaces of  $L^2_\mu[a, b]$ . Define a quadratic form on  $L^2_\mu[a, b]$ ,

$$(1.1) \quad \mathcal{E}(u, v) = \int_a^b u'v' \, dx,$$

with domain  $\text{Dom } \mathcal{E}$  equal to some dense subspace of  $H_0^1(a, b)$  (see Section 2). Since the embedding  $\text{Dom } \mathcal{E} \hookrightarrow L^2_\mu[a, b]$  is compact,  $\mathcal{E}$  is closed and is in fact a Dirichlet form on  $L^2_\mu[a, b]$ . Thus there exists a nonnegative self-adjoint operator  $T$  on  $L^2_\mu[a, b]$  such that  $\text{Dom } \mathcal{E} = \text{Dom}(T^{1/2})$  and

$$\mathcal{E}(u, v) = (T^{1/2}u, T^{1/2}v)_\mu \quad \text{for all } u, v \in \text{Dom } \mathcal{E},$$

where

$$(u, v)_\mu := \int_a^b uv \, d\mu$$

denotes the inner product on  $L^2_\mu[a, b]$ ; we will also let  $\|\cdot\|_\mu$  denote the corresponding norm. We define  $\Delta_\mu := -T$  and call it the *Dirichlet Laplacian with respect to  $\mu$* .

Let  $u \in \text{Dom } \mathcal{E}$  and  $f \in L^2_\mu[a, b]$ . It is known that  $u \in \text{Dom } (\Delta_\mu)$  and  $\Delta_\mu u = f$  if and only if  $\Delta u = f d\mu$  in the sense of distribution, i.e.,

$$\int_a^b u'v' dx = \int_a^b (-\Delta_\mu u)v d\mu \quad \text{for all } v \in C_c^\infty(a, b).$$

It is also known (see, e.g., [1, 19]) that there exists an orthonormal basis of eigenfunctions of  $\Delta_\mu$  and the eigenvalues  $\{\lambda_n\}$  are discrete and satisfy  $0 \leq \lambda_1 < \lambda_2 < \dots$  with  $\lim_{n \rightarrow \infty} \lambda_n = \infty$ .

The operators  $\Delta_\mu$  and their generalizations have been studied in connection with spectral functions of the string and diffusion processes (see [8, 9, 21]). More recently, they have been studied in connection with fractal measures (see [10, 11, 12, 13, 14, 1, 19, 29, 30, 33, 31]).

Our study of the operator  $\Delta_\mu$  is mainly motivated by the effort to extend the current theory of analysis on fractals to include iterated function systems (IFSs) with overlaps. Such IFSs do not satisfy the well-known post-critically finite condition or the open set condition. Nevertheless, by assuming  $\mu$  satisfies a family of second-order self-similar identities, some results concerning  $\Delta_\mu$  can be obtained. In [2], the finite element method is used to compute numerical approximations to the eigenvalues and eigenfunctions, and in [31], formulas defining the spectral dimension of  $\Delta_\mu$  have been obtained for a class of measures that include the infinite Bernoulli convolution associated with the golden ratio and the three-fold convolution of the Cantor measure.

The main purpose of this paper is to study one-dimensional wave equations defined by a class of fractal Laplacians, subject to the Dirichlet boundary condition. More precisely, we study the following non-homogeneous *hyperbolic initial/boundary value problem (IBVP)*:

$$(1.2) \quad \begin{cases} u_{tt} - \Delta_\mu u = f & \text{on } [a, b] \times [0, T], \\ u = 0 & \text{on } \{a, b\} \times [0, T], \\ u = g, u_t = h & \text{on } [a, b] \times \{t = 0\}. \end{cases}$$

The following existence and uniqueness result (see Definition 2.5 for the definition of a *weak solution*) follows easily from the general theory for wave equations in Hilbert spaces (see Section 2).

**Theorem 1.1.** *Assume  $g \in \text{Dom } \mathcal{E}$ ,  $h \in L^2_\mu[a, b]$  and  $f \in L^2(0, T; \text{Dom } \mathcal{E})$ . Then equation (1.2) has a unique weak solution.*

We are mainly interested in fractal measures  $\mu$ . Let  $D$  be a non-empty compact subset of  $\mathbb{R}^d$ . A function  $S : D \rightarrow D$  is called a *contraction* on  $D$  if there is a number  $c$  with  $0 < c < 1$  such that

$$(1.3) \quad |S(x) - S(y)| \leq c|x - y| \quad \text{for all } x, y \in D.$$

An *iterated function system (IFS)* on  $D$  is a finite collection of contractions on  $D$ . Each IFS  $\{S_i\}_{i=1}^q$  defines a unique compact subset  $F \subseteq D$ , called the *invariant set* or *attractor*, satisfying

$$F = \bigcup_{i=1}^q S_i(F).$$

Also, to each set of probability weights  $\{p_i\}_{i=1}^q$ , where  $p_i > 0$  and  $\sum_{i=1}^q p_i = 1$ , there exists a unique probability measure, called the *invariant measure*, satisfying the identity

$$(1.4) \quad \mu = \sum_{i=1}^q p_i \mu \circ S_i^{-1}$$

(see [20, 7]).  $S$  is a contractive *similitude* if equality in (1.3) holds. IFSs studied in this paper consist of contractive similitudes; they are of the form

$$(1.5) \quad S_i(x) = \rho_i R_i x + b_i, \quad i = 1, \dots, q,$$

where  $0 < \rho_i < 1$ ,  $R_i$  is an orthogonal transformation, and  $b_i \in \mathbb{R}^d$ . For such an IFS, we call the corresponding invariant set  $F$  the *self-similar set* and the invariant measure  $\mu$  the *self-similar measure*.

An IFS  $\{S_i\}_{i=1}^q$  is said to satisfy the *open set condition (OSC)* if there exists a non-empty bounded open set  $U$  such that  $\cup_i S_i(U) \subseteq U$  and  $S_i(U) \cap S_j(U) = \emptyset$  if  $i \neq j$ . An IFS that does not satisfy the OSC is said to have *overlaps*. For an IFS of contractive similitudes, it is known that if the linear parts of the IFS maps are commensurable, then the p.c.f. condition implies the OSC [5].

We are interested in one-dimensional self-similar measures defined by IFSs with overlaps. Such IFSs are not p.c.f. and are thus not covered by Kigami's theory. In order to discretize a wave equation and obtain numerical approximations to the weak solution, we will assume that the corresponding self-similar measure satisfies a family of second-order self-similar identities, an idea introduced by Strichartz *et al.* [35]. Let  $\{S_i\}_{i=1}^q$  be an IFS of contractive similitudes on  $\mathbb{R}$  and let  $\mu$  be the corresponding self-similar measure. Assume, in addition, that  $\text{supp}(\mu) = [a, b]$ . Define an auxiliary IFS

$$(1.6) \quad T_j(x) = \rho^{n_j} x + d_j, \quad j = 1, 2, \dots, N,$$

where  $n_j \in \mathbb{N}$  and  $d_j \in \mathbb{R}$ , and let

$$(1.7) \quad \rho := \max\{\rho^{n_j} : 1 \leq j \leq N\}.$$

$\mu$  is said to satisfy a family of *second-order self-similar identities* (or simply *second-order identities*) with respect to  $\{T_j\}_{j=1}^N$  (see [25]) if

- (i)  $\text{supp}(\mu) \subseteq \bigcup_{j=1}^N T_j(\text{supp}(\mu))$ , and
- (ii) for each Borel subset  $A \subseteq \text{supp}(\mu)$  and  $0 \leq i, j \leq N$ ,  $\mu(T_i T_j A)$  can be expressed as a linear combination of  $\{\mu(T_k A) : k = 1, \dots, N\}$  as

$$\mu(T_i T_j A) = \sum_{k=1}^N d_k \mu(T_k A),$$

where  $d_k = d_k(i, j)$  are independent of  $A$ . In matrix form,

$$(1.8) \quad \begin{bmatrix} \mu(T_1 T_j A) \\ \vdots \\ \mu(T_N T_j A) \end{bmatrix} = M_j \begin{bmatrix} \mu(T_1 A) \\ \vdots \\ \mu(T_N A) \end{bmatrix}, \quad j = 1, \dots, N, \quad \text{or equivalently,}$$

$$\mu(T_i T_j A) = \mathbf{e}_i M_j \begin{bmatrix} \mu(T_1 A) \\ \vdots \\ \mu(T_N A) \end{bmatrix}, \quad i, j = 1, \dots, N,$$

where  $e_i$  is the  $i$ th row of the  $N \times N$  identity matrix, and  $M_j$  is some  $N \times N$  matrix independent of  $A$ .

For our purposes, we will assume that  $\{T_j\}_{j=0}^N$  satisfies the OSC. The  $m$ -th level iteration of the auxiliary IFS  $\{T_j\}_{j=0}^N$  induces a partition  $V_m$  of  $\text{supp}(\mu) = [a, b]$ . Moreover, the  $\mu$  measure of each subinterval in the partition can be computed in terms of a matrix product. This provides us with a good way to discretize the wave equation.

By letting  $f(x, t) = 0$ , multiplying the first equation in (1.2) by  $v \in \text{Dom } \mathcal{E}$ , integrating both sides with respect to  $d\mu$ , and then using integration by parts, we obtain

$$(1.9) \quad - \int_a^b u_x(x, t) v'(x) dx = \int_a^b u_{tt}(x, t) v(x) d\mu,$$

where  $u_x(x, t)$  is the weak partial derivative of  $u$  with respect to  $x$  and  $u_{tt}$  is the weak second partial derivative with respect to  $t$ .

**Theorem 1.2.** *Let  $\mu$  be a self-similar measure defined by a one-dimensional IFS of contractive similitudes on  $\mathbb{R}$  as in (1.4) and (1.5). Assume that  $\text{supp}(\mu) = [a, b]$  and that  $\mu$  satisfies a family of second-order self-similar identities. Then the finite element method for the equation (1.9) discretizes it to a system of second-order ordinary differential equations (3.9), which has a unique solution (and can be solved numerically).*

Based on this result, we solve the homogeneous IBVP (1.2) numerically for three different measures, namely, the weighted Bernoulli-type measure, the infinite Bernoulli convolution associated with the golden ratio, and the 3-fold convolution of the Cantor measure. We show that the approximate solutions converge to the actual weak solution and obtain a rate of convergence.

**Theorem 1.3.** *Assume the same hypotheses as in Theorem 1.2. Let  $f = 0$  in equation (1.2) and fix  $t \in [0, T]$ . Then the approximate solutions  $u^m$  obtained by the finite element method converge in  $L^2_\mu[a, b]$  to the actual weak solution  $u$ . Moreover, there exists a constant  $C > 0$  such that for all  $m \geq 1$ ,*

$$\|u^m - u\|_\mu \leq (C\sqrt{T} \|u_{tt}\|_{2, \text{Dom } \mathcal{E}} + 2 \|u\|_{\text{Dom } \mathcal{E}}) \rho^{m/2}.$$

This paper is organized as follows. We summarize some basic classical results, definitions, and notation in Section 2. In Section 3 we use the finite element and central difference methods to obtain numerical approximations to the corresponding homogeneous IBVP (1.2), proving Theorem 1.2. In Section 4 we apply our numerical methods to the above-mentioned measures, and illustrate some numerical results. In Section 5 we prove the convergence of the approximation scheme and obtain an estimate for the convergence rate stated in Theorem 1.3.

## 2. PRELIMINARIES

In this section, we summarize some notation, definitions, and preliminary results that will be used throughout the rest of the paper. For a Banach space  $X$ , we denote its topological dual by  $X'$ . For  $v \in X'$  and  $u \in X$  we let  $\langle v, u \rangle = \langle v, u \rangle_{X', X} := v(u)$  denote the *dual pairing* of  $X'$  and  $X$ .

A function  $s : [0, T] \rightarrow X$  is called *simple* if it has the form

$$(2.1) \quad s(t) = \sum_{m=1}^N \chi_{E_m}(t) u_m \quad \text{for } t \in [0, T],$$

where each  $E_m$  is a Lebesgue measurable subset of  $[0, T]$ ,  $u_m \in X$  for  $m = 1, \dots, N$ , and  $\chi_{E_m}$  is the characteristic function on  $E_m$ . A function  $u : [0, T] \rightarrow X$  is *strongly measurable* if there exist simple functions  $s_n : [0, T] \rightarrow X$  such that

$$s_n(t) \rightarrow u(t) \quad \text{as } n \rightarrow \infty \text{ for Lebesgue a.e. } t \in [0, T].$$

A function  $u : [0, T] \rightarrow X$  is *weakly measurable* if for each  $v \in X'$ , the mapping  $t \mapsto \langle v, u(t) \rangle$  is Lebesgue measurable.

A function  $u : [0, T] \rightarrow X$  is *almost separably valued* if there exists a subset  $E \subseteq [0, T]$  with zero Lebesgue measure such that the set  $\{u(t) : t \in [0, T] \setminus E\}$  is separable. By a theorem of Pettis [32], a function  $u : [0, T] \rightarrow X$  is strongly measurable if and only if it is weakly measurable and almost separably valued. Since any subset of a separable Banach space  $X$  is separable, the two concepts of measurability coincide and we can use the term *measurable* without ambiguity.

**Definition 2.1.** Let  $X$  be a separable Banach space with norm  $\|\cdot\|_X$ . Define  $L^p(0, T; X)$  to be the space of all measurable functions  $u : [0, T] \rightarrow X$  satisfying

- (a)  $\|u\|_{L^p(0, T; X)} := \left( \int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}} < \infty$ , if  $1 \leq p < \infty$ , and
- (b)  $\|u\|_{L^\infty(0, T; X)} := \text{ess sup}_{0 \leq t \leq T} \|u(t)\|_X < \infty$ , if  $p = \infty$ .

If the interval  $[0, T]$  is understood, we will abbreviate these norms as  $\|u\|_{p, X}$  and  $\|u\|_{\infty, X}$ .

**Remark 2.1.** For each  $1 \leq p \leq \infty$ ,  $L^p(0, T; X)$  is a Banach space; moreover,  $L^2(0, T; X) \subseteq L^{p_1}(0, T; X)$  if  $0 \leq p_1 \leq p_2 \leq \infty$ . If  $(X, (\cdot, \cdot)_X)$  is a separable Hilbert space, then  $L^2(0, T; X)$  is a Hilbert space with the inner product

$$(u, v)_{L^2(0, T; X)} := \int_0^T (u(t), v(t))_X dt.$$

**Definition 2.2.** Let  $X$  be a Banach space and  $u \in L^1(0, T; X)$ . We say  $v \in L^1(0, T; X)$  is the weak derivative of  $u$ , written  $u_t = v$ , if

$$\int_0^T \phi_t(t) u(t) dt = - \int_0^T \phi(t) u_t(t) dt$$

for all scalar test functions  $\phi \in C_c^\infty(0, T)$ .

**Definition 2.3.** Let  $X$  be a Banach space and  $X'$  its dual. We say a sequence  $\{u_m\}_{m=1}^\infty \subseteq X$  converges weakly to  $u \in X$ , written  $u_m \rightharpoonup u$ , if

$$\langle v, u_m \rangle \rightarrow \langle v, u \rangle$$

for each bounded linear functional  $v \in X'$ .

For the more general definition of derivatives of distributions with values in a Hilbert space, we refer the reader to [41, Section 25].

The notion of a Gelfand triple [15], defined below, plays an important role in our investigation of the wave equation.

**Definition 2.4.** Let  $V$  and  $H$  be separable Hilbert spaces with the continuous injective dense embedding  $\iota : V \hookrightarrow H$ . By identifying  $H$  with its dual  $H'$ , we obtain the following continuous and dense embedding:

$$V \hookrightarrow H \cong H' \hookrightarrow V'.$$

Assume in addition that the dual pairing between  $V$  and  $V'$  is compatible with the inner product on  $H$ , in the sense that

$$\langle v, u \rangle_{V',V} = (v, u)_H$$

for all  $u \in V \subset H$  and  $v \in H \cong H' \subset V'$ . The triple  $(V, H, V')$  is called a Gelfand triple (The pair  $(H, V)$  is also called a rigged Hilbert space.)

We remark that since  $V$  is itself a Hilbert space, it is isomorphic with its dual  $V'$ . However, this isomorphism is in general not the same as the composition  $\iota^* \iota : V \subset H = H' \hookrightarrow V'$ , where  $\iota^*$  is the adjoint of  $\iota$ .

Throughout the rest of this section we let  $\mu$  be a finite positive Borel measure on  $\mathbb{R}$  with  $\text{supp}(\mu) \subseteq [a, b]$  and  $\mu(a, b) > 0$ , where  $-\infty < a < b < \infty$ . It is known that the following important condition is satisfied (see e.g., [19, 29]): There exists a constant  $C > 0$  such that

$$(2.2) \quad \int_a^b |u|^2 d\mu \leq C \int_a^b |\nabla u|^2 dx \quad \text{for all } u \in C_c^\infty(a, b).$$

This condition implies that each equivalence class  $u \in H_0^1(a, b)$  contains a unique (in  $L_\mu^2[a, b]$  sense) member  $\bar{u}$  that belongs to  $L_\mu^2[a, b]$  and satisfies both conditions below:

- (1) There exists a sequence  $\{u_n\}$  in  $C_c^\infty(a, b)$  such that  $u_n \rightarrow \bar{u}$  in  $H_0^1(a, b)$  and  $u_n \rightarrow \bar{u}$  in  $L_\mu^2[a, b]$ ;
- (2)  $\bar{u}$  satisfies the inequality in (2.2).

We call  $\bar{u}$  the  $L_\mu^2[a, b]$ -representative of  $u$ . Assume condition (2.2) holds and define a mapping  $\iota : H_0^1(a, b) \rightarrow L_\mu^2[a, b]$  by

$$\iota(u) = \bar{u}.$$

$\iota$  is a bounded linear operator.  $\iota$  is not necessarily injective, because it is possible for a non-zero function  $u \in H_0^1(a, b)$  to have an  $L_\mu^2[a, b]$ -representative that has zero  $L_\mu^2[a, b]$ -norm. To deal with this situation, we consider the subspace  $\mathcal{N}$  of  $H_0^1(a, b)$  defined as

$$\mathcal{N} := \{u \in H_0^1(a, b) : \|\iota(u)\|_\mu = 0\}.$$

The continuity of  $\iota$  implies that  $\mathcal{N}$  is a closed subspace of  $H_0^1(a, b)$ . Let  $\mathcal{N}^\perp$  be the orthogonal complement of  $\mathcal{N}$  in  $H_0^1(a, b)$ . It is clear that  $\iota : \mathcal{N}^\perp \rightarrow L_\mu^2[a, b]$  is injective, and we can identify  $\mathcal{N}^\perp$  and  $\iota(\mathcal{N}^\perp)$ .  $\mathcal{N}^\perp$  is dense in  $L_\mu^2[a, b]$  (see [19]). Throughout this paper, we let

$$\text{Dom } \mathcal{E} := \mathcal{N}^\perp \quad \text{and} \quad \|\cdot\|_{\text{Dom}} = \|\cdot\|_{H_0^1(a, b)}.$$

The equivalence classes represented by  $u \in H_0^1(a, b)$  and  $\bar{u} \in L_\mu^2[a, b]$  are in general different. Corollary 2.3 below says that the continuous representative of  $u$  lies in the intersection of these two equivalence classes. We will frequently identify  $\bar{u}$  and  $u$  without mention.

**Proposition 2.2.** *Let  $u \in H_0^1(a, b)$  and  $\{\phi_n\} \subset C_c^\infty(a, b)$  such that  $\phi_n \rightarrow u$  in  $H_0^1(a, b)$ . Then there exists a subsequence  $\{\phi_{n_k}\}$  such that  $\phi_{n_k} \rightarrow u_c$  everywhere in  $[a, b]$ , where  $u_c$  is the continuous representative of the equivalence class of  $u$  in  $H_0^1(a, b)$ .*

*Proof.* Let  $\{\phi_{n_k}\}$  be a subsequence converging pointwise Lebesgue a.e. to  $u_c$  on  $(a, b)$ . Let  $x \in (a, b)$  and  $\epsilon > 0$  be arbitrary. First, since  $\phi_n$  is convergent, there exists  $C > 0$  such that

$$(2.3) \quad \|\phi_n\|_{\text{Dom}} \leq C \quad \text{for all } n \in \mathbb{N}.$$

Next, by the continuity of  $u_c$ , there exists  $0 < \delta_\epsilon < (\epsilon/(3C))^2$  such that for all  $y \in [a, b]$ , with  $|y - x| < \delta_\epsilon$ , we have

$$(2.4) \quad |u_c(x) - u_c(y)| < \epsilon/3.$$

Hence,

$$(2.5) \quad |\phi_{n_k}(x) - u_c(x)| \leq |\phi_{n_k}(x) - \phi_{n_k}(y)| + |\phi_{n_k}(y) - u_c(y)| + |u_c(y) - u_c(x)|.$$

The first term can be estimated by using (2.3) as follows:

$$(2.6) \quad |\phi_{n_k}(x) - \phi_{n_k}(y)| = \left| \int_y^x \phi'_{n_k}(s) ds \right| \leq \left( \int_y^x |\phi'_{n_k}(s)|^2 ds \right)^{1/2} |x - y|^{1/2} \\ \leq \|\phi_{n_k}\|_{\text{Dom } \mathcal{E}} |x - y|^{1/2} \leq C \cdot \epsilon/(3C) \leq \epsilon/3.$$

Substituting (2.4) and (2.6) into (2.5), we get

$$|\phi_{n_k}(x) - u_c(x)| \leq \epsilon/3 + |\phi_{n_k}(y) - u_c(y)| + \epsilon/3$$

for all  $y \in (x - \delta_\epsilon, x + \delta_\epsilon)$ .

Last, let  $y \in (a, b)$  satisfy  $\lim_{k \rightarrow \infty} \phi_{n_k}(y) = u_c(y)$ . Then, for all  $k$  sufficient large, we have  $|\phi_{n_k}(y) - u_c(y)| < \epsilon/3$ , and hence  $|\phi_{n_k}(x) - u_c(x)| < \epsilon$ . Thus,  $\lim_{k \rightarrow \infty} \phi_{n_k}(x) = u_c(x)$  for all  $x \in [a, b]$ .  $\square$

**Corollary 2.3.** *Let  $u \in H_0^1(a, b)$  and let  $\bar{u}$  be its unique  $L_\mu^2[a, b]$  representative. Then  $u_c$  lies in the equivalence class of  $\bar{u}$  in  $L_\mu^2[a, b]$ .*

**Corollary 2.4.** *If  $\text{supp}(\mu) = [a, b]$ , then  $\iota : H_0^1(a, b) \rightarrow L_\mu^2[a, b]$  is injective. Consequently,  $\text{Dom } \mathcal{E} = H_0^1(a, b)$ .*

*Proof.* Let  $u \in H_0^1(a, b)$  such that  $\iota(u) = 0$ . Then we have  $\bar{u} = u_c = 0$  in  $L_\mu^2[a, b]$ . Since  $\text{supp}(u) = [a, b]$ , we have  $u_c \equiv 0$  on  $[a, b]$ . Thus,  $u = 0$  Lebesgue a.e. on  $[a, b]$ .  $\square$

For a function  $\varphi : (a, b) \rightarrow \mathbb{R}$ , we let  $\varphi'$  denote both its classical and weak derivatives. If  $u \in L^2(0, T; X)$ , where  $X$  is  $H_0^1(a, b)$ , or  $L_\mu^2[a, b]$  etc., then for each fixed  $t$  we denote by  $u_x(x, t)$  (or  $\nabla u$ ) the classical or weak derivative of  $u$  with respect to  $x$ .

The spaces  $\text{Dom } \mathcal{E}$ ,  $L_\mu^2[a, b]$ ,  $(\text{Dom } \mathcal{E})'$  form a *Gelfand triple*:

$$\text{Dom } \mathcal{E} \hookrightarrow L_\mu^2[a, b] \cong (L_\mu^2[a, b])' \hookrightarrow (\text{Dom } \mathcal{E})',$$

where we identify  $L_\mu^2[a, b]$  with  $(L_\mu^2[a, b])'$ . The embedding  $L_\mu^2[a, b] \hookrightarrow (\text{Dom } \mathcal{E})'$  is given by

$$w \in L_\mu^2[a, b] \mapsto (w, \cdot)_\mu \in (L_\mu^2[a, b])' \subset (\text{Dom } \mathcal{E})'.$$

We define weak solution of the IBVP (1.2) (see, e.g., [6, 41]).

**Definition 2.5.** *Let  $g \in \text{Dom } \mathcal{E}$ ,  $h \in L_\mu^2[a, b]$ , and  $f \in L^2(0, T; \text{Dom } \mathcal{E})$ . A function  $u \in L^2(0, T; \text{Dom } \mathcal{E})$ , with  $u_t \in L^2(0, T; L_\mu^2[a, b])$  and  $u_{tt} \in L^2(0, T; (\text{Dom } \mathcal{E})')$  is a weak solution of IBVP (1.2) if the following conditions are satisfied:*

- (i)  $\langle u_{tt}, v \rangle + \mathcal{E}(u, v) = (f, v)_\mu$  for each  $v \in \text{Dom } \mathcal{E}$  and Lebesgue a.e.  $t \in [0, T]$ ;
- (ii)  $u(x, 0) = g(x)$  and  $u_t(x, 0) = h(x)$  for all  $x \in [a, b]$ .

Here  $\langle \cdot, \cdot \rangle$  denotes the pairing between  $(\text{Dom } \mathcal{E})'$  and  $\text{Dom } \mathcal{E}$ .



**Remark 2.5.** (a) In (i) above, if  $u_{tt} \in \text{Dom } \mathcal{E}$  or  $u_{tt} \in L^2_\mu[a, b]$ , then  $\langle u_{tt}, v \rangle = (u_{tt}, v)_\mu$  as in the definition of Gelfand triple.

(b) Given the Gelfand triple  $\text{Dom } \mathcal{E} \hookrightarrow L^2_\mu[a, b] \hookrightarrow (\text{Dom } \mathcal{E})'$ , for  $u \in L^2(0, T; \text{Dom } \mathcal{E})$  we also have  $u \in L^2(0, T; L^2_\mu[a, b])$  and  $u \in L^2(0, T; (\text{Dom } \mathcal{E})')$  and thus  $u \in L^1(0, T; L^2_\mu[a, b])$  and  $u \in L^1(0, T; (\text{Dom } \mathcal{E})')$ . Therefore, it makes sense to require that  $u$  has weak derivatives  $u_t \in L^1(0, T; L^2_\mu[a, b])$  and  $u_{tt} \in L^1(0, T; (\text{Dom } \mathcal{E})')$  and to require in addition that  $u_t \in L^2(0, T; L^2_\mu[a, b])$  and  $u_{tt} \in L^2(0, T; (\text{Dom } \mathcal{E})')$ .

Let  $V, H$  be Hilbert spaces, where  $V$  is separable. Assume that the embedding  $V \hookrightarrow H$  is continuous, injective, and dense so that

$$V \hookrightarrow H \hookrightarrow V'$$

form a Gelfand triple (see [41, Section 17]). Let  $0 < T < \infty$ , and assume that for  $t \in [0, T]$ ,  $a(t, \varphi, \psi)$  is a continuous sesquilinear form, i.e.,

$$(2.7) \quad |a(t, \varphi, \psi)| \leq c \|\varphi\|_V \|\psi\|_V, \quad \forall \varphi, \psi \in V,$$

where  $c > 0$  is a constant independent of  $t$ . Then there exists a representation operator

$$L(t) : V \rightarrow V',$$

such that for each  $t$ ,  $L(t)$  is linear and continuous, with

$$a(t; \varphi, \psi) = (L(t)\varphi, \psi)_H.$$

Assume that for all  $\varphi, \psi \in V$  the function  $t \mapsto a(t; \varphi, \psi)$  is continuously differentiable for  $t \in [0, T]$ , i.e.,

$$(2.8) \quad a(t; \varphi, \psi) \in C^1[0, T], \quad \forall \varphi, \psi \in V,$$

where

$$\left| \frac{d}{dt} a(t; \varphi, \psi) \right| \leq c \|\varphi\|_V \|\psi\|_V, \quad \forall t \in [0, T],$$

where  $c$  is independent of  $t$ .

Assume further that  $a(t; \varphi, \psi)$  is *antisymmetric*, i.e.,

$$(2.9) \quad a(t; \varphi, \psi) = \overline{a(t; \psi, \varphi)}, \quad \forall \varphi, \psi \in V.$$

Finally, assume *V-coersion*, i.e., there exist constants  $\alpha, \beta > 0$  such that

$$(2.10) \quad a(t; \varphi, \varphi) + \beta \|\varphi\|_H^2 \geq \alpha \|\varphi\|_V^2, \quad \forall t \in [0, T] \text{ and } \forall \varphi \in V.$$

The proof of Theorems 2.6 and 2.7 stated below can be found in [41, Sections 29–30].

**Theorem 2.6.** *Let  $V, H$  be Hilbert spaces where  $V$  is separable. Assume that the embedding  $V \hookrightarrow H$  is injective, continuous, and dense so that  $V \hookrightarrow H \hookrightarrow V'$  form a Gelfand triple. Assume conditions (2.7)–(2.10) above hold. Then for any  $f \in L^2(0, T; H)$ ,  $0 < T < \infty$ , and initial conditions*

$$u_0 \in V, \quad u_1 \in H,$$

*there exists a unique function  $u(t) \in L^2(0, T; V)$ , with  $du/dt \in L^2(0, T; H)$ , so that*

$$(2.11) \quad \frac{d^2 u}{dt^2} + L(t)u = f \quad \text{for } t \in [0, T], \quad u(0) = u_0, \quad \frac{du(0)}{dt} = u_1,$$

*in the sense that*

$$\left( \frac{d^2 u}{dt^2}, \varphi \right)_H + (L(t)u, \varphi)_H = (f, \varphi)_H, \quad \forall \varphi \in V.$$

**Definition 2.6.** Let  $V$  be a Hilbert space. For each integer  $k \geq 0$ , define the Sobolev space

$$W_2^k(0, T; V) := \left\{ u : (0, T) \rightarrow V \text{ measurable} : \frac{d^n u}{dt^n} \in L^2(0, T; V) \text{ for } 0 \leq n \leq k \right\},$$

where the differentiation is in the distributional sense. Equip  $W_2^k(0, T; V)$  with the norm

$$\|u\|_k^2 := \sum_{n=0}^k \int_0^T \left\| \frac{d^n u}{dt^n} \right\|_V^2 dt.$$

The smoothness of the solution of equation (2.11) increases with that of  $f$ , as shown in the theorem below.

**Theorem 2.7.** Assume the same hypotheses of Theorem 2.6 and assume that  $a(\varphi, \psi)$  and  $L$  are independent of  $t$ . Consider the hyperbolic equation

$$(2.12) \quad \frac{d^2 u}{dt^2} + Lu = f \quad \text{for } t \in (0, T),$$

with the initial conditions

$$(2.13) \quad u(0) = u_0, \quad \frac{du(0)}{dt} = u_1.$$

Assume, in addition, that

$$f \in W_2^k(0, T; H), \quad k \geq 1,$$

and

$$u_0, u_1, f''(0), \dots, f^{(k-3)}(0) \in V \quad \text{and} \quad f^{(k-2)}(0) - Lf^{(k-3)}(0) \in H.$$

Then the solution  $u$  of (2.12) and (2.13) satisfies

$$u \in W_2^{k-1}(0, T; V), \quad \frac{d^k u(t)}{dt^k} \in L^2(0, T; H), \quad \frac{d^{k+1} u(t)}{dt^{k+1}} \in L^2(0, T; V').$$

*Proof of Theorem 1.1.* In order to apply Theorem 2.6, we let  $V = \text{Dom } \mathcal{E}$ ,  $H = L_\mu^2[a, b]$ , and let  $a(t; u, v) = \mathcal{E}(u, v)$ , which independent of  $t$ . Then for all  $u, v \in \text{Dom } \mathcal{E}$ ,

$$|a(t; u, v)| = \left| \int u'v' dx \right| \leq \left( \int |u'|^2 dx \right)^{1/2} \left( \int |v'|^2 dx \right)^{1/2} = \|u\|_{\text{Dom } (\mathcal{E})} \|v\|_{\text{Dom } (\mathcal{E})},$$

and thus condition (2.7) holds. Also,  $\mathcal{E}$  is bilinear. Thus, there exists a representation operator  $L : \text{Dom } \mathcal{E} \rightarrow (\text{Dom } \mathcal{E})'$  such that

$$\mathcal{E}(u, v) = (Lu, v)_{L_\mu^2[a, b]}.$$

$L = -\Delta_\mu$  on  $\text{Dom } (-\Delta_\mu)$ .

Next, since  $t \mapsto a(t; u, v) = \mathcal{E}(u, v)$  is constant in time and real valued, conditions (2.8) and (2.9) clearly hold.

Lastly, for all  $t \in [0, T]$  and  $u \in V$ ,

$$a(t; u, u) + \|u\|_{L_\mu^2[a, b]}^2 \geq \mathcal{E}(u, u) = \|u\|_{\text{Dom } \mathcal{E}}^2,$$

and thus  $\text{Dom } \mathcal{E}$ -coersion (condition (2.10)) holds with  $\alpha = \beta = 1$ . Theorem 1.1 now follows from Theorem 2.6.  $\square$

As a consequence of Theorem 2.7, we have the following regularity result for solutions of homogeneous wave equations in our setting.

**Theorem 2.8.** *Assume the same hypotheses of Theorem 1.1 and assume, in addition, that  $h \in \text{Dom } \mathcal{E}$  and  $f = 0$ . Then the solution of the homogeneous equation (1.2) satisfies:*

$$u \in W_2^{k-1}(0, T; \text{Dom } \mathcal{E}), \quad \frac{d^k u}{dt^k} \in L^2(0, T; L_\mu^2[a, b]), \quad \frac{d^{k+1} u}{dt^{k+1}} \in L^2(0, T; (\text{Dom } \mathcal{E})'), \quad k \geq 1.$$

Theorem 2.8 will be used in proving Theorem 1.3.

### 3. THE FINITE ELEMENT METHOD

In this section, we let  $f = 0$  in equation (1.2) and use the finite element method to solve the homogeneous IBVP. We only consider self-similar measures  $\mu$  (see (1.4)) defined by an IFS  $\{S_i\}_{i=1}^q$  of contractive similitudes of the form

$$S_i(x) = \rho x + b_i, \quad i = 1, \dots, q.$$

We assume in addition that  $\mu$  satisfies a family of second-order self-similar identities with respect to an auxiliary IFS  $\{T_j\}_{j=1}^N$  of the form (1.6). Assume also that  $\{T_j\}_{j=1}^N$  satisfies the OSC.

For each multi-index  $J = (j_1, \dots, j_m) \in \{1, \dots, N\}^m$ , we let  $T_J[a, b]$  be the interval  $[x_{i-1}, x_i]$ , where the index  $i$  is obtained directly from  $J$  as follows (see [2]):

$$i = i(J) := (j_1 - 1)N^{m-1} + (j_2 - 1)N^{m-2} + \dots + (j_m - 1)N^0 + 1.$$

For example, if  $J = (1, \dots, 1)$ , then  $i(J) = 1$ , and if  $J = (N, \dots, N)$ , then  $i(J) = N^m$ . We call  $T_J[a, b]$  a *level- $m$  subinterval*. It follows that

$$(3.1) \quad T_{J_i}[a, b] := T_J[a, b] = [x_{i-1}, x_i] \quad \text{and} \quad T_{J_i}(x) := T_J(x) = (x_i - x_{i-1}) \frac{x - a}{b - a} + x_{i-1}.$$

We apply the finite element method to approximate the weak solution  $u(x, t)$  satisfying (1.9) by

$$(3.2) \quad u^m(x, t) = \sum_{j=0}^{N^m} \beta_j(t) \phi_j(x),$$

where for  $j = 0, 1, \dots, N^m$ ,  $\beta_j(t) = \beta_{m,j}(t)$  are functions to be determined, and  $\phi_j(x) := \phi_{m,j}(x)$  are the standard piecewise linear *finite element basis functions* (also called *tent functions*) defined as

$$(3.3) \quad \phi_j(x) := \begin{cases} \frac{x-x_{j-1}}{x_j-x_{j-1}} & \text{if } x_{j-1} \leq x \leq x_j, \quad j = 1, \dots, N^m \\ \frac{x-x_{j+1}}{x_j-x_{j+1}} & \text{if } x_j \leq x \leq x_{j+1}, \quad j = 0, \dots, N^m - 1 \\ 0 & \text{otherwise.} \end{cases}$$

We require  $u^m(x, t)$  to satisfy the integral form of the homogeneous wave equation

$$(3.4) \quad \int_a^b u_{tt}^m(x, t) \phi_i(x) d\mu = - \int_a^b \nabla u^m(x, t) \phi_i'(x) dx, \quad \text{for } i = 0, 1, \dots, N^m,$$

where  $u_{tt}^m := (u^m)_{tt}$ .

As  $u^m(a, t) = u^m(b, t) = 0$  we have  $\beta_0(t) = \beta_{N^m}(t) = 0$ . Using this and substituting (3.2) into (3.4) gives

$$(3.5) \quad \sum_{j=1}^{N^m-1} \beta_j''(t) \int_a^b \phi_i(x) \phi_j(x) d\mu = - \sum_{j=1}^{N^m-1} \beta_j(t) \int_a^b \phi_i'(x) \phi_j'(x) dx, \quad 1 \leq i \leq N^m - 1.$$

We define the *mass matrix*  $\mathbf{M} = (M_{ij})$  and *stiffness matrix*  $\mathbf{K} = (K_{ij})$  respectively as

$$(3.6) \quad M_{ij} = \int_a^b \phi_i(x) \phi_j(x) d\mu, \quad K_{ij} = - \int_a^b \phi_i'(x) \phi_j'(x) dx, \quad 1 \leq i, j \leq N^m - 1.$$

Both  $\mathbf{M}$  and  $\mathbf{K}$  are tridiagonal and of order  $(N^m - 1) \times (N^m - 1)$ . Let

$$\mathbf{w}(t) =: \begin{bmatrix} w_1(t) \\ \vdots \\ w_{N^m-1}(t) \end{bmatrix} = \begin{bmatrix} \beta_1(t) \\ \vdots \\ \beta_{N^m-1}(t) \end{bmatrix}.$$

be a vector-valued function. Then (3.5) can be put in a matrix form as

$$(3.7) \quad \mathbf{M}\mathbf{w}'' = -\mathbf{K}\mathbf{w}.$$

This gives us a system of second-order linear ODEs with constant coefficients. We need two initial conditions. The initial condition  $u(x, 0) = g(x)$  for  $a \leq x \leq b$  can be approximated by its linear interpolant:

$$\tilde{g}(x) = \sum_{i=1}^{N^m-1} g(x_i) \phi_i(x).$$

Therefore, we set

$$w_i(0) = g(x_i) \quad \text{and} \quad w_i'(0) = h(x_i).$$

These lead to the initial conditions

$$(3.8) \quad \mathbf{w}(0) = \mathbf{w}_0 = \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_{N^m-1}) \end{bmatrix}, \quad \mathbf{w}'(0) = \mathbf{w}'_0 = \begin{bmatrix} h(x_1) \\ \vdots \\ h(x_{N^m-1}) \end{bmatrix}.$$

Consequently, we get the linear system

$$(3.9) \quad \begin{cases} \mathbf{M} \frac{d^2 \mathbf{w}}{dt^2} = -\mathbf{K} \mathbf{w}, & t > 0 \\ \mathbf{w}(0) = \mathbf{w}_0, \quad \mathbf{w}'(0) = \mathbf{w}'_0. \end{cases}$$

We describe how to compute  $\mathbf{M}$ ; the matrix  $\mathbf{K}$  can be computed directly. By using the definition of the  $\phi_i$ 's and (3.1), we have

$$(3.10) \quad \begin{aligned} M_{i,i} &= \frac{1}{(b-a)^2} \left( \int_a^b (x-a)^2 d\mu \circ T_{J_i} + \int_a^b (b-x)^2 d\mu \circ T_{J_{i+1}} \right), \quad 1 \leq i \leq N^m - 1, \\ M_{i,i-1} &= \frac{1}{(b-a)^2} \int_a^b (x-a)(b-x) d\mu \circ T_{J_i}, \quad 2 \leq i \leq N^m - 1, \\ M_{i,i+1} &= \frac{1}{(b-a)^2} \int_a^b (x-a)(b-x) d\mu \circ T_{J_{i+1}}, \quad 1 \leq i \leq N^m - 2. \end{aligned}$$

Define

$$(3.11) \quad \mathcal{I}_{k,j} := \int_a^b x^k d\mu \circ T_j, \quad \mathcal{J}_{k,j} := \int_{T_j[a,b]} x^k d\mu, \quad k = 0, 1, 2, \quad j = 1, \dots, N.$$

We will regard the  $\mathcal{I}_{k,j}$  and  $\mathcal{J}_{k,j}$  as known constants. In fact, for all examples we study, they can be computed exactly (see Section 4). A sufficient condition for computing them explicitly is given in [2].

**Lemma 3.1.** *The matrix  $\mathbf{M}$  is completely determined by the integrals  $\mathcal{I}_{k,j}$ , or equivalently,  $\mathcal{J}_{k,j}$ , where  $k = 0, 1, 2$  and  $j = 1, \dots, N$ .*

*Proof.* For  $J = (j_1, \dots, j_m) \in \{1, \dots, N\}^m$ , iterating (1.8) shows that for any Borel subset  $A \subseteq \text{supp}(\mu)$ ,

$$(3.12) \quad \mu(T_J A) = c_J \begin{bmatrix} \mu(T_1 A) \\ \vdots \\ \mu(T_N A) \end{bmatrix},$$

where  $c_J := [c_J^1, \dots, c_J^N] := \mathbf{e}_{j_1} M_{j_2} \cdots M_{j_m}$ . That is,

$$(3.13) \quad \mu(T_J A) = \sum_{j=1}^N c_J^j \mu(T_j A).$$

In view of the fact that  $\mathbf{M}$  is tridiagonal, and the expressions for  $M_{i,i}$ ,  $M_{i,i-1}$ , and  $M_{i,i+1}$ , the entries of  $\mathbf{M}$  are completely determined by the integrals

$$\int_a^b x^k d\mu \circ T_J, \quad k = 0, 1, 2, \quad J \in \{1, \dots, N\}^m,$$

which, by virtue of (3.13), can be written as

$$\sum_{j=1}^N c_J^j \int_a^b x^k d\mu \circ T_j.$$

This proves that  $\mathbf{M}$  is determined by the  $\mathcal{I}_{k,j}$ . Lastly, since

$$\int_a^b x^k d\mu \circ T_j = \int_{T_j[a,b]} (T_j^{-1}x)^k d\mu \quad \text{and} \quad \int_{T_j[a,b]} x^k d\mu = \int_a^b (T_j x)^k d\mu \circ T_j,$$

$\mathbf{M}$  is also determined by the  $\mathcal{J}_{k,j}$ . □

The system in (3.9) has a unique solution if  $\mathbf{M}$  is invertible.

**Proposition 3.2.** *Assume that  $\text{supp}(\mu) = [a, b]$ . Then the mass matrix  $\mathbf{M}$  is invertible. Consequently, (3.9) has a unique solution  $\mathbf{w}(t)$ ; moreover,  $\beta_j(t) \in C^2(0, T)$  for  $j = 1, \dots, N^m - 1$ .*

*Proof.* If the mass matrix  $\mathbf{M}$  is not invertible, then there exists a nonzero piece-wise linear function with zero  $L_\mu^2$  norm, which implies that the measure  $\mu$  does not have a full support. □

*Proof of Theorem 1.2.* This follows by combining the derivations above, Lemma 3.1, and Proposition 3.2. □

We now give another sufficient condition for the matrix  $\mathbf{M}$  to be invertible. If we define

$$(3.14) \quad \begin{aligned} p_1(x) &:= \frac{(x-a)^2}{(b-a)^2}, & p_2(x) &:= \frac{(x-a)(2x-a-b)}{(b-a)^2}, \\ p_3(x) &:= \frac{(b-x)(a+b-2x)}{(b-a)^2}, & p_4(x) &:= \frac{(b-x)^2}{(b-a)^2}, \end{aligned}$$

then

$$(3.15) \quad \begin{aligned} M_{1,1} - M_{1,2} &= \int_a^b p_1 d\mu \circ T_{J_1} + \int_a^b p_2 d\mu \circ T_{J_2}, \\ M_{i,i} - M_{i,i-1} - M_{i,i+1} &= \int_a^b p_2 d\mu \circ T_{J_i} + \int_a^b p_3 d\mu \circ T_{J_{i+1}}, \quad 2 \leq i \leq N^m - 2, \\ M_{N^m-1, N^m-1} - M_{N^m-1, N^m-2} &= \int_a^b p_3 d\mu \circ T_{J_{N^m-1}} + \int_a^b p_4 d\mu \circ T_{J_{N^m}}. \end{aligned}$$

We recall that an  $n \times n$  complex matrix  $A = (a_{ij})$  is *strictly diagonally dominant* if

$$(3.16) \quad |a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}| \quad \text{for } 1 \leq i \leq n.$$

It is well known that any  $n \times n$  strictly diagonally dominant complex matrix is invertible (see e.g., [40]).

**Proposition 3.3.** *Let  $\mathbf{M}$  be the mass matrix defined in (3.6) and  $p_i, i = 1, \dots, 4$ , be defined as in (3.14). Assume that*

$$\begin{aligned} \int_a^b p_1 d\mu \circ T_{J_1} + \int_a^b p_2 d\mu \circ T_{J_2} > 0, \quad \int_a^b p_3 d\mu \circ T_{J_{N^m-1}} + \int_a^b p_4 d\mu \circ T_{J_{N^m}} > 0, \quad \text{and} \\ \int_a^b p_2 d\mu \circ T_{J_i} + \int_a^b p_3 d\mu \circ T_{J_{i+1}} > 0, \quad \text{for all } 2 \leq i \leq N^m - 2. \end{aligned}$$

*Then  $\mathbf{M}$  is strictly diagonally dominant and thus invertible. Hence the same conclusions of Proposition 3.2 hold.*

For the infinite Bernoulli convolution associated with the golden ratio, as well as the 3-fold convolution of the Cantor measure (see Section 4), we can verify that  $\mathbf{M}$  is strictly diagonally dominant; we omit the details.

Next, we discuss the solution of the linear system (3.7). We let  $\mathbf{w}_n := \mathbf{w}(t_n)$ ,  $n \geq -1$ , and use the *central difference method* to solve the IVP (3.9). (The value of  $\mathbf{w}_{-1}$  is defined below.)

We approximate the derivatives as follows:

$$(3.17) \quad \frac{d^2 \mathbf{w}(t_n)}{dt^2} \approx \frac{\mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1}}{(\Delta t)^2} \quad \text{and} \quad \mathbf{w}'(t_n) \approx \frac{\mathbf{w}_{n+1} - \mathbf{w}_{n-1}}{2\Delta t}.$$

Substituting (3.17) into (3.7) yields

$$\frac{\mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1}}{(\Delta t)^2} = -\mathbf{M}^{-1} \mathbf{K} \mathbf{w}_n, \quad \text{i.e.,} \quad \mathbf{w}_{n+1} = (2\mathbf{I} - (\Delta t)^2 \mathbf{M}^{-1} \mathbf{K}) \mathbf{w}_n - \mathbf{w}_{n-1}.$$

Moreover, using

$$\mathbf{w}_1 = (2\mathbf{I} - (\Delta t)^2 \mathbf{M}^{-1} \mathbf{K}) \mathbf{w}_0 - \mathbf{w}_{-1} \quad \text{and} \quad \mathbf{w}'_0 = \frac{\mathbf{w}_1 - \mathbf{w}_{-1}}{2\Delta t},$$

we get

$$(3.18) \quad \mathbf{w}_1 = \left( \mathbf{I} - \frac{(\Delta t)^2}{2} \mathbf{M}^{-1} \mathbf{K} \right) \mathbf{w}_0 + (\Delta t) \mathbf{w}'_0.$$

Therefore, equation (3.7) becomes:

$$(3.19) \quad \begin{cases} \mathbf{w}_{n+1} = (2\mathbf{I} - (\Delta t)^2 \mathbf{M}^{-1} \mathbf{K}) \mathbf{w}_n - \mathbf{w}_{n-1}, & n = 1, 2, \dots \\ \mathbf{w}_0 = \mathbf{w}(t_0) = \mathbf{w}(0) \\ \mathbf{w}_1 = \mathbf{w}(t_1) = \left( \mathbf{I} - \frac{(\Delta t)^2}{2} \mathbf{M}^{-1} \mathbf{K} \right) \mathbf{w}_0 + (\Delta t) \mathbf{w}'_0 \\ t_n = n\Delta t. \end{cases}$$

To solve this system, we fix  $\Delta t$  and substitute the initial conditions  $\mathbf{w}_0$  and  $\mathbf{w}'_0$  from (3.8) into (3.18) to get  $\mathbf{w}_1$ . Then substitute  $\mathbf{w}_0$  and  $\mathbf{w}_1$  into the first equation in (3.19) to find  $\mathbf{w}_2$ .  $\mathbf{w}_{n+1}$  can then be computed recursively.

#### 4. FRACTAL MEASURES DEFINED BY ITERATED FUNCTION SYSTEMS

In this section, we solve the homogeneous IBVP (1.2) numerically for three different measures, namely, a weighted Bernoulli-type measure, the infinite Bernoulli convolution associated with the golden ratio, and the 3-fold convolution of the Cantor measure. The first one is defined by a p.c.f. IFS, while the second and third are defined by IFSs with overlaps.

We assume the same hypotheses of Section 3. In order to solve (3.9) or (3.19), we need to compute the matrix  $\mathbf{M}$  (the matrix  $\mathbf{K}$  can be computed easily). According to Lemma 3.1, it suffices to compute the integrals  $\mathcal{I}_{k,j}$ ,  $k = 0, 1, 2$ ,  $j = 1, \dots, N$ , as defined in (3.11). We find the exact values of these integrals for the measures in this section. The following integration formula will be used repeatedly: for any continuous function  $\varphi$  on  $\text{supp}(\mu) = [a, b]$ ,

$$(4.1) \quad \int_a^b \varphi d\mu = \sum_{i=1}^q p_i \int_a^b \varphi \circ S_i d\mu.$$

By substituting the values of  $\mathcal{I}_{k,j}$  into (3.10), we obtain the matrix  $\mathbf{M}$ . This allows us to solve equation (3.19).

**4.1. Weighted Bernoulli-type measure.** A weighted Bernoulli-type measure  $\mu$  is defined by the IFS

$$S_1(x) = \frac{1}{2}x, \quad S_2(x) = \frac{1}{2}x + \frac{1}{2},$$

together with probability weights  $p, 1 - p$ . Thus,

$$\mu = p\mu \circ S_1^{-1} + (1 - p)\mu \circ S_2^{-1}.$$

For any Borel subset  $A \subseteq [0, 1]$ , we have:

$$\begin{bmatrix} \mu(S_1 S_i A) \\ \mu(S_2 S_i A) \end{bmatrix} = M_i \begin{bmatrix} \mu(S_1 A) \\ \mu(S_2 A) \end{bmatrix}, \quad i = 1, 2,$$

where

$$M_1 = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1-p & 0 \\ 0 & 1-p \end{bmatrix}.$$

Let  $J = j_1 j_2 \cdots j_m$ ,  $j_i = 1$  or  $2$ . Then

$$\mu(S_J A) = c_J \begin{bmatrix} \mu(S_1 A) \\ \mu(S_2 A) \end{bmatrix}, \quad \text{where} \quad c_J = \mathbf{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2).$$

Since the IFS satisfies the open set condition, it is straightforward to evaluate the integrals  $\mathcal{I}_{k,j}$ ; we omit the details. In view of [1], we choose the weight  $p = 2 - \sqrt{3}$  in Figure 1.

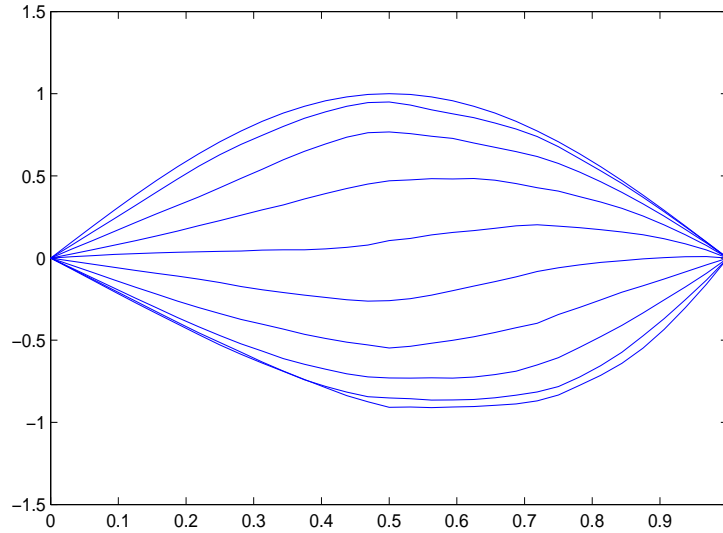
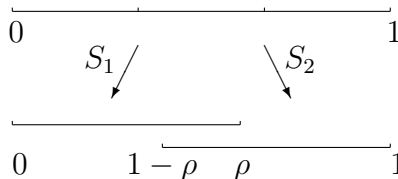


FIGURE 1. The weighted Bernoulli-type measure associated with the weights  $p = 2 - \sqrt{3}$  and  $1 - p = \sqrt{3} - 1$ . The initial data  $g = \sin(\pi x)$  and  $h = 0$  are used, and the time step  $\Delta t$  in equation (3.19) is taken to be 0.001. From top to bottom, the values of  $t$  are 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9. Animations for this and other graphs in the paper are created and uploaded to the webpage <http://homepages.uconn.edu/fractals/wave/>.

#### 4.2. Infinite Bernoulli convolution associated with the golden ratio.

The infinite Bernoulli convolution associated with the golden ratio is defined by the IFS

$$S_1(x) = \rho x, \quad S_2(x) = \rho x + (1 - \rho), \quad \rho = \frac{\sqrt{5} - 1}{2}.$$



For each  $0 < p < 1$ , we call the corresponding self-similar measure

$$\mu = p\mu \circ S_1^{-1} + (1 - p)\mu \circ S_2^{-1},$$



a *weighted infinite Bernoulli convolution associated with the golden ratio*. If  $p = 1/2$ , we get the classical one.

The measure  $\mu_p$  satisfies a family of second-order identities. This was first pointed out by Strichartz et al. [35]. Define

$$T_1(x) = \rho^2 x, \quad T_2(x) = \rho^3 x + \rho^2, \quad T_3(x) = \rho^2 x + \rho.$$

Then  $\mu$  satisfies the following second-order identities (see [25]): for any Borel subset  $A \subseteq [0, 1]$ ,

$$\begin{bmatrix} \mu(T_1 T_i A) \\ \mu(T_2 T_i A) \\ \mu(T_3 T_i A) \end{bmatrix} = M_i \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix}, \quad i = 1, 2, 3,$$

where  $M_1, M_2, M_3$  are, respectively,

$$\begin{bmatrix} p^2 & 0 & 0 \\ (1-p)p^2 & (1-p)p & 0 \\ 0 & 1-p & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & p^2 & 0 \\ 0 & (1-p)p & 0 \\ 0 & (1-p)^2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & p & 0 \\ 0 & (1-p)p & (1-p)^2 p \\ 0 & 0 & (1-p)^2 \end{bmatrix}.$$

We can make use of this to compute the measure of suitable subintervals of  $[0, 1]$ . In fact, if we let  $J = j_1 \cdots j_m$ ,  $j_i = 1, 2$  or  $3$ , then for any Borel subset  $A \subseteq [0, 1]$ ,

$$\mu(T_J A) = c_J \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix}, \quad \text{where } c_J = \mathbf{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2, c_J^3).$$

Moreover, by using (4.1) we can evaluate the integrals  $\mathcal{I}_{k,j}$  in (3.11). For  $p = 1/2$ , the results are summarized below:

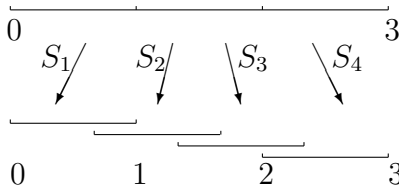
$$(4.2) \quad \begin{aligned} \int_0^1 d\mu \circ T_1 &= \frac{1}{3} & \int_0^1 d\mu \circ T_2 &= \frac{1}{3} & \int_0^1 d\mu \circ T_3 &= \frac{1}{3} \\ \int_0^1 x d\mu \circ T_1 &= \frac{1}{6(3\rho-1)} & \int_0^1 x d\mu \circ T_2 &= \frac{1}{6} & \int_0^1 x d\mu \circ T_3 &= \frac{1}{6(3\rho^2+3)} \\ \int_0^1 x^2 d\mu \circ T_1 &= \frac{5\rho+4}{6(\rho+8)} & \int_0^1 x^2 d\mu \circ T_2 &= \frac{\rho+5}{6(\rho+8)} & \int_0^1 x^2 d\mu \circ T_3 &= \frac{2-\rho}{6(\rho+8)}. \end{aligned}$$

We can thus calculate the entries of the mass matrix  $\mathbf{M}$  and solve the linear system (3.9). The result is shown in Figure 2.

**4.3. 3-fold convolution of the Cantor measure.** The 3-fold convolution of the Cantor measure  $\mu$  also satisfies a family of second-order identities. It is defined by the IFS

$$S_i(x) = \frac{1}{3}x + \frac{2}{3}(i-1), \quad \text{for } i = 1, 2, 3, 4,$$

which does not satisfy the OSC.



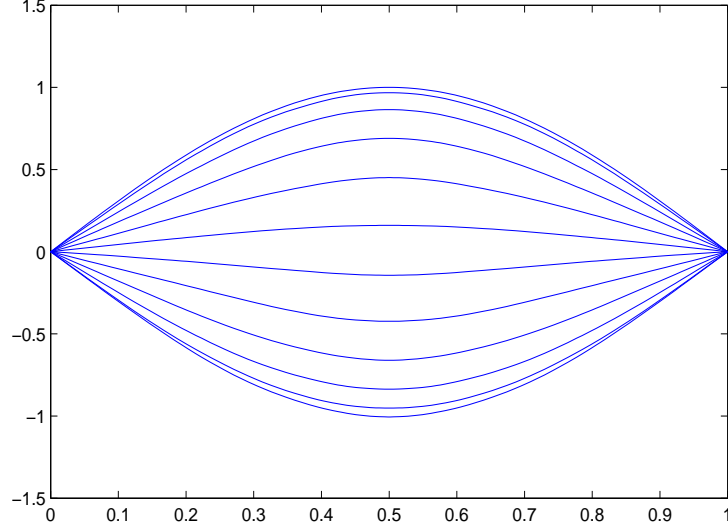


FIGURE 2. Infinite Bernoulli convolution associated with the golden ratio. The initial data  $g = \sin(\pi x)$  and  $h = 0$  are used. The time step  $\Delta t$  in equation (3.19) is taken to be 0.001. From top to bottom, the values of  $t$  are 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1.

The measure  $\mu$  satisfies the following self-similar identity:

$$\mu = \frac{1}{8}\mu \circ S_1^{-1} + \frac{3}{8}\mu \circ S_2^{-1} + \frac{3}{8}\mu \circ S_3^{-1} + \frac{1}{8}\mu \circ S_4^{-1}$$

Define

$$T_1(x) = \frac{1}{3}x, \quad T_2(x) = \frac{1}{3}x + 1, \quad T_3(x) = \frac{1}{3}x + 2.$$

Then  $\mu$  satisfies the following second-order identities (see [25]): for any Borel subset  $A \subseteq [0, 3]$ ,

$$\begin{bmatrix} \mu(T_{1j}A) \\ \mu(T_{2j}A) \\ \mu(T_{3j}A) \end{bmatrix} = M_j \begin{bmatrix} \mu(T_1A) \\ \mu(T_2A) \\ \mu(T_3A) \end{bmatrix}, \quad j = 1, 2, 3,$$

where the coefficient matrices  $M_j$  are given by

$$M_1 = \frac{1}{8} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad M_2 = \frac{1}{8} \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \frac{1}{8} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let  $J = j_1 \cdots j_m$ ,  $j_i = 1, 2$  or  $3$ . Then

$$\mu(T_J A) = c_J \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix}, \quad \text{where } c_J = \mathbf{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2, c_J^3).$$

The integrals  $\mathcal{I}_{k,j}$  in (3.11) are given below:

$$(4.3) \quad \begin{aligned} \int_0^3 d\mu \circ T_1 &= \frac{1}{5} & \int_0^3 d\mu \circ T_2 &= \frac{3}{5} & \int_0^3 d\mu \circ T_3 &= \frac{1}{5} \\ \int_0^3 x d\mu \circ T_1 &= \frac{27}{70} & \int_0^3 x d\mu \circ T_2 &= \frac{9}{10} & \int_0^3 x d\mu \circ T_3 &= \frac{3}{14} \\ \int_0^3 x^2 d\mu \circ T_1 &= \frac{5517}{6440} & \int_0^3 x^2 d\mu \circ T_2 &= \frac{11943}{6440} & \int_0^3 x^2 d\mu \circ T_3 &= \frac{63}{184}. \end{aligned}$$

Again, using these values we can compute  $\mathbf{M}$  and solve (3.9) (see Figure 3).

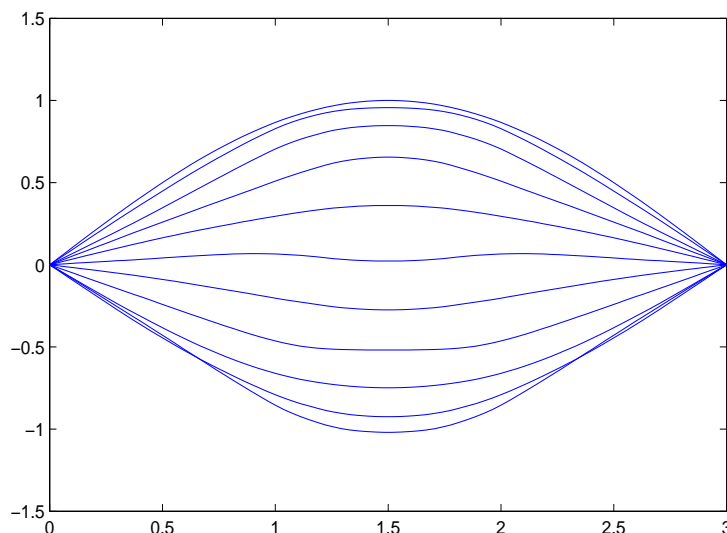


FIGURE 3. Three-fold convolution of the Cantor measure. The initial data  $g = \sin(\pi x/3)$  and  $h = 0$  are used, and  $\Delta t = 0.001$ . From top to bottom, the values of  $t$  are 0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0.

## 5. CONVERGENCE OF NUMERICAL APPROXIMATIONS

In this section we prove the convergence of the numerical approximations of the homogeneous IBVP (1.2). Some of our results are obtained by modifying similar ones in [34] (see also [2]).

We assume the same setup of Section 3 unless stated otherwise. Let  $V_m$  be the set of end-points of all the level- $m$  intervals, and arrange its elements so that  $V_m = \{x_i : i = 0, 1, \dots, N^m\}$  with  $x_i < x_{i+1}$  for  $i = 0, 1, \dots, N^m - 1$ ,  $x_0 = a$  and  $x_{N^m} = b$ . Let  $S^m$  be the space of continuous piecewise linear functions on  $[a, b]$  with nodes  $V_m$ , and let

$$S_D^m := \{u \in S^m : u(a) = u(b) = 0\}$$

be the subspace of  $S^m$  consisting of functions satisfying the Dirichlet boundary condition. Then

$$\dim S^m = \#V_m = N^m + 1 \quad \text{and} \quad \dim S_D^m = \#V_m - 2 = N^m - 1.$$

We choose the basis of  $S^m$  consisting of the tent functions  $\{\phi_i\}_{i=0}^{N^m}$  defined in (3.3) and choose the basis  $\{\phi_i\}_{i=1}^{N^m-1}$  for  $S_D^m$ .

**Definition 5.1.** Let  $V_m$  be defined as above and  $\{\phi_i\}_{i=0}^{N^m}$  be defined as in (3.3). The linear map  $\mathcal{P}_m : \text{Dom } \mathcal{E} \rightarrow S_D^m$  defined by

$$\mathcal{P}_m v := \sum_{i=1}^{N^m-1} v(x_i) \phi_i(x), \quad v \in \text{Dom } \mathcal{E},$$

is called the Rayleigh-Ritz projection with respect to  $V_m$ .

$\mathcal{P}_m v$  is the piecewise linear interpolant of the values of  $v$  on  $V_m$ .

**Lemma 5.1.** For any  $m \geq 1$ , let  $V_m$  and  $\mathcal{P}_m$  be the Rayleigh-Ritz projection defined as in Definition 5.1. Then for any  $v \in \text{Dom } \mathcal{E}$ ,  $\mathcal{P}_m v$  is the component of  $v$  in the subspace  $S_D^m$ ,  $v - \mathcal{P}_m v$  vanishes on the boundary  $\{a, b\}$ , and

$$\mathcal{E}(v - \mathcal{P}_m v, w) = 0 \quad \text{for all } w \in S_D^m.$$

*Proof.* See, e.g., [34]. □

**Lemma 5.2.** Assume the same hypotheses of Lemma 5.1. Then for any  $v \in \text{Dom } \mathcal{E}$ ,

$$v|_{V_m} = \mathcal{P}_m v|_{V_m}.$$

*Proof.* Similar to that of [2, Lemma 5.3]. □

Let  $\|V_m\| := \max\{x_i - x_{i-1} : 1 \leq i \leq m\}$  denote the norm of the partition  $V_m$ .

**Lemma 5.3.** Assume the same hypotheses of Lemma 5.1 and let  $v \in \text{Dom } \mathcal{E}$ . Then

$$|\mathcal{P}_m v(x) - v(x)| \leq 2 \|V_m\|^{1/2} \|v\|_{\text{Dom } \mathcal{E}} \quad \text{for all } x \in [a, b].$$

In particular,

$$\|\mathcal{P}_m v - v\|_{\mu} \leq 2 \|V_m\|^{1/2} \|v\|_{\text{Dom } \mathcal{E}}.$$

*Proof.* We first note that since  $v$  is absolutely continuous and belongs to  $\text{Dom } \mathcal{E}$ ,

$$(5.4) \quad |v(x) - v(y)| = \left| \int_y^x v'(s) ds \right| \leq |x - y|^{1/2} \|v\|_{H_0^1(a,b)}, \quad \forall x, y \in [a, b].$$

Now Let  $i \in \{1, \dots, N^m\}$  so that  $x \in [x_{i-1}, x_i]$ . Then by (5.4) and Lemma 5.2, we get

$$\begin{aligned} |\mathcal{P}_m v(x) - v(x)| &\leq |\mathcal{P}_m v(x) - v(x_{i-1})| + |v(x_{i-1}) - v(x)| \\ &\leq |v(x_i) - v(x_{i-1})| + |v(x_{i-1}) - v(x)| \\ &\leq 2 \|V_m\|^{1/2} \|v\|_{\text{Dom } \mathcal{E}}. \end{aligned}$$

□

Throughout the rest of this section we let

$$(5.5) \quad g, h \in \text{Dom } \mathcal{E} \quad \text{and} \quad f = 0,$$

and let  $u$  be the solution of the corresponding homogeneous IBVP (1.2). According to Theorem 2.8,

$$(5.6) \quad u \in W_2^k(0, T; \text{Dom } \mathcal{E}) \quad \text{for all } k \geq 0.$$

In particular,  $u_{tt} \in \text{Dom } \mathcal{E}$  and

$$(5.7) \quad (u_{tt}, v)_\mu + \mathcal{E}(u, v) = 0 \quad \text{for all } v \in \text{Dom } \mathcal{E}.$$

As in Section 3, we let

$$u^m(x, t) = \sum_{i=1}^{N^m-1} \beta_i(t) \phi_i(x).$$

Lastly, we define

$$e(x, t) = e^m(x, t) := \mathcal{P}_m u(x, t) - u^m(x, t).$$

**Lemma 5.4.** *Let  $g, h, f, u, u^m, e$  be defined as above.*

(a)  $u^m$  satisfies:

$$(i) \quad (u_{tt}^m, v^m)_\mu + \mathcal{E}(u^m, v^m) = 0 \quad \text{for all } v^m \in S_D^m,$$

$$(ii) \quad u^m(x, 0) = \sum_{i=1}^{N^m-1} g(x_i) \phi_i(x) \quad \text{and} \quad u_t^m(x, 0) = \sum_{i=1}^{N^m-1} h(x_i) \phi_i(x).$$

(b) The following identity holds:

$$(5.8) \quad (e_{tt}, e_t)_\mu + \mathcal{E}(e, e_t) = (\mathcal{P}_m u_{tt} - u_{tt}, e_t)_\mu.$$

*Proof.* (a) The proof of part (a) follows from the derivations in Section 3; we omit the details.

(b) By definition and the fact that  $u \in W_2^k(0, T; \text{Dom } \mathcal{E})$  for  $k \geq 0$ , the functions  $e_t, e_{tt}$ , and  $(\mathcal{P}_m u)_{tt} = \mathcal{P}_m u_{tt}$  all belong to  $S_D^m$ .

Substitute  $e_t$  for  $v$  in (5.7) and for  $v^m$  in (a)(i), and then subtracting the resulting equations, we get

$$(u_{tt} - u_{tt}^m, e_t)_\mu + \mathcal{E}(u - u^m, e_t) = 0.$$

Equivalently,

$$(u_{tt} - \mathcal{P}_m u_{tt} + \mathcal{P}_m u_{tt} - u_{tt}^m, e_t)_\mu + \mathcal{E}(u - \mathcal{P}_m u + \mathcal{P}_m u - u^m, e_t) = 0,$$

which implies

$$(\mathcal{P}_m u_{tt} - u_{tt}^m, e_t)_\mu + \mathcal{E}(\mathcal{P}_m u - u^m, e_t) = (\mathcal{P}_m u_{tt} - u_{tt}, e_t)_\mu,$$

because  $\mathcal{E}(u - \mathcal{P}_m u, e_t) = 0$  (Lemma 5.1). Identity (5.8) now follows from the definition of  $e(t)$ .  $\square$

**Theorem 5.5.** *Assume the same hypotheses of Lemma 5.4 and let  $\rho$  be as in (1.7). Then there exists a constant  $C > 0$  such that*

$$\|\mathcal{P}_m u - u^m\|_\mu \leq C \sqrt{T} \rho^{m/2} \|u_{tt}\|_{2, \text{Dom } \mathcal{E}}.$$

*Proof.* Let  $E(t) := \frac{1}{2}(e_t, e_t)_\mu + \frac{1}{2}\mathcal{E}(e, e) = \frac{1}{2}\|e_t\|_\mu^2 + \frac{1}{2}\|e\|_{\text{Dom } \mathcal{E}}^2$ . Then

$$(5.9) \quad \|e_t\|_\mu \leq \sqrt{2} \sqrt{E(t)},$$

$$(5.10) \quad \|e\|_{\text{Dom } \mathcal{E}} \leq \sqrt{2} \sqrt{E(t)},$$

$$(5.11) \quad E(t) \leq \frac{1}{2} (\|e_t\|_\mu + \|e\|_{\text{Dom } \mathcal{E}})^2.$$

The left-hand side of (5.8) is equal to

$$(5.12) \quad \frac{1}{2} (\|e_t\|_\mu^2)_t + \frac{1}{2} (\|e\|_{\text{Dom } \mathcal{E}}^2)_t = E_t(t).$$

For the right-hand side of (5.8), we apply Cauchy-Schwarz inequality and (5.9) to get

$$(5.13) \quad E_t(t) = (\mathcal{P}_m u_{tt} - u_{tt}, e_t)_\mu \leq \|\mathcal{P}_m u_{tt} - u_{tt}\|_\mu \|e_t\|_\mu \leq \|\mathcal{P}_m u_{tt} - u_{tt}\|_\mu \sqrt{2} \sqrt{E(t)}.$$

Since  $E(t) \geq 0$  with  $E(0) = 0$ , we can assume that  $E(t) > 0$  on some interval  $(\alpha, \beta) \subset [0, T]$  with  $\alpha < \beta$  and  $E(\alpha) = 0$ . (Otherwise, by the continuity  $E(t)$ , we have  $E(s) = 0$  for all  $s \in [0, T]$  and (5.14) below still holds.) It follows from (5.13) that

$$\frac{E_t(t)}{\sqrt{E(t)}} \leq \sqrt{2} \|\mathcal{P}_m u_{tt} - u_{tt}\|_\mu, \quad \alpha < s < \beta,$$

and thus

$$(5.14) \quad 2\sqrt{E(s)} \leq \sqrt{2} \int_\alpha^\beta \|\mathcal{P}_m u_{tt} - u_{tt}\|_\mu dt, \quad \alpha \leq s \leq \beta.$$

From (5.10) and (5.14), we have

$$\|e(s)\|_{\text{Dom } \mathcal{E}} \leq \sqrt{2} \sqrt{E(s)} \leq \sqrt{2} \int_\alpha^\beta \|\mathcal{P}_m u_{tt} - u_{tt}\|_\mu dt \leq \sqrt{2} \int_0^T \|\mathcal{P}_m u_{tt} - u_{tt}\|_\mu dt,$$

which actually holds for all  $s \in [0, T]$ . Thus by combining condition (2.2), Lemma 5.3, and the above estimations, we have

$$\begin{aligned} \|e(s)\|_\mu &\leq C \|e(s)\|_{\text{Dom } \mathcal{E}} \leq C \sqrt{T} \left( \int_0^T \|\mathcal{P}_m u_{tt} - u_{tt}\|_\mu^2 dt \right)^{1/2} \\ &\leq C \sqrt{T} \left( \int_0^T (2\|V_m\|^{1/2} \|u_{tt}\|_{\text{Dom } \mathcal{E}})^2 dt \right)^{1/2} \quad (\text{Lemma 5.3}) \\ &\leq 2C \sqrt{T} \rho^{m/2} \|u_{tt}\|_{2, \text{Dom } \mathcal{E}}, \end{aligned}$$

which holds for all  $s \in [0, T]$ . This completes the proof.  $\square$

*Proof of Theorem 1.3.* For fixed  $t \in [0, T]$ ,

$$\|u^m - u\|_\mu \leq \|u^m - \mathcal{P}_m u\|_\mu + \|\mathcal{P}_m u - u\|_\mu.$$

Theorem 1.3 now follows by combining Lemma 5.3 and Theorem 5.5.  $\square$

**Acknowledgements.** The authors thank Scott Kersey, Yin-Tat Lee, Frederic Mynard, Po-Lam Yung, Shijun Zheng, and, especially, Robert Strichartz, for valuable discussions and suggestions.

## REFERENCES

- [1] E. J. Bird, S.-M. Ngai and A. Teplyaev, Fractal Laplacians on the unit interval, *Ann. Sci. Math. Québec* **27** (2003), no. 2, 135–168.
- [2] J. Chen and S.-M. Ngai, Eigenvalues and eigenfunctions of one-dimensional fractal Laplacians defined by iterated function systems with overlaps, *J. Math. Anal. Appl.* **364** (2010), 222 – 241.
- [3] K. Coletta, K. Dias, R. S. Strichartz, Numerical analysis on the Sierpinski gasket, with applications to Schrödinger equations, wave equation, and Gibbs' phenomenon, *Fractals* **12** (2004), 413–449.
- [4] K. Dalrymple, R. S. Strichartz and J. P. Vinson, Fractal differential equations on the Sierpinski gasket, *J. Fourier Anal. Appl.* **5** (1999), 203–284.
- [5] Q.-R. Deng and K.-S. Lau, Open set condition and post-critically finite self-similar sets, *Nonlinearity* **21** (2008), 1227–232.
- [6] L. C. Evans, *Partial Differential Equations: Second Ed. Graduate Studies in Mathematics. Vol. 19*, American Math Society, 2010.

- [7] K. J. Falconer, *Fractal geometry. Mathematical foundations and applications*, John Wiley & Sons, Ltd., Chichester, 1990.
- [8] W. Feller, On second order differential operators, *Ann. of Math. (2)* **61** (1955), 90–105.
- [9] W. Feller, Generalized second order differential operators and their lateral conditions, *Illinois J. Math.* **1** (1957), 459–504.
- [10] U. Freiberg, Analytical properties of measure geometric Krein-Feller-operators on the real line, *Math. Nachr.* **260** (2003), 34–47.
- [11] U. Freiberg, Dirichlet forms on fractal subsets of the real line, *Real Anal. Exchange* **30** (2004/05), no. 2, 589–603.
- [12] U. Freiberg, Spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets, *Forum Math.* **17** (2005), no. 1, 87–104.
- [13] U. Freiberg and J.-U. Löbus, Zeros of eigenfunctions of a class of generalized second order differential operators on the Cantor set, *Math. Nachr.* **265** (2004), 3–14.
- [14] T. Fujita, A fractional dimension, self-similarity and a generalized diffusion operator, *Probabilistic methods in mathematical physics (Katata/Kyoto, 1985)*, 83–90, Academic Press, Boston, MA, 1987.
- [15] I. M. Gel'fand and N. Ya. Vilenkin, *Generalized functions. Vol. 4. Applications of harmonic analysis*, Academic Press, New York-London, 1964.
- [16] A. Grigor'yan and A. Telcs, Two-sided estimates of heat kernels on metric measure spaces, *Ann. Prob.* **40** (2012) 1212–1284.
- [17] A. Grigor'yan and J. Hu, Heat kernels and Green functions on metric measure spaces, to appear in *Canad. J. Math.*
- [18] K.E. Hare, B.A. Steinhurst, A. Teplyaev, D. Zhou, Disconnected Julia sets and gaps in the spectrum of Laplacians on symmetric finitely ramified fractals, *Math. Res. Lett.* **19** (2012), 537–553.
- [19] J. Hu, K.-S. Lau and S.-M. Ngai, Laplace operators related to self-similar measures on  $\mathbb{R}^d$ , *J. Funct. Anal.* **239** (2006), 542–565.
- [20] J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747.
- [21] I. S. Kac and M. G. Krein, On the spectral functions of the string, *Amer. Math. Soc. Transl. (2)* **103** (1974), 19–102.
- [22] J. Kigami, *Analysis on fractals*, Cambridge Tracts in Mathematics, **143**, Cambridge University Press, Cambridge, 2001.
- [23] J. Kigami, Volume doubling measures and heat kernel estimates on self-similar sets, *Mem. Amer. Math. Soc.* **199** (2009), no. 932.
- [24] J. Kigami, Resistance forms, quasisymmetric maps and heat kernel estimates, *Mem. Amer. Math. Soc.* **216** (2012), no. 1015.
- [25] K.-S. Lau and S.-M. Ngai, Second-order self-similar identities and multifractal decompositions, *Indiana Univ. Math. J.* **49** (2000), 925–972.
- [26] Yin-Tat Lee, Infinite propagation speed for wave solutions on some P.C.F. fractals, arXiv:1111.2938
- [27] R. Meyers, R. S. Strichartz, and A. Teplyaev, Dirichlet forms on the Sierpiński gasket, *Pacific J. Math.* **217** (2004), 149–174.
- [28] S. Molchanov, B. Vainberg, Wave propagation in periodic networks of thin fibers, *Waves Random Complex Media* **20** (2010), 260–275.
- [29] K. Naimark and M. Solomyak, On the eigenvalue behaviour for a class of operators related to self-similar measures on  $\mathbb{R}^d$ , *C. R. Acad. Sci. Paris Sér. I Math.* **319** (1994), 837–842.
- [30] K. Naimark and M. Solomyak, The eigenvalue behaviour for the boundary value problems related to self-similar measures on  $\mathbb{R}^d$ , *Math. Res. Lett.* **2** (1995), 279–298.
- [31] S.-M. Ngai, Spectral asymptotics of Laplacians associated with one-dimensional iterated function systems with overlaps, *Canad. J. Math.* **63** (2011), 648–688.
- [32] B. J. Pettis, On integration in vector spaces, *Trans. Amer. Math. Soc.* **44** (1938), 277–304.
- [33] M. Solomyak and E. Verbitsky, On a spectral problem related to self-similar measures, *Bull. London Math. Soc.* **27** (1995), 242–248.
- [34] G. Strang, G. J. Fix, *An analysis of the finite element method*, Prentice-Hall Series in Automatic Computation, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1973.
- [35] R. S. Strichartz, A. Taylor and T. Zhang, Densities of self-similar measures on the line, *Experiment. Math.* **4** (1995), 101–128.

- [36] R. S. Strichartz and A. Teplyaev, Spectral analysis on infinite Sierpiński fractafolds, *J. Anal. Math.* **116** (2012), 255–297.
- [37] R. S. Strichartz, Analysis on fractals, *Notices Amer. Math. Soc.* **46** (1999), 1199–208.
- [38] R. S. Strichartz, Laplacians on fractals with spectral gaps have nicer Fourier series, *Math. Res. Lett.* **12** (2005), 269–274.
- [39] R. S. Strichartz and M. Usher, Splines on fractals, *Math. Proc. Cambridge Philos. Soc.* **129** (2000), 331–360.
- [40] R. S. Varga, *Matrix iterative analysis*, Second revised and expanded edition, Springer Series in Computational Mathematics, **27**, Springer-Verlag, Berlin, 2000.
- [41] J. Wloka, *Partial differential equations*, Cambridge University Press, Cambridge, 1987.
- [42] P.-L. Yung, Doubling properties of self-similar measures, *Indiana Univ. Math. J.* **56** (2007), 965–990.

DEPARTMENT OF MATHEMATICAL SCIENCES, GEORGIA SOUTHERN UNIVERSITY, STATESBORO, GA 30460-8093, USA.

*E-mail address:* fchan@udel.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, GEORGIA SOUTHERN UNIVERSITY, STATESBORO, GA 30460-8093, USA, AND COLLEGE OF MATHEMATICS AND COMPUTER SCIENCE, HUNAN NORMAL UNIVERSITY, CHANGSHA, HUNAN 410081, CHINA.

*E-mail address:* smngai@georgiasouthern.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CT 06269, USA.

*E-mail address:* alexander.teplyaev@uconn.edu