Disconnected Julia sets and gaps in the spectrum of Laplacians on symmetric finitely ramified fractals

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Abstract. It is known that Laplacian operators on many fractals have gaps in their spectra. This fact precludes the possibility that a Weyl-type ratio can have a limit and is also a key ingredient in proving that the Fourier series on such fractals can have better convergence results than in the classical setting. In this paper we prove that the existence of gaps is equivalent to the total disconnectedness of the Julia set of the spectral decimation function for the class of fully symmetric p.c.f. fractals, and for self-similar fully symmetric finitely ramified fractals with regular harmonic structure. We also formulate conjectures related to geometry of finitely ramified fractals with spectral gaps, to complex spectral dimensions, and to convergence of Fourier series on such fractals.

1. Introduction

In recent years, there have been extensive studies of Laplacian operators on self-similar fractals, both as normalized limits of discrete Laplacians on finite graphs and as generators of a diffusion process. One prominent feature of these Laplacian operators is that there can be gaps in their spectrum. Examples include the standard Laplacian on the Sierpinski gasket [**13, 14, 40**], the n-branch tree-like fractals and the Vicsek sets [**12, 43, 44**], and a number of other cases [**2, 11, 26, 27**]. The existence of gaps is an interesting phenomenon in itself as this does not happen in the classical cases. It is also a significant issue to analysis on fractals. For instance, the existence of gaps precludes the possibility that a Weyl-type limit can exist as the ratio must drop by a constant factor when passing through a gap. (Note however that, alternatively, oscillations in the spectrum can occur when there are large multiplicities of eigenavalues but no gaps, see [**1, 2, 6**, and references therein].) Furthermore, the existence of gaps, together with a suitable heat kernel estimate allows one to show that Fourier series on these fractals can have better convergence

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than the classical case. This was first observed by Strichartz in [**38**] for the Sierpinski gasket and, as Strichartz points out, "... is the first kind of example which improves on the corresponding results in smooth analysis."

We remark that there is an extensive general theory of heat kernel estimates on fractals, which imply that the fractals we consider have Laplacians (and corresponding diffusion processes) whose heat kernel (transition probability density) satisfy the so called sub-Gaussian heat kernel estimates. The post-critically finite (p.c.f.) case is covered in [**19**], and the latest developments and further references can be found in [**3, 4, 18, 24**]. The heat kernel estimates are not used in our work, but are assumed in [**38**].

In [**43**], one of the authors gave general criteria for the existence of gaps in the spectrum of the Laplacian on fractals that admit spectral decimation and used this to establish the existence of gaps in many examples. All of the examples exhibit a common feature that the Julia set of the spectral decimation function (a rational function associated with the Laplacian) is totally disconnected. This has raised the question of whether the existence of gaps in the spectrum of the Laplacian is equivalent to the total disconnectedness of the Julia set of the spectral decimation function. The purpose of this paper is to prove that this is indeed true under mild and natural assumptions on the fractal (a self-similar fully symmetric finitely ramified fractals with regular harmonic structure). Our main aim is to supply a large, natural class of Laplacians with spectral gaps. Our results can be generalized for a larger class of finitely ramified symmetric fractals with weights, and with more elaborate combinatorial structure, but doing so would require dealing with many technical details and auxiliary results which are not available in the existing literature. The main ideas and techniques behind these results come from [**2, 30, 33, 43**]. One of the newest examples of fractals we consider can be found in [**15**]. We also give results about the location of gaps which generalize the results in [**43**].

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2. Preliminary background

2.1. Finitely ramified and p.c.f. fractals with full symmetry. The following is a definition of a class of fractals where spectral gaps appear naturally.

Definition 1. *We say that K is a fully symmetric finitely ramified self-similar set if K is a compact connected metric space with injective contraction maps $\{\psi_j\}_{j=1}^N$ such that*

$$K = \bigcup_{j=1}^{N} \psi_j(K)$$

and the following three conditions hold:

(1) *there exists a finite subset V_0 of K such that*

$$\psi_j(K) \cap \psi_k(K) = \psi_j(V_0) \cap \psi_k(V_0)$$

for $j \neq k$ (this intersection may be empty);

(2) *if $v_0 \in V_0 \cap \psi_j(K)$ then v_0 is the fixed point of ψ_j;*

(3) *there is a group $\mathfrak{G}$ of isometries of K that has a doubly transitive action on V_0 and is compatible with the self-similar structure $\{\psi_j\}_{j=1}^N$, which means (*[**32**, Proposition 4.9] *and also* [**2, 30**]*) that for any j and any $g \in \mathfrak{G}$ there exists k such that*

$$g^{-1} \circ \psi_j \circ g = \psi_k.$$

One can see that V_0 contains at least two points if K is not a singleton, which we always assume.

Post critically finite (p.c.f.) self-similar sets are defined in [**21, 22**]. We will not repeat the definition, which in general does not assume any symmetries. For examples of p.c.f. fully symmetric self-similar sets see [**11, 12, 13, 26, 27, 30, 32, 33, 43, 44**], while some examples of fully symmetric finitely ramified fractals that are not p.c.f. can be found in [**1, 2, 42**]. In the fully symmetric case one can easily obtain the following proposition.

Proposition 1. *A fully symmetric finitely ramified self-similar set K is a p.c.f. self-similar set if and only if for any $v_0 \in V_0$ there is a unique j such that $v_0 \in \psi_j(K)$.*

2.2. Spectral decimation for graph Laplacians. This subsection follows [**2, 30, 40**], and more information about self-similar graphs can be found in [**26, 27**].

Following Definition 1, we define recursively

$$V_n = \bigcup_{j=1}^{N} \psi_j(V_{n-1})$$

and call these sets the vertices of level or depth n. Note that $V_n \subset V_{n+1}$ and the sets V_n approximate K in the sense that $K = \overline{\cup_{n=0}^{\infty} V_n}$.

There is an associated recursively defined sequence of self-similar graphs G_n that have V_n as their vertex set. We define G_0 to be the complete graph on V_0. Then two elements, $x, y \in V_n$ are connected by an edge in G_n if $\psi_j^{-1}(x)$ and $\psi_j^{-1}(y)$ are connected in G_{n-1}. Note that $G_n \not\subset G_{n+1}$ and, in fact, one can deduce that $E(G_n) \cap E(G_{n+1}) = \varnothing$, where $E(G_n)$ denotes the set of edges of the graph G_n. In particular, in G_1 no two elements of V_0 are neighbors.

Definition 2. *Let the operator Δ_n, called the discrete probabilistic Laplacian on V_n, be defined by*

$$\Delta_n f(x) = f(x) - \frac{1}{\deg_n(x)} \sum_{(x,y) \in E(G_n)} f(y)$$

where $\deg_n(x)$ is the degree of x in the graph G_n, which may depend on n.

In this paper we define all Laplacians to be non-negative operators which is different by a minus sign from the usual probabilistic convention that generators of random walks and diffusion processes are non-positive.

The matrix of Δ_n with respect to the standard basis for functions on V_n, ordered so that the basis elements representing V_{n-1} are listed first, will be denoted M_n. We also denote by $I_{1,0}$ the identity matrix of size $|V_0|$, and by $I_{n+1,n}$ the identity matrix of size $|V_{n+1} \backslash V_n|$.

The matrix M_1 can be decomposed in the following block form

$$M_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A = I_0$ is the identity matrix of size $|V_0|$ that corresponds to the vertices $V_0 \subset V_1$. The Schur complement of M_1 is $S = A - BD^{-1}C$. The spectral decimation function $R(z)$ will be calculated from the Schur complement of the matrix $M_1 - zI_1$ which is the matrix valued function

$$S(z) = (1-z)A - B(D - zI_{1,0})^{-1}C. \tag{2.1}$$

Proposition 2 ([**2, 30**]). *For a fully symmetric self-similar structure on a finitely ramified fractal K there are unique rational scalar-valued functions $\phi(z)$ and $R(z)$ that satisfy*

$$S(z) = \phi(z)\,(M_0 - R(z)) \tag{2.2}$$

and are given by

$$\begin{aligned} \phi(z) &= -(|V_0| - 1)S_{1,2}(z) \\ R(z) &= 1 - \frac{S_{1,1}}{\phi(z)}. \end{aligned}$$

Corollary 1. $\phi(z) = O\left(\frac{1}{|z|}\right)_{z\to\infty}$ *and* $R(z) \geqslant O\left(|z|^2\right)_{z\to\infty}$.

Proof. By (2.1), the diagonal terms of $S(z)$ grows linearly at infinity, and the off-diagonal terms tend to zero. □

The initial step in spectral decimation is to relate the eigenvalues of M_1 back to those of M_0 with the help of the rational function $R(z)$, and after that continue iteratively by induction in n. If v is an eigenvector of M_1 with eigenvalue z, then we can write $v = (v_0, v_1')^T$ and

$$M_1 v = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} v_0 \\ v_1' \end{pmatrix} = z\begin{pmatrix} v_0 \\ v_1' \end{pmatrix},$$

which can be rewritten as two equations

$$\begin{aligned} Av_0 + Bv_1' &= zv_0 \\ Cv_0 + Dv_1' &= zv_1' \end{aligned}$$

This can be solved to give $v_1' = -(D-z)^{-1}Cv_0$, provided that $z \notin \sigma(D)$, which then implies that $S(z)v_0 = 0$. Note that v_0 is an eigenvector of M_0 with eigenvalue z_0 if and only if $(M_0 - z_0)v_0 = 0$, which we relate to the Schur complement $S(z)$ by Proposition 2, obtaining $z_0 = R(z)$. This calculation is possible if $D - z$ is invertible and $\phi(z) \neq 0$, which motivates the following definition.

Definition 3. *We denote the set $\sigma(D) \cup \{z : \phi(z) = 0\}$ by $E(M, M_0)$ and call it the exceptional set for the sequence of discrete Laplacians Δ_n on G_n.*

If we suppose that $z \notin E(M, M_0)$ and apply the above argument, then z is an eigenvalue of M_1 with eigenvector v if and only if $R(z)$ is an eigenvalue of M_0 with eigenvector v_0 and $v = (v_0, v_1')^T$, where v_1' is given by

$$v_1' = -(D-z)^{-1}Cv_0.$$

This implies the existence of a one-to-one map from the eigenspace of M_0 corresponding to $R(z)$ onto the eigenspace of M_1 corresponding to z

$$v_0 \mapsto v = T_0(z)v_0 = \begin{pmatrix} I_0 \\ -(D-z)^{-1}C \end{pmatrix} v_0,$$

which is called the eigenfunction extension map. This calculation gives a part of the spectrum of M_1 using only the spectrum M_0. It does leave open whether any of the elements of $E(M, M_0)$ are in the spectrum of M_1 (see [**2**] for more detail). Also note that while $z \notin E(M, M_0)$ it can happen that $R(z)$ is.

The previous argument serves as the base case in the inductive nature of spectral decimation, and the next lemma gives the inductive steps, which use the function $R(z)$ as the spectral decimation function that relate eigenvalues from one level to the next higher level. The function $R(z)$ and the exceptional set $E(M, M_0)$ have the property that if $\Delta_{m+1} f = \lambda f$ and $\lambda \notin E(M, M_0)$, then $\Delta_m f|_{V_m} = R(\lambda) f|_{V_m}$ and vice versa. We denote M_n the matrix representation for Δ_n given in a block form as

$$M_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$$

corresponding to the representation of V_n as $V_{n-1} \cup (V_n \setminus V_{n-1})$, and let $P_{n-1} : V_n \to V_{n-1}$ be the restriction operator.

Lemma 1 ([**2, 30, 40**])**.** *For all $n > 0$ we have*

$$P_{n-1}(M_n - z)^{-1} P^*_{n-1} = \frac{1}{\phi(z)} (M_{n-1} - R(z))^{-1}.$$

Suppose that $z_n \notin E(M, M_0)$. Then z_n is an eigenvalue of M_n with an eigenvector v_n if and only if $z_{n-1} = R(z_n)$ is an eigenvalue of M_{n-1} with eigenvector v_{n-1} and $v_n = (v_{n-1}, v'_n)^T$ where

$$v'_n = -(D_n - z_n)^{-1} C_n v_{n-1}. \tag{2.3}$$

We will refer to v_n as an extension of v_{n-1} from V_{n-1} to V_n. This lemma does not explain the status of potential eigenvalues from the set of exceptional values $E(M, M_0)$. The question of when exceptional values are eigenvalues is resolved in [**2**].

Remark 1. *It is known, by* [**30**, Lemma 4.9] *and* [**33**]*, that*

$$R(0) = 0 \quad \textit{and} \quad c_{\vartriangle} = R'(0) > 1.$$

Therefore, 0 is a repulsive fixed point of R, which is important for the complex dynamics associated with this function.

Moreover, $c_{\vartriangle}$ is also called the Laplacian normalization constant*, which appears in Subsection 2.3, and there are relations*

$$c_{\vartriangle} = Nc \quad \textit{and} \quad d_R = \frac{\log N}{\log c},$$

where $c = \frac{1}{r}$ is the conductance scaling factor, r is the resistance scaling factor, and d_R is the Hausdorff dimension in the effective resistance metric.

Often in the literature there is the relation

$$d_S = \frac{2 \log N}{\log Nc} = \frac{2 d_R}{d_R + 1},$$

where d_S is the so-called spectral dimension (for more detail see Subsection 2.4 below and [**25, 37**]*). In particular,* [**37**] *mentions that the notion of spectral dimension is a misnomer because the coefficient 2 implicitly assumes that the Laplacian is an operator of order two, which may not be justified in fractal case. Furthermore, Strichartz argues that a natural notion of the order of the Laplacian is $d_R + 1$.*

2.3. Spectral decimation for the Laplacian on K. The standard Laplacian operator Δ is defined by

$$\Delta u = \lim_{n\to\infty} c_{\Delta}^{n} \Delta_n u(x) \tag{2.4}$$

if the limit exists. Here $x \in V_*$, $c_{\Delta} = R'(0)$ is the same as above, and the sequence of difference operators $\{\Delta_n\}_{n=0}^{\infty}$ acting on functions defined on V_m is defined in Definition 2.

Definition 4. *The* continuous Neumann Laplacian *is defined for all functions u for which the limit* (2.4) *exists for all $x \in V_*$, and there is a continuous function f such that $\Delta u(x) = f(x)$ for $x \in V_*$.*

The continuous Dirichlet Laplacian *is defined for all functions u, vanishing on V_0, for which the limit* (2.4) *exists for all $x \in V_* \backslash V_0$, and there is a continuous function f, vanishing on V_0, such that $\Delta u(x) = f(x)$ for $x \in V_* \backslash V_0$.*

Standard references for the Laplacian on p.c.f. fractals are [**21, 22**], [**39**] for an introduction, and [**23**] for a very general context. It will be explained below in Subsection 2.4 why the Neumann and Dirichlet Laplacians are self-adjoint.

Let $\phi_0, ..., \phi_L$ denote the partial inverses of R, where ϕ_0 is the partial inverse of R with 0 in its range, and $\phi_0^{(n)}$ be the n-th composition power of ϕ_0. Note that L, the degree of the rational function $R(z)$, does not have to be equal to N, which is the number of contraction maps in Definition 1. Given $w = w_n...w_1$, a word of length $n = |w|$ on the letters $0, ..., L$, we put $\phi_w = \phi_{w_n} \circ \cdots \circ \phi_{w_1}$. Since $R'(0) > 1$, $\phi_0(x) < x$ for $x < \epsilon$, and so $\lim_{n\to\infty} c_{\Delta}^{n} \phi_w(x)$ with $w = w_n...w_1$ exists if and only if there is a word v and an integer n_0 such that for all $n \geq n_o$, $\phi_w = \phi_0^{(n)} \circ \phi_v$ (see [**33, 43, 44**] formore detail).

Definition 5. *A Laplacian Δ is said to admit* spectral decimation *if all its eigenvalues are of the form*

$$\lambda = c_{\Delta}^{i} \lim_{n\to\infty} c_{\Delta}^{n+j} \phi_0^{(n)} \circ \phi_w(x), \tag{2.5}$$

where $x \in \sigma(\Delta_i) \cup E(M, M_0)$, $i \in \mathbb{N} \cup \{0\}$ and $|w| = j$.

In other words, the spectrum of the Laplacian, $\sigma(\Delta)$, is approximated by the spectrum of Δ_n, $\sigma(\Delta_n)$, scaled by the Laplacian renormalization constant c_{Δ}^{n}. The spectrum of Δ_n can be computed by spectral decimation (see Subsection 2.2). The Sierpinski gasket, n-branch tree-like fractals, the fractal 3-tree and Viscek sets are all examples of fractals that admit spectral decimation. For further background on spectral decimation we refer the reader to [**2, 13, 33, 39, 43**].

2.4. Self-similar resistance forms and self-adjoint Laplacians. This discussion of resistance forms is largely taken from [**22, 23**].

Definition 6. *Let X be a set. A pair $(\mathcal{E}, \mathcal{F})$ is a resistance form on X if it satisfies the following conditions.*

(1) *$\mathcal{F}$ is a linear subspace of $l(X)$ containing constants and $\mathcal{E}$ is a non-negative symmetric quadratic form on $\mathcal{F}$. Moreover $\mathcal{E}(u, u) = 0$ if and only if u is constant on X.*
(2) *Let $\sim$ be an equivalence relation on $\mathcal{F}$ defined by $u \sim v$ if and only if $u - v$ is constant on X. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.*

(3) *For any finite subset* $V \subset X$ *and for any* $v \in l(V)$ *there exists* $u \in \mathcal{F}$ *such that* $u|_V = v$.

(4) *For any* $p, q \in X$,

$$R_{(\mathcal{E},\mathcal{F})}(p,q) = \sup\left\{\frac{|u(p)-u(q)|^2}{\mathcal{E}(u,u)} : u \in \mathcal{F}, \mathcal{E}(u,u) > 0\right\} \tag{2.6}$$

is finite.

(5) *If* $u \in \mathcal{F}$, *then* $\bar{u} \in \mathcal{F}$ *and* $\mathcal{E}(\bar{u},\bar{u}) \leq \mathcal{E}(u,u)$, *where* $\bar{u} = (u \wedge 0) \vee 1$ *is the normal contraction of* u.

Note that conditions $(1), (2)$, and (5) are the conditions that are also used in the defintion of a Dirichlet form, but here there is no reference L^2 space in the background. The quantity $R_{(\mathcal{E},\mathcal{F})}(p,q)$ is a metric called the effective resistance metric associated with the resistance form $(\mathcal{E},\mathcal{F})$. In this metric elements of $\mathcal{F}$ are 1/2-Hölder continuous. Moreover, if $B \subset X$ is chosen as the set where zero (Dirichlet) boundary conditions are imposed, then the Green's function $g_B(x,y)$ is always continuous outside the diagonal (see [**23**, Theorem 4.5]).

The definition above does not involve any self-similarity. If we deal with a self-similar fractal, such as in Definition 1, then, according to [**21, 22**], a resistance form on $V_* = \cup_{n\geq 0} V_n$ is self-similar with resistance weights r_j if

$$\mathcal{E}(u,u) = \sum_{j=1}^{N} \frac{1}{r_j}\mathcal{E}(\psi_j^*(u), \psi_j^*(u)) \tag{2.7}$$

for any $u \in \mathcal{F}$. For such resistance forms the maps ψ_j are asymptotic contraction maps in the effective resistance metric with contraction ratio r_j.

In our paper we always assume that all resistance weights are equal, that is

$$r_i = r_j = r \quad \text{for all} \quad i,j = 1,...,N.$$

In addition, we assume that the resistance form is *regular*, that is

$$r < 1,$$

which corresponds to the case

$$d_S = \frac{2\log N}{\log N - \log r} < 2$$

according to [**21, 22, 25, 28**], where d_S is the so called spectral dimension (see Remark 1 and [**37**] for a discussion of this).

Theorem (Kigami [**23**]). *If* $r < 1$ *then the Dirichlet and Neumann Laplacians are self-adjoint, have discrete spectra and, moreover, the Dirichlet Laplacian has a continuous Green's function* $g(x,y)$.

This result relies primarily on Theorem 8.13 in [**23**], which says that Δ has compact resolvent under the conditions that measure on K is non-atomic and that the effective resistance metric is integrable over K. Thus Δ has a discrete spectrum. Finitely ramified fractals with regular harmonic structure have a natural non-atomic self-similar probability measure and have finite diameter in the effective resistance metric. Thus the spectrum of the Dirichlet or Neumann Laplacian Δ can be written as $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$. Moreover, in the Dirichlet case we have $0 = \lambda_0 < \lambda_1$.

In the case of a fully symmetric p.c.f. with equal resistance weights, one always has $r < 1$ and Shima [**33**] has shown that the spectral decimation method will produce the spectrum of Δ, which is described in Subsection 2.3. Here we extend this result for finitely ramified fractals with regular harmonic structure.

Theorem 1. *If the harmonic structure on a finitely ramified self-similar fractal with equal resistance weights is regular, that is $r < 1$, then every eigenvalue of the Laplacian has a form*

$$\lambda_k = \lim_{n\to\infty} c_{\vartriangle}^{n} \lambda_k^{(n)} = \lim_{n\to\infty} c_{\vartriangle}^{n_0+m} \phi_0^m \lambda_k^{(n_0)} \tag{2.8}$$

where $k \in \mathbb{N}$, $\lambda_k^{(n)}$ is the k-th eigenvalue of the Laplacian Δ_n, $n = n_0 + m$. The second equality holds for any $n_0 = n_0(k)$ large enough, depending on k and λ_k. Moreover, the multiplicities can be computed according to the formulas in [**2**, Proposition 4.1].

Proof. We prove this result in four steps to clearly delineate where the assumption on the Green's function and spectral dimension is used, also because the first two steps have been discussed before but not collected into a single result.

Step 1: We show that $\lambda_k = \lim_{n\to\infty} c_{\vartriangle}^{n} \lambda_k^{(n)}$. Using Lemma 1 if f_n, a function on V_n, is an eigenfunction of Δ_n with eigenvalue $\lambda_k^{(n)}$ there is an extension f_{n+1} on V_{n+1} that is an eigenfunction of Δ_{n+1} with eigenvalue $\lambda_k^{(n+1)}$ and by induction there is a continuous function f on K with the property that $f|_{V_n}$ is an eigenfunction of Δ_n with eigenvalue $\lambda_k^{(n)}$. Then

$$\Delta f = \lim_{n\to\infty} c_{\vartriangle}^{n} \Delta_n f|_{V_n} \tag{2.9}$$

$$= \lim_{n\to\infty} c_{\vartriangle}^{n} \lambda_k^{(n)} f|_{V_n} \tag{2.10}$$

$$= \lambda_k f. \tag{2.11}$$

Where $\lambda_k = \lim_{n\to\infty} c_{\vartriangle}^{n} \lambda_k^{(n)}$. So any sequence $\lambda_k^{(n)}$ of eigenvalues of Δ_n will produce an eigenvalue of Δ along this scaled limit, if it exists. These are called "raw eigenvalues" by Shima [**33**] and Kigami [**21, 23**]. It is in this step that we use the assumption on the Green's function. If the Green's function is continuous then Theorem 3.5 in [**33**] states that all the eigenvalues of Δ are "raw eigenvalues" in the post critically finite case. Kigami [**23**] provides enough background for the extension of the claim to self-similar sets where the Laplacian defines a resistance form, such as fully symmetric finitely ramified fractals with a harmonic structure is regular.

Step 2: Let $\Lambda \geq 0$. Using the mechanics of spectral decimation given above, and in more detail in [**2**], $\sigma(\Delta_n)$ is calculated using inverse images of $\sigma(\Delta_0)$ and $E(M, M_0)$ under $R(z)$. There may be several branches of of R^{-1} denoted ϕ_i but for $n = n_0 + m$ for n_0 large enough only $c_{\vartriangle}^{n_0} c_{\vartriangle}^{m} \phi_0^m(\sigma(\Delta_{n_0}))$ will intersect $[0, \Lambda)$. The value of n_0 can be computed directly from R, and depends on Λ. But elements of $\sigma(\Delta_{n_0})$ are given by spectral decimation to be $c_{\vartriangle}^{n_0} \phi_w(\Delta_0)$ where w is any word of length n_0. This gives eigenvalues for Δ of the form of (2.8) which is equivalent to (2.5). So the eigenvalues less than Λ are of the form claimed.

Step 3: Let $\epsilon > 0$. Observe that $c_{\vartriangle}^{-n} \sigma(c_{\vartriangle}^{n} \Delta_n) = \sigma(\Delta_n)$. Thus $\sigma(\Delta) \cap c_{\vartriangle}^{-n}[0, \Lambda) \subset [0, \epsilon)$. For n large enough ϵ can be taken to be less than the least exceptional value.

And for any Λ the starting level n_0 can be chosen high enough. Hence all eigenvalues are of the claimed form since Λ is arbitrary.

Step 4: The multiplicities are computed in [**2**, Theorem 1.1] and, in particular, this theorem shows that, for a given k, the multiplicity of λ_k is the same as that of $\lambda_k^{(n)}$ for all n large enough (depending on k as above). $\square$

It is often convenient to assume that n_0 is the smallest integer with the properties described above, but the claims are true if n_0 is replaced with any integer between the smallest possible n_0 and n in the proof above.

2.5. The Julia set and the graph Laplacians. According to the classical theory [**7, 8, 31**], the Julia set of the spectral decimation function R, denoted $\mathcal{J}_R$, is given by

$$\mathcal{J}_R = closure \bigcup_{n\geq 0} R^{-n}(0),$$

where $R^{-n}(0)$ are pre-images of 0 of order n (because 0 is a repulsive fixed point). Furthermore, according to [**30**], we have that

$$\mathcal{J}_R \subseteq \sigma(\Delta_\infty) \subseteq \mathcal{J}_R \cup \mathcal{D}_\infty \cap \mathbb{R}^+,$$

where $\Delta_\infty = \lim_{n\to\infty} \Delta_n$ is the discrete probabilistic Laplacian on an infinite self-similar graph and $\mathcal{D}_\infty \setminus \mathcal{J}_R$ contains only isolated points. Here

$$\mathcal{D}_n = \bigcup_{m=0}^{n} R^{-m}(E(M, M_0) \cup \sigma(\Delta_0)),$$

and

$$\mathcal{D}_\infty = \bigcup_{n=0}^{\infty} \mathcal{D}_n,$$

where the unions are increasing in n. Moreover

$$\sigma(\Delta_n) \subseteq \mathcal{D}_n \quad \text{and} \quad \sigma(\Delta_n) \subseteq \sigma(\Delta_\infty).$$

3. Gaps in the spectrum

In this section we prove that the existence of gaps can be characterized by the Julia set.

It is known that spectra of Laplacian operators on many fractals have gaps. Investigation of the existence of gaps is important to analysis on fractals because of its many interesting applications, as mentioned in the introduction. Some criteria and examples are given in [**43**] and [**44**], although the verification can be tedious. In the next section we will derive a simple and easy to apply criterion based on the total disconnectedness of the Julia set of the spectral decimation function, generalizing the results of [**43**].

Definition 7. *For a given infinite sequence* $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq \cdots$, *we say that there exist gaps in the sequence if* $\limsup_{k\geq 1} \dfrac{\alpha_{k+1}}{\alpha_k} > 1$.

By Theorem 1 the study of ratios of eigenvalues of the Laplacian on fully symmetric finitely ramified fractals can be reduced to the corresponding limit of ratios of eigenvalues of the discrete Laplacians on finite graphs G_n that approximate the fractal. This is because any ratio of eigenvalues in $\sigma(\Delta)$ will have the form

$$\frac{c_\Delta^i \lim_{n\to\infty} c_\Delta^{n+|w|} \phi_0^{(n)} \phi_w(\beta_j)}{c_\Delta^l \lim_{n\to\infty} c_\Delta^{n+|v|} \phi_0^{(n)} \phi_v(\beta_k)} = \lim_{n\to\infty} \frac{\phi_0^{(n-i-|w|)} \phi_w(\beta_j)}{\phi_0^{(n-l-|v|)} \phi_v(\beta_k)} \tag{3.1}$$

$$= \lim_{n\to\infty} \frac{\phi_0^{(n)} \phi_w(\beta_j)}{\phi_0^{(n+r)} \phi_v(\beta_k)} \tag{3.2}$$

for some integer r and $\beta_j,\ \beta_k \in \sigma(\Delta_0)$. Both the numerator and denominator in the last ratio are eigenvalues of discrete Laplacians on graphs approximating the fractal. Hence it is sufficient to consider all ratios of eigenvalues in $\bigcup_n \sigma(\Delta_n)$ (without powers of c_Δ) in order to show the existence of gaps in the spectrum of Laplacians on fractals.

According to Subsection 2.5 and Corollary 1, the following lemma applies for any spectral decimation function of a finitely ramified fully symmetric fractal, always a rational function.

Lemma 2. *Let R be a spectral decimation function, rational and of degree at least 2 with an attractive fixed point at infinity and such that $R(0) = 0$, $R'(0) > 1$, and the Julia set $\mathfrak{J}_R$ of R is real and nonnegative. If $I = [0, a]$ is the convex hull of $\mathfrak{J}_R$ then the following is true:*

1. *$R^{-n-1}(I) \subset R^{-n}(I)$ for all $n \geq 0$;*
2. *$\mathfrak{J}_R = \bigcap_{n=0}^{\infty} R^{-n}(I)$;*
3. *either $\mathfrak{J}_R = I$ or $\mathfrak{J}_R$ is totally disconnected;*
4. *$\mathfrak{J}_R = I$ if and only if $R^{-1}(I)$ is connected;*
5. *either $R(a) = 0$ or a is the largest fixed point of R.*

Proof. By the classical complex dynamics theory (see [**7, 8, 31**]), the assumptions of this lemma imply that there is a single Fatou component attracted to infinity, and the action of R on its Julia set is hyperbolic. Let b be the smallest positive number such that $R^{-1}([0, b]) \subseteq [0, b]$, which exists because infinity is an attractive fixed point. We have that $R^{-1}([0, b])$ is a finite collection of intervals, which easily imply that either $R(b) = 0$ or b is a repulsive or indifferent fixed point of R. However the Sullivan classification of Fatou components excludes the possibility of an indifferent fixed point in the Fatou component, and hence $a = b$ and all the claims of the lemma follow. □

Our main result is the following theorem.

Theorem 2. *Under the conditions of Theorem 1, there exist gaps in $\sigma(\Delta)$ if and only if $\mathfrak{J}_R$ is totally disconnected.*

Proof. First, suppose $\mathfrak{J}_R$ is totally disconnected. Then, following Lemma 2, the set $I_n = R^{-n}(I)$ is a finite collection of closed intervals which cover the Julia set and decrease as n increases (because of the hyperbolicity). Hence there is an n_0 such that one of the intervals that constitute I_{n_0} has the form $[0, \epsilon]$, where ϵ is smaller than any point in the exceptional set. Moreover, if n_0 is large enough then $[0, \epsilon]$ is contained in the domain of the branch ϕ_0 of the partial inverses of R. It is easy to see that $[0, \epsilon] \cap I_{n+1}$ is a union of at least two closed nonintersecting intervals. We have that

$$\bigcup_{m=0}^{n} R^{-m}\big(\sigma(\Delta_0) \cup E(M, M_0)\big) \setminus \mathfrak{J}_R \tag{3.3}$$

consists of isolated points that accumulate to $\mathfrak{J}_R$, unless the set in (3.3) is empty, because $\mathbb{C} \setminus \mathfrak{J}_R$ is attracted to infinity by the iterations of R.

Therefore there are positive numbers $\alpha_0, \beta_0 \in (0, \epsilon]$ such that $\alpha_0 < \beta_0$ and the interval (α_0, β_0) does not intersect $\sigma(\Delta_\infty)$. But then there is another interval $(\phi_0(\alpha_0), \phi_0(\beta_0)) = (\alpha_1, \beta_1)$ that does not intersect $\sigma(\Delta_\infty)$ and so forth.

The ratio $\frac{\alpha_k}{\beta_k}$ is for large enough k bounded below uniformly in k, since $\alpha_k, \beta_k \to 0$ as $k \to 0$ and $\phi_0(z)$ is asymptotically linear near zero. (The starting interval could have been chosen close enough to zero so that the final fraction bounding the ratios is at least $\frac{1}{2}$ since $\mathfrak{J}_R$ is totally disconnected and compact.) Then we multiply the spectrum by c_Δ^n to obtain gaps in the spectra of $c_\Delta^n \Delta_n$ which are uniform in n. Therefore the spectrum of Δ has gaps because of Theorem 1.

In the opposite direction, suppose there are gaps in the spectrum but $\mathfrak{J}_R = I$ is an interval (Lemma 2 implies that if $\mathfrak{J}_R \neq I$ then it is totally disconnected). By the reasoning very similar to that given above, the gaps in the spectrum imply that $\sigma(\Delta_\infty)$ is not dense in I, which contradicts to the main result, Theorem 5.8, in [**30**]. □

Conjecture 1. *We conjecture that, under the conditions of Theorems 1 and 2, there are no gaps (which means $\mathfrak{J}_R = I$ is an interval) if and only if $R(z)$ is a Chebyshev polynomial, up to trivial constants. Furthermore, we conjecture that this happens if and only if the fractal K is one of the so-called Barlow-Evans fractals based on a unit interval (see* [**5, 34, 35**] *for more detail). This means that the self-similar structure of K is based on an interval, such as in the case of Diamond fractals of* [**2**, Section 7]*,* [**1**]*, and* [**20**]*. Relevant information can be found in* [**26**, Section 9]*,* [**27**, Section 8] *and* [**17, 16, 10, 9**]*. Note that our results already show that the complex dynamics of R is conjugate to that of a Chebyshev polynomial if there are no gaps.*

Conjecture 2. *We conjecture that, under the conditions of Theorems 1 and 2, there are nontrivial complex dimensions if and only if K is not homeomorphic to an interval (see references in Conjecture 1 and* [**29, 36**] *for more detail).*

Conjecture 3. *We conjecture that the results of* [**38**] *are true under the conditions of Theorems 1 and 2 even without heat kernel estimates (see* [**19, 3, 4, 18, 24**] *for more detail).*

4. An improved criterion for gaps in the spectrum

We conclude with a generalization of the criteria for gaps found in [**43**], which also demonstrates where gaps are located. In this section the members of the exceptional set are also called forbidden eigenvalues.

Theorem 3. *Suppose b is a real number dominating all the forbidden eigenvalues and suppose $R^{-1}[0, b] \subseteq [0, b]$. Let $\{\phi_j\}_{j=0}^{L}$ be the partial inverses of R ordered so that $\max \phi_i \leq \min \phi_{i+1}$. Assume $\phi_0(b) < b$ and ϕ_0 is strictly convex. There exist gaps in the spectrum of the Laplacian if for some $0 \leq J < L$,*

$$M \equiv \max \phi_J \mid_{[0,b]} < \min \phi_{J+1} \mid_{[0,b]} \equiv m.$$

In fact, there are a bounded number of elements of the spectrum in the intervals (A_k, B_k) *where*

$$\begin{aligned} A_k &= c_\Delta^k \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-1)}(M) = \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-1)}(M) \\ B_k &= c_\Delta^k \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-1)}(m) = \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-1)}(m). \end{aligned}$$

Proof. Lemma 12 of [**43**] shows that the strict convexity of ϕ_0 and the assumption that $\phi_0(b) < b$ imply that A_k and B_k are well defined and $A_k < B_k$. Of course, $B_k/A_k = B_0/A_0 > 1$.

Thus each eigenvalue is of the form $x = c_\Delta^i \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-j)} \phi_v(z)$, where $i,\ j \in \mathbb{N} \cup \{0\},\ z \in E(M, M_0)$, $|v| = j$ and if $j \neq 0$, then $v = v_j \dots v_1$ where $v_j \neq 0$. Equivalently $x = \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-i-j)} \phi_v(z)$.

Case 1: $j \neq 0$, say $i + j = k + 1$ for $k \in \mathbb{N} \cup \{0\}$. If $v_j \geq J + 1$, then $\phi_{v_j} \circ \dots \circ \phi_{v_1}(z) = \phi_{v_j}(z') \geq \min \phi_{J+1}$. Hence

$$x = \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-1)} \phi_{v_j}(z') \geq \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-1)}(m) = B_k.$$

We can also write

$$x = \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-2)} \phi_0(\phi_{v_j}(z')) \leq \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-2)} \max \phi_0,$$

and this is clearly bounded by

$$\lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-2)}(M) = A_{k+1}.$$

Thus $x \in [B_k, A_{k+1}]$.

Otherwise, $v_j \leq J$. Then $\phi_{v_j} \circ \dots \circ \phi_{v_1}(z) = \phi_{v_j}(z') \leq \max \phi_{v_j} \leq \max \phi_J$, so

$$x \leq \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-1)}(M) = A_k.$$

Furthermore, because $v_j \geq 1$,

$$\begin{aligned} B_{k-1} &= \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-1)} \phi_0(m) \leq \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-1)}(\max \phi_0) \\ &\leq \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-1)}(\min \phi_1) \leq x. \end{aligned}$$

Hence $x \in [B_{k-1}, A_k]$. Consequently, if x is of the first type then $x \notin \cup(A_k, B_k)$.

Case 2: $j = 0$. We first note that if $z = 0$, then $x = 0$ and $x \notin (A_k, B_k)$, so we assume otherwise. The strict convexity of ϕ_0 ensures

$$\frac{\phi_0(x)}{x} \leq \frac{\phi_0(b)}{b} \leq \lambda < 1,$$

for some λ. Thus $\phi_0^{(n)}(z) \leq \lambda^n z \to 0$ for all $z \in [0, b]$ and therefore we can choose n such that $\phi_0^{(n)}(m) < z^*$, where $z^* := \min E(M, M_0)$. We claim that $x = \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-i)}(z) \notin (A_k, B_k)$, if $i \leq k$ or $i > k + n$.

To prove the claim, we first suppose $i = k - s$ for some $s \geq 0$. Then

$$x = \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-1)} \phi_0^{(s+1)}(z) \leq \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-1)} \phi_0(z) \leq A_k.$$

If $i = k + n + s$ for some $s \geq 1$, then by the definition of z^*,

$$x \geq \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-n-s)}(z^*).$$

By our choice of n, the last term above dominates

$$\lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-n-s)}\phi_0^{(n)}(m) \geq \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-k-n-s)}\phi_0^{(n+s-1)}(m) = B_k.$$

This proves our claim and therefore the only forbidden eigenvalues that can lie in the interval (A_k, B_k) are those of the form $\lim_{m\to\infty} c_\Delta^m \phi_0^{(m-i)}(z)$, where $k < i \leq k+n$, and $z \in E(M, M_0)$.

It follows that the theorem holds with the bound on the number of eigenvalues in any interval (A_k, B_k) being at most $n|E(M, M_0)|$, where $|E(M, M_0)|$ is the cardinality of $E(M, M_0)$. Hence there exists a subinterval $(c_k,\ d_k) \subseteq (A_k, B_k)$ containing no elements of the spectrum having $d_k - c_k \geq \frac{1}{N}|B_k - A_k|$. □

Remark 2. *Note that the proof actually shows that the only numbers of the form* $c^i \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-j)}\phi_v(z)$*, for* $z \in [0, b]$*, which may be contained in* (A_k, B_k)*, are those equal to* $\lim_{m\to\infty} c_\Delta^m \phi_0^{(m-i)}(z)$*, where* $k < i \leq k+n$*.*

Next, we show that the three theorems in [**43**] can be deduced from Theorem 3.

Corollary 2. [**43**, Thm. 13] *Suppose* b *is the largest forbidden eigenvalue, and that* $R^{-1}[0,\ b] \subseteq [0,\ b]$*,* ϕ_1 *is decreasing on* $[0, b]$*,* ϕ_0 *is strictly convex and* $\phi_0(b) < \phi_1(b)$*. Then there are gaps in the spectrum of the Laplacian.*

Proof. We apply the theorem with $J = 0$. Note that $\phi_1(b) \leq b$, and hence $\phi_0(b) < b$. □

Corollary 3. [**43**, Thm. 16] *Suppose* $\alpha < \beta$ *are two consecutive forbidden eigenvalues. Let* b *be the largest forbidden eigenvalue and suppose that* $c \geq b$ *satisfies* $R^{-1}([0, c]) \subseteq [0, c]$[1]*. Assume* ϕ_0 *is strictly convex,* $\phi_0(c) \leq \alpha$ *and* $\phi_1(x) \geq \beta$ *for all* $x \in [0, c]$*. Then there are gaps in the spectrum of the Laplacian.*

Proof. This also follows easily from Theorem 3 since

$$\max \phi_0 \mid_{[0,c]} \leq \alpha < \beta \leq \min \phi_1 \mid_{[0,c]} \leq c.$$

□

Corollary 4. [**43**, Thm. 15] *Suppose* $a < b$ *are the two largest forbidden eigenvalues,* $R^{-1}[0,\ b] \subseteq [0, a]$*,* ϕ_1 *is decreasing on* $[0, b]$ *and* ϕ_0 *is strictly convex. Then there are gaps in the spectrum of the Laplacian.*

Proof. Let $E(M, M_0)' = \{\beta_t\}$ consist of all the elements in $E(M, M_0) \setminus \{b\}$ together with the real numbers $\phi_0(b), \cdots, \phi_L(b)$. As $R^{-1}[0, b] \subseteq [0, a]$, $\phi_j(b) \leq a < b$ for all $j = 0, \cdots, L$. Hence the largest member of $E(M, M_0)'$ is a and $R^{-1}[0, a] \subseteq R^{-1}[0, b] \subseteq [0, a]$.

All eigenvalues of the Laplacian are of the form $c^i \lim_{m\to\infty} c_\Delta^m \phi_0^{(m-j)}\phi_v(\beta_t)$, thus we may apply the same arguments as in Theorem 3, but with $E(M, M_0)'$ taking the place of the forbidden eigenvalues $E(M, M_0)$.

By the monotonicity assumptions, $\phi_0(a) = \max\limits_{[0,a]} \phi_0$ and $\phi_1(a) = \min\limits_{[0,a]} \phi_1$. Moreover, $\phi_0(a) < \phi_0(b) \leq \phi_1(b) < \phi_1(a) \leq a$, thus Theorem 3 implies there are a

[1]In the statement of Theorem 16 of [**43**] it states $R^{-1}[0, b] \subseteq [0, c]$, but it is clear from the proof that the assumption $R^{-1}[0, c] \subseteq [0, c]$ was intended.

bounded number of elements in (A_k, B_k), where

$$\begin{aligned} A_k &= c_\vartriangle^k \lim_{m\to\infty} c_\vartriangle^m \phi_0^{(m-1)} \phi_0(a) \text{ and} \\ B_k &= c_\vartriangle^k \lim_{m\to\infty} c_\vartriangle^m \phi_0^{(m-1)} \phi_1(a). \end{aligned}$$

□

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