Energy and Laplacian on the Sierpiński Gasket

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Abstract. This is an expository paper which includes several topics related to the Dirichlet form analysis on the Sierpiński gasket. We discuss the analog of the classical Laplacian; approximation by harmonic functions that gives a notion of a gradient; directional energies and an equipartition of energy; analysis with respect to the energy measure; harmonic coordinates; and non self-similar Dirichlet forms on the Sierpiński gasket, one of which is defined by the Apollonian packing.

Contents

1. Introduction: what is not in the domain of the Laplacian

The Sierpiński triangle was introduced in an influential paper [**53**] by W. Sierpiński (reprinted in [**54**]), and plays an important role in the theory of curves. More information on the history of the Sierpiński gasket can be found in an expository paper [**56**]. It was in the paper [**32**] by B. Mandelbrot that the Sierpiński triangle like shape was used for the first time to describe an object besides the world of pure mathematics. The now familiar name "Sierpiński gasket" was coined by B. Mandelbrot in his celebrated book [**34**], where it is used to illustrate many ideas. The residual set of the Apollonian packing (see Example 5.16) was also used several times in [**33**] and [**34**]. In this paper we discuss a type of analysis which does not depend on how the Sierpiński gasket is embedded into $\mathbb{R}^2$, but rather

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on its inner structure. However some special and somewhat unusual embeddings defined by harmonic coordinates will play an important role in Sections 4 and 5. In particular, there exists a non self-similar energy form on the Sierpiński gasket such that the Apollonian packing represents the gasket in harmonic coordinates, thus giving another point of view on this construction.

The study of the Laplacian on fractals was originated in the Physics literature, largely motivated by [**33**] and [**34**], and the series of papers [**13**]. In particular, the so-called spectral decimation method was developed for the Laplacian on the Sierpiński lattice ([**2, 43, 44**]). The Laplacian on the Sierpiński gasket was first constructed as the generator of a diffusion process by S. Kusuoka and S. Goldstein in [**14, 25**], later studied in detail and extended to other fractals in [**3, 4, 5, 7, 12, 24, 26, 27, 31**] and many other papers. An analytic approach was developed by J. Kigami, who constructed the Laplacian on the Sierpiński gasket in [**17**] using the theory of Dirichlet forms. This construction was extended to a large class of p.c.f. self-similar sets, or finitely ramified fractals, in [**18**]. The eigenvalue distribution and eigenfunctions for the Laplacian on the Sierpiński gasket were studied in detail in [**11**]. Certain questions related to completeness of localized eigenfunctions distribution of eigenvalues, and general "calculus" on the Sierpiński gasket and other fractals were addressed in [**1, 6, 9, 15, 22, 23, 29, 30, 42, 50, 57, 58, 59, 60, 61, 62**]. Recently there were two books [**21, 28**] published on the analysis on fractals. In particular, the reader is referred to [**21**] for an extensive background related to this article, and for definitions and propositions given here without specific reference.

In this paper we mainly review results of [**8, 40, 55, 63**]. First we consider the analog of the classical Laplacian on the Sierpiński gasket. It is shown that the square of any non constant function from the domain of the Laplacian does not belong to this domain. This is remarkably different from the properties of the Laplacian on a manifold. There are two proofs of this fact. One is based on a dichotomy for the local behavior of a function in the domain of the Laplacian. The other proof uses the theory of random matrices. It shows, in particular, that the natural analog of the norm of the gradient is not a function but a singular measure. Also it gives a simpler and more general proof of Kusuoka's result (see [**26, 27**]) of the singularity of the energy measures with respect to the Hausdorff (Bernoulli) measure.

In Section 2 we consider harmonic tangents to functions defined on the Sierpiński gasket. Analogously to the linear tangents to functions defined on $\mathbb{R}^n$, these harmonic tangents give a notion of a gradient on the Sierpiński gasket. It is shown that for a C^1–function on the Sierpiński gasket the gradient considered here and Kusuoka's gradient essentially coincide with a gradient considered by J. Kigami (see [**26, 27, 19**]). The gradient at a junction point was studied by R. Strichartz in relation to the Taylor approximation on fractals ([**61**]). In this paper we present certain continuity properties of the gradient for a function in the domain of the Laplacian.

The energy of a function defined on a post-critically finite self-similar fractal can be written as a sum of directional energies. It is shown in Section 3 for a self-similar Dirichlet form under mild hypotheses that each directional energy is a fixed multiple of the total energy, and we compute the multiple for a one-parameter family of energy forms on the Sierpiński gasket. For the standard energy form, the result is an equipartition of energy principle.

Then, in Section 4, we discuss analysis with respect to the energy measure on the Sierpiński gasket, in particular, the properties of the energy measure and the related energy measure Laplacian. Harmonic coordinates, which were first introduced by J. Kigami in [**19**], play an important role. For example, in harmonic coordinates, the energy measure is in some sense a tangential second derivative. We use harmonic coordinates to define a structure of a generalized one dimensional manifold, vector fields and harmonic tangents. Although tangent vector fields can be defined naturally, one can show that there are no nonzero continuous tangent vector fields.

In Section 5 we discuss not necessarily self-similar Dirichlet forms on the Sierpiński gasket that can be described as limits of compatible resistance networks on the sequence of graphs approximating the gasket. However, this description is not particularly useful because the compatibility conditions are difficult to analyze. We present an alternative geometric description, and discuss the associated effective resistance topology. In particular, the residual set of the Apollonian packing (see Example 5.16) defines a non self-similar Dirichlet forms on the Sierpiński gasket. In addition, we show how to parameterize all the Dirichlet forms by infinite sequences of independent variables.

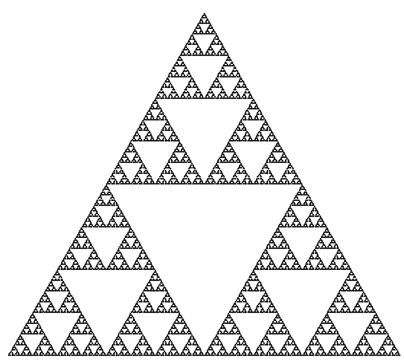

FIGURE 1. Sierpiński gasket.

To define the Sierpiński gasket one can use an iteration function system as follows. We fix three contractions $\Psi_j(x) = \frac{1}{2}(x + v_j)$, $x \in \mathbb{R}^2$. The **Sierpiński gasket** is the unique compact set S such that $S = \Psi_1(S) \cup \Psi_2(S) \cup \Psi_3(S)$.

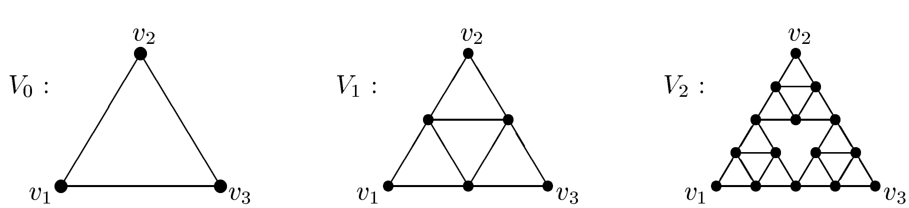

FIGURE 2. Approximations V_n to the Sierpiński gasket.

DEFINITION 1.1. For each $n \geqslant 0$ we define V_n inductively by

$$V_{n+1} = \Psi_1(V_n) \bigcup \Psi_2(V_n) \bigcup \Psi_3(V_n)$$

where the initial set of vertices $V_0 = \{v_1, v_2, v_3\} = \partial S$ also plays the role of the boundary of S. For $x, y \in V_n$ we write $y \sim x$ if x and y are neighbors in V_n (on Figure 2 the neighboring vertices of V_n are connected by line segments).

DEFINITION 1.2. The **discrete Laplacian** on $\ell^2(V_n \backslash \partial S)$ is defined by

$$\Delta_n f(x) = \tfrac{1}{4} \sum_{\substack{y \sim x \\ y \in V_n}} f(y) - f(x)$$

For a continuous function f on S, we define the **Laplacian** on the Sierpiński gasket

$$\Delta f(x) = \lim_{n \to \infty} 5^n \Delta_n f(x)$$

if this limit exists. We write $f \in \mathcal{D}om(\Delta)$ if Δf is continuous.

The factor 5 in this definition (and not 4 as one might expect based on the Euclidean distance between neighboring points) is the product of the scaling factor $\frac{5}{3}$ for the energy on the Sierpiński gasket (see Section 4) and the scaling factor 3 for the Hausdorff measure.

THEOREM 1.3 ([**8**]). *If $f \in \mathcal{D}om(\Delta)$ and $f \not\equiv const$ then $f^2 \notin \mathcal{D}om(\Delta)$.*

DEFINITION 1.4. The **normal derivative** of f at a junction point x is $\partial_n f(x) = \lim_{n \to \infty} \left(\frac{5}{3}\right)^n (2f(x) - f(a_n) - f(b_n))$

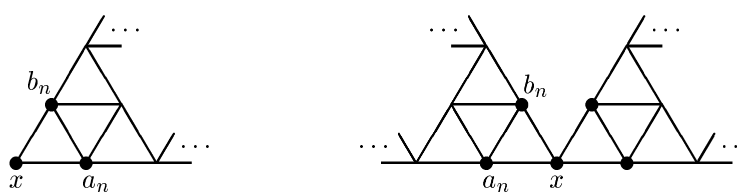

FIGURE 3. Approximation of the normal derivative on the Sierpiński gasket.

The following proposition follows elementary form Definitions 1.2 and 1.4.

PROPOSITION 1.5. *If $f \in \mathcal{D}om(\Delta)$ then $\partial_n f(x)$ exists at any junction point x. If $\partial_n f(x) \equiv 0$ then $f \equiv const$. If $\partial_n f(x) \neq 0$ then $\Delta f^2(x) = \infty$.*

The next theorem and corollary give more information on this phenomenon.

THEOREM 1.6 ([**8**]). *Let $f \in \mathcal{D}om(\Delta)$.*
(i) *If $\partial_n f(x) \neq 0$ then*

$$C_1|x-y|^\alpha \leqslant |f(x) - f(y)| \leqslant C_2|x-y|^\alpha$$

where $\alpha = \frac{\log 5 - \log 3}{\log 2} \approx .737$
(ii) *If $\partial_n f(x) = 0$ then for any $\varepsilon > 0$ there exists C_3 such that*

$$|f(x) - f(y)| \leqslant C_3|x-y|^{\beta+\varepsilon}$$

where $\beta = \frac{\log 5}{\log 2} \approx 2.322$.

COROLLARY 1.7. *If $f \in \mathcal{D}om(\Delta)$ and $\partial_n f(a) \neq 0$ then $g(x) = (f(x) - f(a))^2$ satisfies neither* **(i)** *nor* **(ii)** *and so $f^2 \notin \mathcal{D}om(\Delta)$.*

An additional explanation of the "no squares in the domain of the Laplacian" property based on the singularity of the energy measures is given in Section 4.

2. Harmonic tangents

A continuous function h on S is called **harmonic** if $\Delta f(x) = 0$ at every junction point $x \in S$. A harmonic function is uniquely determined by its three boundary values on ∂S.

DEFINITION 2.1. A **harmonic tangent** of f at x is $T_x f = \lim_{n\to\infty} h_{n,x}$ where $h_{n,x}$ is a unique harmonic function which coincides with f on the vertices of a triangle $S_{n,x} \ni x$ of the size 2^{-n}. That is, $T_x f$ is a harmonic approximation to f at x.

We can consider $T_x f$ as a 3-dimensional vector, which is determined by its three boundary values on ∂S. We also can define a 2-dimensional **gradient** vector

$$\nabla_x f = T_x f \quad \bmod\ (constants)$$

If x is a junction point then there can be two different "directional" harmonic tangents at x, each corresponds to one of the two small triangles that meet at x.

Let μ be the normalized Hausdorff measure on S.

THEOREM 2.2 ([**63**]). *If $f \in \mathcal{D}om(\Delta)$ then $T_x f$ exists for μ–almost all x. Moreover, $x \mapsto T_x f$ is continuous for μ–almost all x.*

If Δf is Hölder continuous then $T_x f$ exists at every non junction point x. Moreover, $x \mapsto T_x f$ is continuous at every non junction point x, but can be discontinuous at every junction point x where $\Delta f(x) \neq 0$.

THEOREM 2.3 ([**63**]). *Let $\vec{h} = (h_1, ..., h_m)$, where h_k are harmonic functions, and*

$$f = F(\vec{h}) : S \to \mathbb{R}.$$

If $F \in C^2(\mathbb{R}^m)$ then $T_x f$ exists for μ–almost all x.
If $F \in C^4(\mathbb{R}^m)$ then $T_x f$ exists at every junction point x.
In both cases

$$T_x f(y) = f(x) + \nabla F(\vec{h}(x)) \cdot (\vec{h}(y) - \vec{h}(x))$$

and so T_x is continuous in x.

Note that any non constant f considered in Theorem 2.3 is not in $\mathcal{D}om(\Delta)$.

3. Energy partition

The ***discrete Dirichlet (energy) form*** on V_n is

$$\mathcal{E}_n(f,f) = \sum_{\substack{x,y\in V_n \\ y\sim x}} (f(y) - f(x))^2$$

and the ***Dirichlet (energy) form*** on the Sierpiński gasket S is defined by

$$\mathcal{E}(f,f) = \lim_{n\to\infty} \left(\tfrac{5}{3}\right)^n \mathcal{E}_n(f,f)$$

with the domain

$$Dom(\mathcal{E}) = \{f : \mathcal{E}(f,f) < \infty\}.$$

A function h is **harmonic** if and only if it minimizes $\mathcal{E}(h,h)$ given the three boundary values. (Also we have $\Delta h = 0$ for a harmonic h.)

PROPOSITION 3.1. *For any function on S and any $n \geq 0$ we have*

$$\tfrac{5}{3}\mathcal{E}_{n+1}(f,f) \geqslant \mathcal{E}_n(f,f).$$

Moreover, $\frac{5}{3}\mathcal{E}_{n+1}(h,h) = \mathcal{E}_n(h,h) = \left(\frac{5}{3}\right)^{-n}\mathcal{E}(h,h)$ *for any harmonic function h.*

The next proposition summarizes several results due to J. Kigami.

PROPOSITION 3.2. *$\mathcal{E}$ is a local regular Dirichlet form on S which is self-similar in the sense that*

$$\mathcal{E}(f,f) = \tfrac{5}{3}\sum_{j=1,2,3}\mathcal{E}(f\circ F_j, f\circ F_j).$$

We have $Dom(\Delta) \subsetneqq Dom(\mathcal{E}) \subsetneqq C(S)$, and $\mathcal{E}(f,f) = 0$ if and only if f is constant. There is a Gauss–Green formula

$$\mathcal{E}(f,f) = -\tfrac{3}{2}\int_S f\Delta f d\mu + \sum_{p\in\partial S} f(p)\partial_n f(p).$$

Since there are three "special" directions on the Sierpiński gasket corresponding to the three sides of the largest triangle, we can define discrete "directional" energies as follows.

$$\mathcal{E}_m(u,u) = \sum_{i=1,2,3}\mathcal{E}^{(i)}_m(u,u)$$

$$\mathcal{E}^{(i)}_m(u,u) = \sum_{|w|=m}\left(u(\Psi_w v_{i+1}) - u(\Psi_w v_i)\right)^2$$

where indices of v_j are considered $\mod (3)$, that is $v_0 = v_3$ etc.

DEFINITION 3.3. The **directional energy** is

$$\mathcal{E}^{(i)}(u,u) = \lim_{m\to\infty}\left(\tfrac{5}{3}\right)^m\mathcal{E}^{(i)}_m(u,u)$$

if the limit exists.

PROPOSITION 3.4. *For any function u of finite energy,* ***equipartition of energy*** *holds in the sense that*

$$\mathcal{E}^{(i)}(u,u) = \tfrac{1}{3}\mathcal{E}(u,u).$$

In fact for a harmonic function h

$$\mathcal{E}^{(i)}_m(h,h) - \tfrac{1}{3}\mathcal{E}(h,h) = \left(\tfrac{4}{5}\right)^m\Big(\mathcal{E}^{(i)}_0(h,h) - \frac{1}{3}\mathcal{E}(h,h)\Big).$$

There are matrices A_1, A_2, A_3, which can be computed easily, such that for any harmonic function h

$$h|_{\Psi_j(\partial S)} = A_j\left(h|_{\partial S}\right).$$

Define

$$TQ = \tfrac{5}{3}\sum_{j=1,2,3}A_j^* Q A_j.$$

for any 3×3 symmetric matrix Q such that $Q1 = 0$. Then the largest eigenvalue of T is 1, with the eigenvector

$$Q_E = \binom{2}{} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

that is $TQ_E = Q_E$. The other two eigenvalues are $\frac{4}{5}$.

REMARK 3.5. The partition of energy by directions should be contrasted with the distribution of energy by location. The latter can be described by the energy measures (see Section 4), which are singular with respect to the Hausdorff measure. We can paraphrase the results as follows: ***energy distribution is geographically wild but directionally tame***. Note also that the distribution of energy by the directions defined in terms of harmonic functions, rather than in terms of the sides of the triangle as in this paper, is studied in detail in [**42**]. This distribution is given by an invariant measure on a circle.

Consider the self–similar energy forms on the Sierpiński gasket with bilateral symmetry, that is

$$\mathcal{E}(u,u) = \sum_{j=1,2,3} \frac{1}{r_j} \mathcal{E}(u \circ \Psi_j, u \circ \Psi_j)$$

with $r_1 = r_2$. If the conductances defining $\mathcal{E}_0$ are 1, 1, b, then

$$r_1 = r_2 = \frac{1+c+b}{1+2c+2b}$$

where $r_3 = cr_1$, and c is a solution to $3b^2 + 2b = c^2 + 2c^2b + 2cb^2$ with the restriction $0 < c < 3/2$.

THEOREM 3.6 ([**55**]). *For any function u of finite energy,*

$$\mathcal{E}^{(j)}(u,u) = a_j \mathcal{E}(u,u),$$

where

$$a_1 = a_2 = \frac{\eta}{2(\eta+b)}, \quad a_3 = \frac{b}{\eta+b}, \quad \text{and} \quad \eta = \tfrac{1}{2} + \left(b + \tfrac{1}{2}\right)\left(\tfrac{1+c+b+2cb}{1+2c+2b}\right)^2.$$

As $b \to 0$, $a_3 \to 1/2$ and $a_1 \to 0$, while $a_3 \to 1/5$ and $a_2 \to 2/5$ when $b \to \infty$.

Note that in Figure 4 the graphs of a_0 and a_1 cross in two different points, and so the equipartition of energy holds for two different values of b. Explanation for one intersection point is simple: it corresponds to $b = 1$ (the most symmetric case, Proposition 3.4). We do not have any explanation why the equipartition of energy holds for another value of b, where it can not be explained by the symmetry argument.

Next we will briefly explain how partition of energy occurs for other p.c.f. (finitely ramified) fractals. The reader is referred to the book [**21**] for the precise definition and more information.

Suppose we have a p.c.f. self–similar fractal K:

$$K = \bigcup_{i=1}^{N} \Psi_i K$$

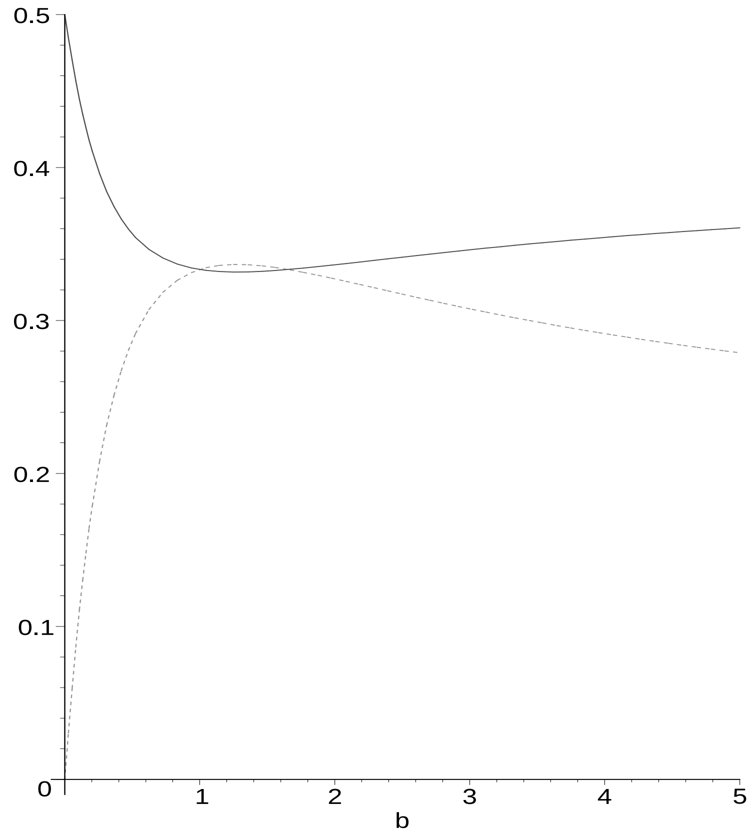

FIGURE 4. The graphs of a_0 (solid line) and a_1 (dotted line) as a function of the parameter b in the range $0 < b < 5$.

where Ψ_i are contractive injections, and there is a finite subset $V_0 = \{v_1, \dots, v_{N_0}\}$ called the **boundary** of K such that for $i \neq j$

$$\Psi_i K \cap \Psi_j K \subseteq \Psi_i V_0 \cap \Psi_j V_0,$$

and the same for all the distinct compositions $\Psi_w = \Psi_{i_1} \cdots \Psi_{i_k}$.

Suppose also that there is a self–similar Dirichlet form $\mathcal{E}$ on K:

$$\mathcal{E}(u,u) = \lim_{m\to\infty} \mathcal{E}_m(u,u)$$

where

$$\mathcal{E}_m(u,u) = \sum_{|w|=m} r_w^{-1} \mathcal{E}_0(u \circ \Psi_w, u \circ \Psi_w)$$

for a choice of $\mathcal{E}_0$ and r_j with the "decimation property" that $\mathcal{E}_m(u,u)$ is independent of m for harmonic functions.

The self–similar identity is

$$\mathcal{E}(u,u) = \sum_{j=1}^{N} r_j^{-1} \mathcal{E}(u \circ \Psi_j, u \circ \Psi_j)$$

Let $A_1, \dots, A_N$ denote the harmonic extension matrices, so that $h\big|_{\Psi_i V_0} = A_i h\big|_{V_0}$ for any harmonic function h.

We define an operator T on $N \times N$ quadratic forms Q such that $Q1 = 0$ by

$$TQ = \sum_{j=1}^{N} r_j^{-1} A_j^* Q A_j.$$

The decimation property is

$$TQ_E = Q_E.$$

where Q_E is the matrix of $\mathcal{E}_0$.

THEOREM 3.7 ([**55**]). *Assume that*

(i) Each point $v_j \in V_0$ is the fixed point of one of the contractions Ψ_i.

(ii) (Irreducibility) There is no proper subspace invariant mod (*constants*) *under all A_i.*

Then 1 is a simple eigenvalue of T with the eigenvector Q_E, and every other eigenvalue λ satisfies $|\lambda| < 1$. Moreover, for any nonzero $Q \geqslant 0$ there exists a constant $\alpha(Q) > 0$ such that

$$\lim_{m\to\infty} T^m(Q) = \alpha(Q) Q_E.$$

This result is related to the partition of energy as follows. We can write the "initial" energy form $\mathcal{E}_0$ as a sum of directional energy forms $\mathcal{E}_0^{(i)}$. For each such form $\mathcal{E}_0^{(i)} = Q$ we apply Theorem 3.7 to obtain that $\mathcal{E}^{(j)}(u,u) = \alpha(\mathcal{E}_0^{(i)})\mathcal{E}(u,u)$.

REMARK 3.8. To put results of this section in a broader perspective, we note that Theorem 3.7 is a particular and somewhat simpler case of the study of the existence, uniqueness and stability of self-similar Dirichlet forms on fractals (see [**37, 38, 49**] and references therein). The map T plays a prominent role because it is the derivative (that is, linearization) of the generally nonlinear renormalization map Λ involved in this study. Theorem 3.7 follows from more general results of V. Metz (a more straightforward proof, based on the standard Perron-Frobenius type argument, is given in [**55**]). Note that Theorem 3.7 implies uniqueness and stability of a self-similar Dirichlet form for a given set of weights r_j, but it says nothing about the existence of such a form.

EXAMPLE 3.9. On the Vicsek set (Figure 5) there ***are*** nontrivial subspaces of harmonic functions, for example, generated by a harmonic function with boundary values (0,1,0,-1). The analog of Theorem 3.7 fails because for this harmonic function the energy in the direction of one diagonal is zero and in the other diagonal direction is not. See [**36**] for more detailed study of this case.

EXAMPLE 3.10. Consider a self-similar harmonic structure on the Sierpiński gasket without any symmetry assumptions. Then there are no nontrivial subspaces of harmonic functions and the hypotheses of Theorem 3.7 are satisfied.

The absence of invariant subspaces of harmonic functions seems to be related in many cases to the singularity of the energy measures, although this relation has not yet been clarified. For any $f \in Dom\mathcal{E}$ we can define the measure ν_f in the same way as for the Sierpiński gasket (see Section 4). Then there is a semi–norm $\|\cdot\| = \langle \,\cdot\,, Q_E \,\cdot\, \rangle$ such that for any harmonic function h

$$\nu_h(\Psi_w K) = r_w^{-1} \|A_w h|_{\partial K}\|^2$$

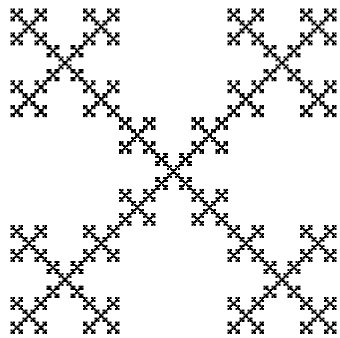

FIGURE 5. Vicsek set.

For any harmonic function h we have

$$h|_{\partial K_{\omega_1\cdots\omega_n}} = A_{\omega_n}\cdots A_{\omega_1}h|_{\partial K}$$

where $K_{\omega_1\cdots\omega_n} = \Psi_{\omega_n}\cdots\Psi_{\omega_1}(K)$. Then

$$T_x f = \lim_{n\to\infty} A_{\omega_1}^{-1}\cdots A_{\omega_n}^{-1} f|_{\partial K_{\omega_1\cdots\omega_n}}$$

and

$$Z_x = \lim_{n\to\infty} \frac{W_{n,x}^* W_{n,x}}{\mathbf{Tr}(W_{n,x}^* W_{n,x})}$$

where $W_{n,x} = A_{\omega_n}\cdots A_{\omega_1}$ (T_x and Z_x are introduced in Sections 2 and 4 respectively). For all $x \in K$, except a countable subset, there is a unique sequence $\{\omega_m\}_{m\geq 1}$ such that

$$\{x\} = \bigcap_{m\geq 1} K_{\omega_1\cdots\omega_m}.$$

Then we denote $M_m(x) = A_{\omega_m}$. Let μ be a Bernoulli measure on K such that $\mu_i = \mu(\Psi_i K) = (\lambda r_i)^{-1}$. The matrices $M_m(x)$ are statistically independent with respect to μ with $\mathrm{Prob}\{M_m(x) = A_i\} = \mu_i$.

THEOREM 3.11 ([**8**]). *Suppose that for any non constant harmonic function h there exists m such that the function*

$$x \mapsto \|M_m(x)\cdots M_1(x)h|_{\partial K}\|$$

is not constant. Then the measure ν_f is singular with respect to μ for any $f \in \mathcal{D}_{\mathcal{E}}$.

REMARK 3.12. If we have irreducibility (no proper invariant subspaces) then either $\nu_h = \mathcal{E}(h,h)\mu$ for each harmonic function h and each $\nu_f \ll \mu$, or each $\nu_f \perp \mu$. We conjecture that an interval is essentially the only situation when ν_h is not singular with respect to μ. The singularity of the measures ν_f has been proved by S. Kusuoka in [**26**] under the assumption that the matrices $\{A_1,\ldots,A_N\}$ are invertible and strongly irreducible, and an additional assumption on a certain index.

4. Analysis with respect to energy measure

The **energy measure** ν_f is defined for an open set O by

$$\nu_f(O) = \lim_{n\to\infty} \left(\tfrac{5}{3}\right)^n \sum_{\substack{y\sim x\\ x,y\in V_n\cap O}} (f(y)-f(x))^2$$

Indeed, $\mathcal{E}(f,f) = \nu_f(S)$.

The following fundamental result is due to S. Kusuoka.

THEOREM 4.1 ([**26, 27**]). *Let* $\{h_1, h_2\}$ *be an* $\mathcal{E}(\cdot,\cdot)$*–orthonormal basis of the two dimensional space of harmonic functions* $\mod (constants)$. *Define* $\nu = \nu_{h_1} + \nu_{h_2}$ *(it does not depend on the choice of the basis). Then*

(i) *The measure* ν_f *is absolutely continuous with respect to* ν *for any* $f \in Dom\mathcal{E}$.

(ii) *The measures* ν *and* μ *are mutually singular.*

A simpler proof of this theorem is given in [**8**].

Note that in the case of a domain in $\mathbb{R}^n$ or a manifold, one has the usual formula

$$\Delta f^2 = 2f\Delta f + |\nabla f|^2 \ .$$

In a more general setting of abstract Dirichlet form theory it should be replaced with

$$\Delta f^2 = 2f\Delta f d\mu + d\nu_f.$$

Thus we have "no squares in the domain of the Laplacian" property (see Section 1) because of the singularity of the measures μ and ν_f. The latter formula is closely related to the "square of the field operators" which often appear in the theory of Dirichlet forms.

THEOREM 4.2 ([**26, 27**]). *For* ν*–almost all* x *there is an* x*–dependent inner product* $\langle \ \cdot\ , Z_x\ \cdot\ \rangle$ *such that*

$$\mathcal{E}(f,f) = \int_S \langle T_x f, Z_x T_x f\rangle \nu(dx).$$

In particular $T_x f$ *exists for* ν*–almost all* x *in the sense of* $L^2(S, \nu, \langle\ \cdot\ , Z_x\ \cdot\ \rangle)$*–convergence. The rank of* Z_x *is one for* ν*–almost all* x.

By factoring out constants and choosing $\{h_1, h_2\}$ as the $\mathcal{E}(\cdot,\cdot)$–orthonormal basis of the two dimensional space of harmonic functions $\mod (constants)$ we can consider Z_x as a 2×2 matrix.

THEOREM 4.3 ([**19**]). *Let* $F \in C^1(\mathbb{R}^2)$ *and* $f = F(h_1, h_2) : S \to \mathbb{R}$.
Then $f \in Dom\mathcal{E}$ *and*

$$\mathcal{E}(f,f) = \int_S \langle \nabla f, Z\nabla f\rangle d\nu$$

where $\nabla f(x) = \nabla F(h_1(x), h_2(x))$. *Thus* Z *can be considered as a "harmonic Riemannian metric" on the Sierpiński gasket.*

REMARK 4.4. By definition, $Z = Z^* \geqslant 0$, $TrZ = 1$. Then ν–almost everywhere $P_x = Z_x$ is an orthogonal projection. Therefore we can define an **essential gradient** $\nabla_{ess} f(x) = P_x T_x f$. Then

$$\mathcal{E}(f,f) = \int_S \|\nabla_{ess} f(x)\|^2 \nu(dx).$$

In the situation of Kigami's theorem, $\nabla f(x) \neq \nabla_{ess} f(x)$ since the former is continuous and the latter is everywhere discontinuous. In fact, P_x can be interpreted as an orthogonal projection, in harmonic coordinates, onto the tangent line to the Sierpiński gasket at x.

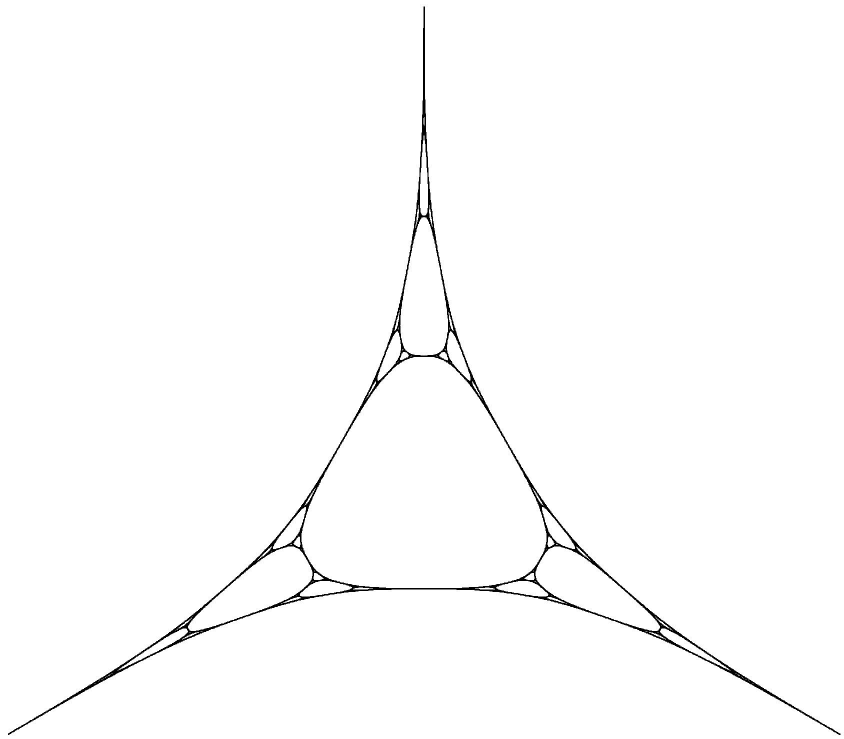

FIGURE 6. Sierpiński gasket in harmonic coordinates.

THEOREM 4.5 ([**19**]). *Let $\psi : S \to \mathbb{R}^2$ be defined by $\psi(x) = (h_1(x), h_2(x))$ and let $S_H = \psi(S)$. Then $\psi : S \to S_H$ is a homeomorphism.*

If $F_1, F_2 \in C^1(\mathbb{R}^2)$ and $F_1|_{S_H} = F_2|_{S_H}$ then $\nabla F_1|_{S_H} = \nabla F_2|_{S_H}$.

DEFINITION 4.6. Let τ be a parameterization of the boundary of a connected component of $\mathbb{R}^2 \backslash S_H$. Then τ is called a **boundary curve** of the Sierpiński gasket in harmonic coordinates.

THEOREM 4.7 ([**64**]). *If τ is a boundary curve then it is concave. Moreover $\tau \in C^1$ but $\tau \notin C^2$. For any x such that $\psi(x) \in \tau$ the projection P_x is, in harmonic coordinates, the orthogonal projection onto the tangent line to τ.*

REMARK 4.8. There are no nonzero continuous tangent vector fields on the Sierpiński gasket because in any neighborhood of any point of S there are three (topological) smooth circles which can not be oriented simultaneously (see Figure 7). Thus the Sierpiński gasket is not locally orientable.

REMARK 4.9. In [**45, 46, 47, 48**] G. de Rham considered a curve that has properties somewhat similar to the boundary curves we consider here. Given a convex polygon P in the plane, we denote by $P'(P)$ the polygon whose vertices are the points which divide the sides of P into 3 equal parts (if P has n vertices, then $P'(P)$ has $2n$ vertices). If P_0 is a square and $P_n = P'(P_{n-1})$, then P_n tends to an interesting convex curve C. In particular, G. de Rham showed that C has everywhere a tangent, but every subarc of C contains both points with vanishing

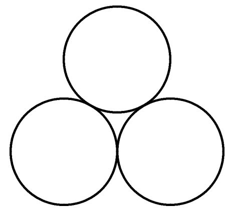

FIGURE 7. Three circles which can not be oriented simultaneously.

and infinite radius of curvature. Thus the properties of de Rham's curves are similar to the properties of the boundary curves τ considered in this paper. In [**41**] P. Nikitin considered the harmonic measure of this curve C and computes the Hausdorff dimension of this measure. We conjecture that P. Nikitin's method, which is based on the ideas of A. M. Vershik *et al.* [**16, 52**], is applicable for the computation of the Hausdorff dimension of the harmonic measure of the boundary curves τ.

THEOREM 4.10 ([**64**]). *Let Δ_ν be the ν–Laplacian, that is a densely defined linear operator such that*

$$\mathcal{E}(f,f) = -\int\limits_S f\Delta_\nu f d\nu + \sum_{p\in\partial S} f(p)\partial_n f(p).$$

Suppose that $F \in C^2(\mathbb{R}^2)$ and $f = F(h_1,h_2) : S \to \mathbb{R}$.

Then

$$f \in Dom\Delta_\nu$$

and

$$\Delta_\nu f = Tr(ZD^2 f)$$

where

$$D^2 f(x) = \left\{\frac{\partial^2}{\partial h_i \partial h_j} F((h_1(x), h_2(x)))\right\}_{i,j=1}^2.$$

REMARK 4.11. Thus in harmonic coordinates the Laplacian Δ_ν is the second derivative along the tangent line (that is, in the direction of P_x). We conjecture that the ν–heat kernel has Gaussian asymptotics (compare Theorems 4.12 and 4.13).

THEOREM 4.12 ([**39**]). *The transition density $p_t^\nu(x,y)$ of the ν–diffusion on the Sierpiński gasket admits the following Gaussian type upper bound*

$$\int_A\int_B p_t^\nu(x,y)d\nu(x)d\nu(y) \leqslant (\nu(A)\nu(B))^{\frac{1}{2}} \exp\left\{-\frac{d_{\mathbb{R}^2}(\psi(A),\psi(B))^2}{4t}\right\}$$

for any two compact sets $A, B \subset S$.

THEOREM 4.13 ([**7, 39**]). *The transition density $p_t^\mu(x,y)$ of the μ–diffusion on the Sierpiński gasket has the following non Gaussian behavior*

$$p_t^\mu(x,y) \asymp C\ t^{-\beta} \exp\left\{-C\, d_S(x,y)^\gamma t^{-\alpha}\right\}$$

where $\alpha = \frac{\log 2}{\log 5 - \log 2}$, $\beta = \frac{\log 3}{\log 5}$ and $\gamma = \frac{\log 5}{\log 5 - \log 2}$.

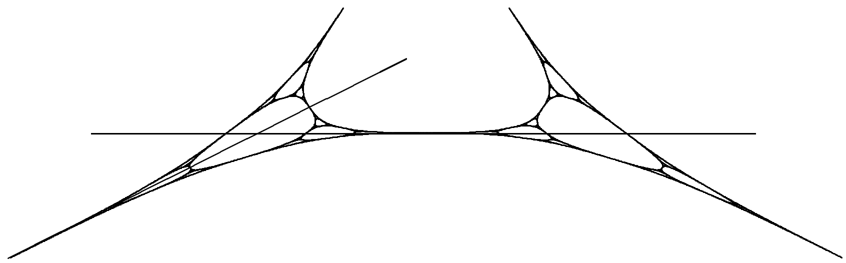

FIGURE 8. Two tangent lines to the Sierpiński gasket in harmonic coordinates.

The next proposition, which holds true for any electrical network ([**64**]), gives an idea of how to "see" the energy measure in harmonic coordinates. Specifically in our case, note that in harmonic coordinates the Sierpiński gasket is "generically one-dimensional" and so there is a well defined tangent line at a generic point, and also at every junction point (see Figure 8). If h corresponds to the direction of the tangent line at such a point, then locally in harmonic coordinates ν looks like the measure $\widetilde{\nu}_h$ defined below in Proposition 4.14.

PROPOSITION 4.14. *Let ν_h be the energy measure of a harmonic function h. Let $\widetilde{\nu}_h$ be the projection, in harmonic coordinates, of ν_h onto the direction of h. Then $\widetilde{\nu}_h$ is absolutely continuous with respect to Lebesgue measure. The density is constant between the projections of the boundary points. The jump of the density is the normal derivative of h at the boundary point.*

5. Non self–similar Dirichlet forms

In this section we discuss not necessarily self-similar Dirichlet forms on the Sierpiński gasket that give positive capacity to junction points, that is points in the union of all V_n. Any such Dirichlet form can be described as a limit of compatible discrete Dirichlet forms on the sequence V_n approximating the gasket. (We call Dirichlet forms compatible if they corresponds to compatible resistance networks on the sequence of graphs G_n, with set of vertices V_n, as in Definition 5.1.)

DEFINITION 5.1. Resistance networks on G_n and G_m, $n \leqslant m$, are compatible if the effective resistance between any $x, y \in V_n$ is the same in G_n and G_m.

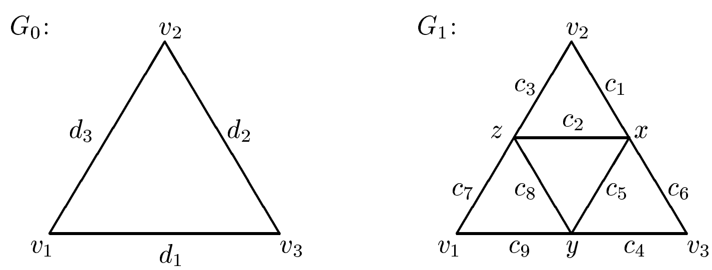

FIGURE 9. Notation for conductances on the initial and first level electrical networks approximating the Sierpiński gasket.

The highly nonlinear and multivariable compatibility conditions are difficult to analyze and so we present an alternative description which uses the three initial conductances d_1, d_2, and d_3 on G_0 (see Figure 9) and the space of harmonic functions.

THEOREM 5.2 ([**40**]). *A Dirichlet form on the Sierpiński gasket that gives positive capacity to junction points is uniquely determined by the initial conductances on G_0 and the space of harmonic functions.*

In particular, the space of Dirichlet forms can be parametrized by a subset of the space of 6–tuples $(d_1, d_2, d_3, h_1, h_2, h_3)$ where $d_j > 0$, and h_j are harmonic functions $S \to \mathbb{R}$ satisfying $h_j(v_k) = \delta_{jk}$ and $h_1 + h_2 + h_3 = 1$.

THEOREM 5.3 ([**40**]). *Every resistance network on G_n is the restriction of a regular Dirichlet form on S which is locally self–similar.*

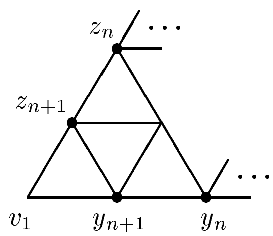

FIGURE 10. Sequences z_n and y_n in Definition 5.4.

DEFINITION 5.4. We say that a uniform local Harnack inequality holds at v_1 if there exists $\delta > 0$ such that if h is harmonic, $h(v_1) = 0$, $h(z_n) > 0$ and $h(y_n) > 0$ then

$$\delta < \frac{h(z_{n+1})}{h(y_{n+1})} < \frac{1}{\delta}$$

for all $n \geq 0$ (see Figure 10 for the notation).

THEOREM 5.5 ([**40**]). *If the uniform local Harnack inequality holds at v_1 then the ratio $\dfrac{d_3}{d_2}$ is uniquely determined by the space of harmonic functions.*

Indeed, if the uniform local Harnack inequality of Theorem 5.5 holds at all the three vertices v_1, v_2, and v_3, then the space of harmonic functions determines the initial conductances d_1, d_2, and d_3 uniquely up to a constant multiple. In geometric terms it means that in harmonic coordinates we have a cusp at each vertex of the Sierpiński gasket as in Figures 6 and 16. Otherwise, instead of a cusp there can be a non trivial angle.

A compatible sequence of resistance networks on the sequence of graphs G_n can be defined inductively. To study further the transition from $n = 0$ to $n = 1$ we introduce the following notation for the values of the three harmonic functions at the three first level junction points of V_1. It is convenient to write these nine values as a three by three matrix.

NOTATION 5.6.

$$H = \begin{pmatrix} h_1(x) & h_2(x) & h_3(x) \\ h_1(y) & h_2(y) & h_3(y) \\ h_1(z) & h_2(z) & h_3(z) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

LEMMA 5.7. *The following three determinants are positive*

$$\begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix}, \quad \begin{vmatrix} z_2 & y_2 \\ z_3 & y_3 \end{vmatrix}, \quad \begin{vmatrix} x_3 & z_3 \\ x_1 & z_1 \end{vmatrix}.$$

This lemma has an interesting geometric interpretation if we draw the Sierpiński gasket in harmonic coordinates, that is coordinates defined by the functions h_1 and h_2. Namely, the positivity of the three determinants of Lemma 5.7 is equivalent to a certain orientation (see Figure 11) of the triangles of G_1 in harmonic coordinates.

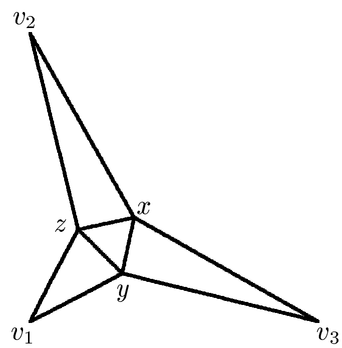

FIGURE 11. Orientation of the first level triangles approximating the Sierpiński gasket in harmonic coordinates.

The complete algebraic description of the compatibility condition in terms of the harmonic functions is given by the following theorem.

THEOREM 5.8 ([**40**]). *Given positive conductances $\{d_j\}$ and a matrix H with positive entries satisfying $H1 = 1$ there exists a compatible choice of positive conductances $\{c_j\}$ (see Figure 11) if and only if the following 13 determinants are all positive:*

$$\det H,$$

$$\begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix}, \quad \begin{vmatrix} z_2 & y_2 \\ z_3 & y_3 \end{vmatrix}, \quad \begin{vmatrix} x_3 & z_3 \\ x_1 & z_1 \end{vmatrix},$$

$$\begin{vmatrix} d_1 & z_3 \\ d_3 & z_1 \end{vmatrix}, \quad \begin{vmatrix} x_3 & d_1 \\ x_1 & d_3 \end{vmatrix}, \quad \begin{vmatrix} d_2 & x_1 \\ d_1 & x_2 \end{vmatrix}, \quad \begin{vmatrix} y_1 & d_2 \\ y_2 & d_1 \end{vmatrix}, \quad \begin{vmatrix} d_3 & y_2 \\ d_2 & y_3 \end{vmatrix}, \quad \begin{vmatrix} z_2 & d_3 \\ z_3 & d_2 \end{vmatrix},$$

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ d_1x_1 & d_2y_2 & d_3z_3 \end{vmatrix}, \quad \begin{vmatrix} x_1 & y_1 & z_1 \\ d_1x_1 & d_2y_2 & d_3z_3 \\ x_3 & y_3 & z_3 \end{vmatrix}, \quad \begin{vmatrix} d_1x_1 & d_2y_2 & d_3z_3 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

The algebraic description in Theorem 5.8 is not particularly useful because of the many complicated inequalities to be verified, and so we present an alternative geometric description (see Theorem 5.9 and Figure 12). In particular, it will be used for the residual set of the Apollonian packing in Example 5.16.

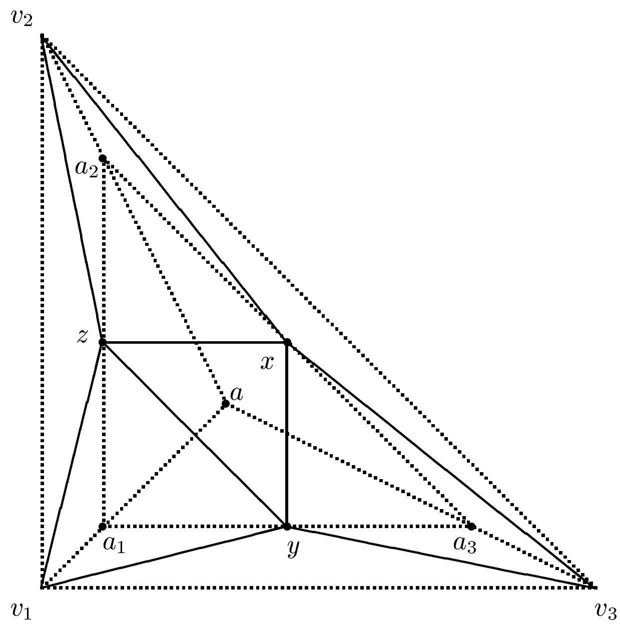

FIGURE 12. Geometric construction of the first level triangles approximating the Sierpiński gasket in harmonic coordinates.

THEOREM 5.9 ([**40**]). *In harmonic coordinates, let distinct points x, y, z correspond to a matrix H as described above. Then the conductances $c_1, \ldots, c_9$ are positive if and only if there are three points a_j inside of triangles $T(v_1, y, z)$, $T(v_2, x, z)$, $T(v_3, x, y)$ such that*

(1) $x \in [a_2, a_3]$, $y \in [a_1, a_3]$, $z \in [a_2, a_3]$

(2) three straight lines from v_j to a_j intersect in a single point a.

These geometric conditions also hold for the transition from G_n to G_{n+1} (in harmonic coordinates, for each small triangle in G_n separately). In addition, there is a **Monotonicity Lemma** that the triangles of level $n+1$ are "nested" inside of those of level n (see Figure 13).

As we have already mentioned, the compatibility conditions are difficult to analyze. The following algorithm allows to avoid this difficulty completely by introducing a new set of parameters which can be chosen from the open unit interval $(0, 1)$ independently of one another, and each choice corresponds to a different Dirichlet form. Thus, we show how to parametrize all the Dirichlet forms by infinite sequences of independent variables. The steps of Algorithm 5.10 are illustrated by Figures 14 and 15 below.

ALGORITHM 5.10. Given any positive conductances $\{d_j\}$ on G_0, for each choice of

$$(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \in (0, 1)^6$$

perform the following steps to obtain a distinct compatible Dirichlet form on G_1:

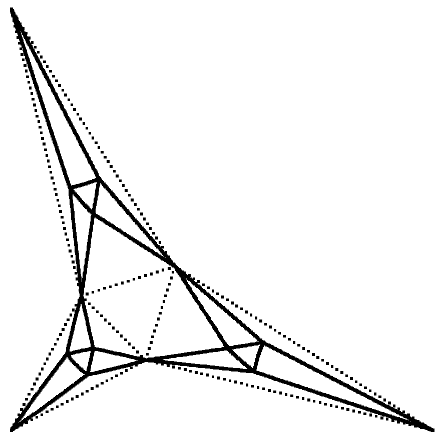

FIGURE 13. Illustration for the Monotonicity Lemma.

Step 1. Compute initial resistances $R_1, R_2, R_3 > 0$ by the $\Delta - Y$ transform of the conductances $\{d_j\}$:

$$R_j = \frac{d_j}{d_1 d_2 + d_2 d_3 + d_1 d_3}.$$

Step 2. Split the resistances R_j into $\alpha_j R_j$ and $(1-\alpha_j)R_j$.

Step 3. Do a $Y - \Delta$ transform of resistances $\{\alpha_j R_j\}$:

$$\widetilde{R}_j = \frac{\alpha_1\alpha_2 R_1 R_2 + \alpha_1\alpha_3 R_1 R_3 + \alpha_2\alpha_3 R_2 R_3}{\alpha_j R_j}.$$

Step 4. Split the resistances $\widetilde{R}_j$ into $\beta_j \widetilde{R}_j$ and $(1-\beta_j)\widetilde{R}_j$. Then for each j do a $Y - \Delta$ transform on the triple of resistances

$$(R_1^j, R_2^j, R_3^j) = \big((1-\alpha_j)R_j, (1-\beta_{j+1})\widetilde{R}_{j+1}, \beta_{j-1}\widetilde{R}_{j-1}\big).$$

THEOREM 5.11 ([**40**]). *The space of all Dirichlet forms on G_1 compatible with a fixed Dirichlet form on G_0 is a manifold of dimension 6 (the direct product $(0,1)^6$).*

More generally, the space of all Dirichlet forms on G_k compatible with a fixed Dirichlet form on G_0 is a manifold $(0,1)^{3(3^k-1)}$ of dimension $3(3^k-1)$.

Dirichlet forms on S giving junction points positive capacity can be parametrized by an infinite dimensional manifold $(0,1)^{\mathbb{N}}$.

To make this parameterization explicit we choose a six dimensional vector

$$(\alpha_1^w, \alpha_2^w, \alpha_3^w, \beta_1^w, \beta_2^w, \beta_3^w) \in (0,1)^6$$

for each finite word w of symbols 1,2,3 (each such word encodes a small triangle in the appropriate graph G_n). This allows, in particular, to consider random Dirichlet forms on the Sierpiński gasket. The continuity of harmonic functions is important because of Remark 5.14.

THEOREM 5.12 ([**40**]). *(1) Suppose that*

$$\xi_w = (\alpha_1^w, \alpha_2^w, \alpha_3^w, \beta_1^w, \beta_2^w, \beta_3^w) \in (0,1)^6$$

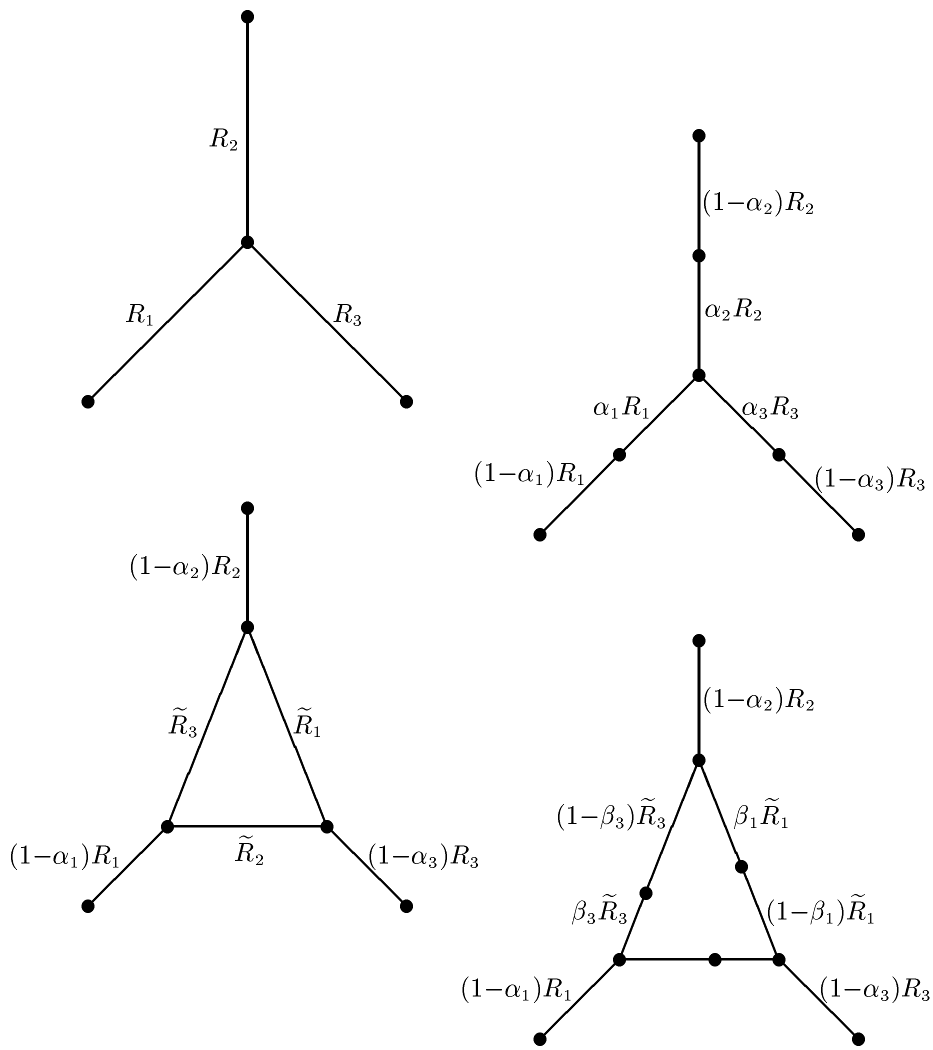

FIGURE 14. Building electrical networks in Steps 1–4 using Y-Δ and Δ-Y tranforms (see Algorithm 5.10 and Theorem 5.11).

are independent identically distributed random 6-dimensional vectors indexed by the words w of finite length. Then with probability one harmonic functions are continuous.
(2) Suppose that there is $\varepsilon > 0$ such that

$$\alpha_j^w, \beta_j^w \in [\varepsilon, 1-\varepsilon]$$

for all w, j. Then the harmonic functions are Hölder continuous with Hölder exponent $1-\varepsilon^2$.

DEFINITION 5.13. The **effective resistance** metric is

$$R(x,y) = \Big(\min_u \{ \mathcal{E}(u,u) \mid u(x) = 1,\ u(y) = 0\} \Big)^{-1}.$$

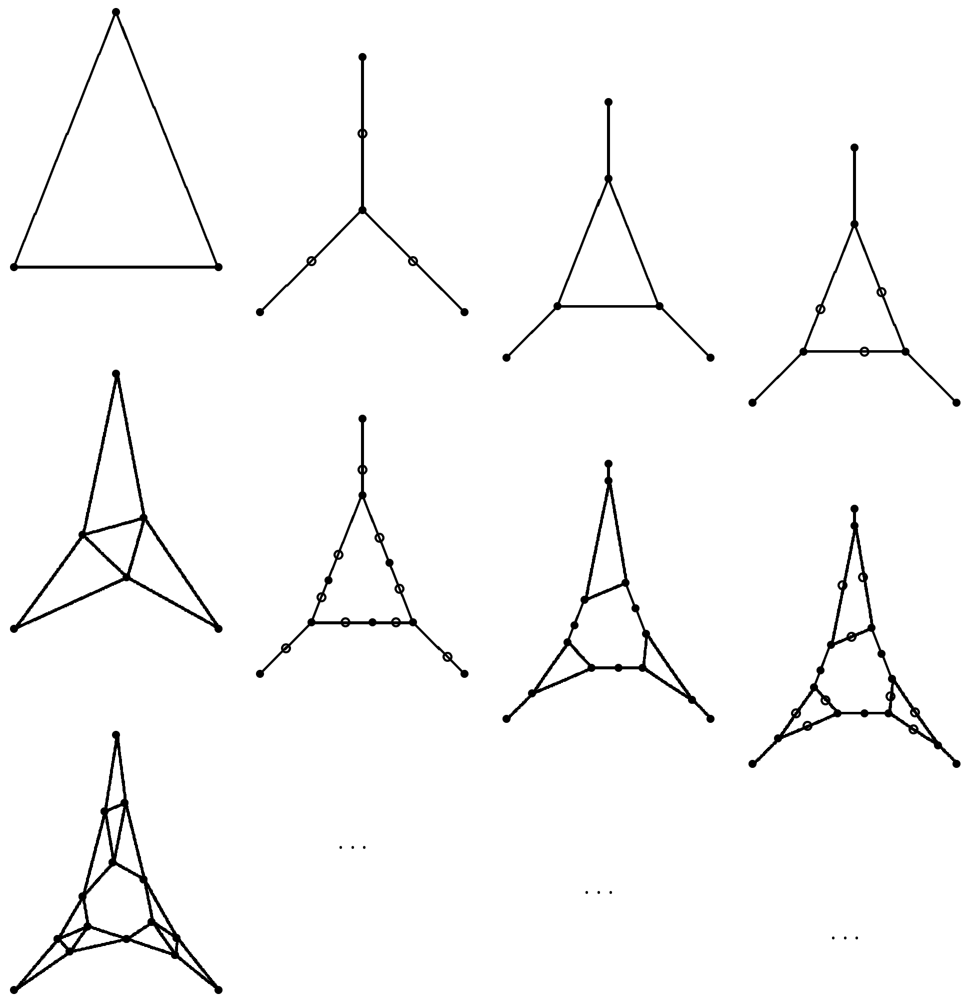

FIGURE 15. Approximation the Sierpiński gasket in harmonic coordinates using Y-Δ and Δ-Y transforms (see Algorithm 5.10 and Theorem 5.11).

It is known that if $\mathcal{E}(u,u) < \infty$ then u is R–continuous. Let Ω be the R–completion of $V_* = \bigcup_{n=0}^{\infty} V_n$.

REMARK 5.14. A general result by J. Kigami in [**21**] implies in our setting that any Dirichlet form we constructed is a local regular Dirichlet form on Ω with respect to the effective resistance metric. In a sense, Ω is the natural set where the Dirichlet form "lives". It also has been proved by J. Kigami (see [**21**]) that if harmonic functions are continuous then there is a natural continuous injective map $\theta : \Omega \to S$. Therefore in this case one can consider Ω as a subset of S.

An important question is whether Ω is equal to S. The answer is positive if all the conductances tend to infinity. This happens, for example, in the case of a regular self-similar harmonic structure. Thus it is natural to say that a harmonic structure is **regular** if $\Omega = S$ and non regular otherwise. However, there is a non regular harmonic structure such that harmonic functions are Hölder continuous (see [**40**]).

THEOREM 5.15 ([**21**]). *If $x \in \Omega$, then $\{x\}$ has positive capacity. Moreover, there exists Green's function $g(\cdot,\cdot)$ such that $g(x,x) < \infty$, $g(x,\cdot) \in Dom\mathcal{E}_0$ and $\mathcal{E}(g(x,\cdot),u) = u(x)$ for any $u \in Dom\mathcal{E}_0$.*

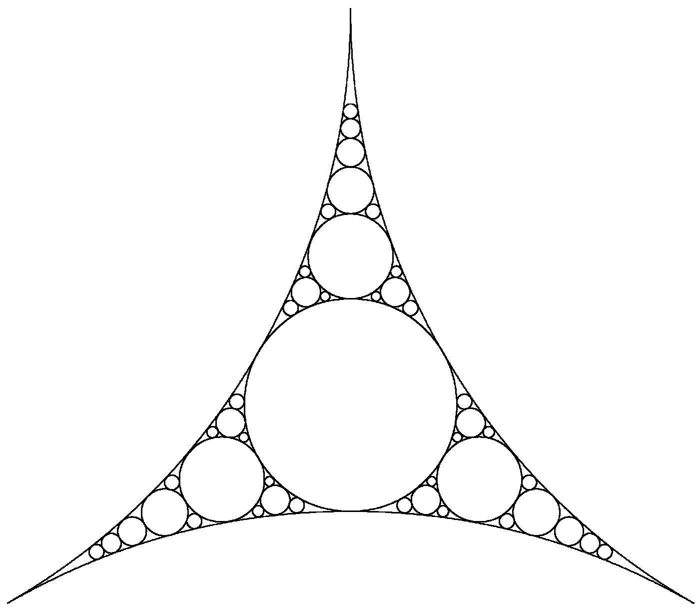

FIGURE 16. Residue set of the Apollonian packing.

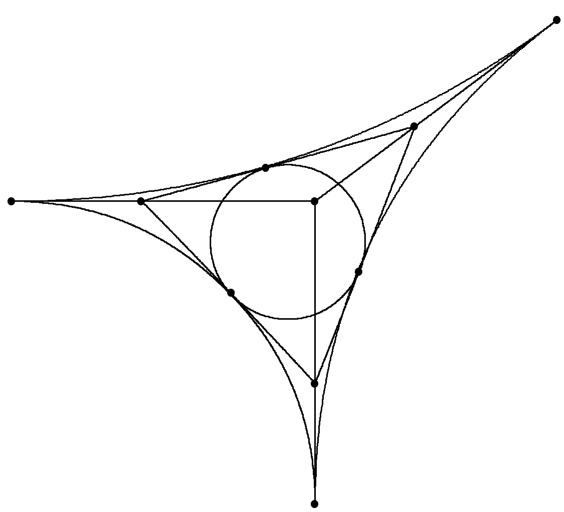

FIGURE 17. Geometric construction of Theorem 5.9 and Figure 12 for the Apollonian packing.

Example 5.16. The Apollonian packing has an ancient history and is extensively studied, both in the Mathematics and the Physics literature (see, for instance, [**10, 35, 33, 34, 51**] and references therein). In this example we will introduce one more point of view on this familiar construction.

It is an elementary geometrical fact that if three circles are tangent to one another, then the three tangent lines at the points where the circles meet intersect at a single point. If a fourth circle is tangent to each of these three circles, then all the common tangent lines shown on Figure 17 will satisfy the conditions of Theorem 5.9 (see Figure 12). Thus we have the following theorem.

Theorem 5.17. *There exists a Dirichlet form on the Sierpiński gasket giving junction points positive capacity such that the residue set of the Apollonian packing is the Sierpiński gasket in harmonic coordinates with respect to this form.*

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