INFINITE DIMENSIONAL I.F.S. AND SMOOTH FUNCTIONS ON THE SIERPIŃSKI GASKET

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ABSTRACT. We describe the infinitesimal geometric behavior of a large class of intrinsically smooth functions on the Sierpiński gasket in terms of the limit distribution of their local *eccentricity*, which is essentially the direction of the gradient. The distribution of eccentricities is codified as an infinite dimensional perturbation problem for a suitable iterated function system, which has the limit distribution as an invariant measure. Continuity properties of the gradient are used to define a class of *nearly harmonic* functions which are well approximated by harmonic functions. The gradient is also used to identify the part of the Sierpiński gasket where a smooth function is nearly harmonic locally. We prove that for nearly harmonic functions the limit distribution is the same as that for harmonic functions found by Öberg, Strichartz and Yingst. In particular, we prove convergence in the Wasserstein metric. We consider uniform as well as energy weights.

1. Introduction and notation

There is an extensive theory of analysis on fractals, see for example the books by Kigami [3] and Strichartz [9], and the survey article [7]. For the most part of the analytic theory (there is also a probabilistic theory) one is concerned with fractals which are not too complicated. In the present paper we consider the Sierpiński gasket, which is the example of two-dimensional fractal theory which is best understood from an analytic point of view.

In classical analysis the study of the local structure of smooth functions is fundamental and has many important consequences. For instance it gives rise to such a basic notion as the tangent space. In analysis on fractals the local stucture of smooth functions is not yet understood well enough to make

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it clear what conclusions can be drawn. In this paper we will address some questions concerning the local structure of smooth functions on the Sierpiński gasket. We actually show that they inherit a property of the local structure of harmonic functions, which could be seen as the analogues of linear functions on an interval, proven in [6].

The Sierpiński gasket K is the invariant set for the iterated function system (i.f.s.) in the plane given by

$$F_i x = \frac{1}{2}(x - q_i) + q_i \qquad i = 0, 1, 2,$$

where q_i are the vertices of an equilateral triangle. More specifically, K is the unique compact subset of $\mathbb{R}^2$ such that $K = F_0(K) \cup F_1(K) \cup F_2(K)$.

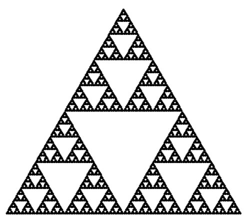

FIGURE 1. Sierpiński gasket.

One reason why the Sierpiński gasket is not 'too complicated', is that it is an example of a fractal which is post-critically finite (p.c.f.). In this particular case, the p.c.f. condition says that for the boundary $V_0 := \{q_0, q_1, q_2\}$ of K we have

$$F_i K \cap F_j K \subseteq F_i V_0 \cap F_j V_0,$$

for $i \neq j$. The general definition can be found in [3].

We regard K as the limit of graphs Γ_n with vertices V_n and edge relations $x \sim_n y$ defined inductively as follows. Let Γ_0 be the complete graph on $V_0 = \{q_0, q_1, q_2\}$. Then $V_n = \bigcup_i F_i V_{n-1}$ with $x \sim_n y$ if and only if there exists i such that $x = F_i x'$, $y = F_i y'$ and $x' \sim_{n-1} y'$. Note that $V_{n-1} \subseteq V_n$. We regard $V_0 = \partial K = \{q_0, q_1, q_2\}$ as the boundary of each of the graphs Γ_n, so that $V_n \setminus V_0$ consists of all non-boundary vertices in Γ_n. Note that every such vertex has exactly four neighbors in V_n. Points in $V_n \setminus V_0$ are called *junction points*.

We define W_n as the space of finite sequences, or words, $w = w_1 \cdots w_n$ of length $|w| = n$, $W_* = \bigcup_{n \geqslant 0} W_n$ as the space of finite words of all lengths, and Ω as the space of infinite sequences $\omega = w_1 w_2 \cdots$, $w_j \in W_1 = \{0, 1, 2\}$. For

$\omega = w_1 w_2 \cdots \in \Omega$, let $[\omega]_k = w_1 \cdots w_k \in W_k$ and likewise for $w \in W_*$ and $k < |w|$. We denote

$$F_w = F_{w_1} \circ \cdots \circ F_{w_n} \qquad \text{and} \qquad K_w = F_w(K).$$

For any function f on K and $w \in W_*$ we will use notation f_w for the function $f_w = f \circ F_w$ defined on K.

We will denote by m the standard self-similar measure on K defined by

$$m(K_w) = \frac{1}{3^{|w|}}.$$

Note that there is a natural continuous projection $\pi : \Omega \to K$ defined by

$$\pi(\omega) = \bigcap_{n \geqslant 0} K_{[\omega]_n}.$$

We will abuse notation and define a measure m on Ω as the pullback of the measure m on K under the projection map π, that is $m(\pi^{-1}(\cdot)) = m(\cdot)$. Then m is the product Bernoulli measure.

A continuous function h on K is said to be harmonic if for all n its restriction to V_n is graph-harmonic: its value at every nonboundary vertex $x \in V_n$ is equal to the average of its values at the four neighboring points in V_n,

$$h(x) = \frac{1}{4} \sum_{y \sim_n x} h(y). \tag{1.1}$$

We say that f is n-harmonic if all restrictions f_w, $w \in W_n$ are harmonic.

We will need the concept of energy for functions defined on K. Define graph energy forms

$$\mathcal{E}_n(u, v) = \left(\frac{5}{3}\right)^n \sum_{y \sim_n x} (u(x) - u(y))(v(x) - v(y)).$$

Then the sequence of graph energies $\mathcal{E}_n(u) = \mathcal{E}_n(u, u)$ is nondecreasing for every u and the harmonic functions are the only ones for which the sequence is constant. The *energy* of a continuous function u can thus be defined as

$$\mathcal{E}(u) = \lim_{n \to \infty} \mathcal{E}_n(u),$$

and we will say that $u \in \mathrm{Dom}\mathcal{E}$ if and only if u has finite energy. The energy form is defined on $\mathrm{Dom}\mathcal{E}$ through

$$\mathcal{E}(u, v) = \lim_{n \to \infty} \mathcal{E}_n(u, v).$$

Constant functions are the only ones with zero energy and $\mathrm{Dom}\mathcal{E}$ modulo constants is a Hilbert space with the energy form as inner product. Functions with finite energy are continuous and form a dense subspace of $C(K)$. To every function $f \in \mathrm{Dom}\mathcal{E}$ we associate its energy measure ν_f through

$$\nu_f(K_w) = \left(\frac{3}{5}\right)^{-|w|} \mathcal{E}(f_w),\ w \in W_*,$$

and, as with m, we denote also by ν_f the measure on Ω that is the pullback under π of ν_f.

There is an unbounded Laplacian denoted Δ for which the domain of definition, $\mathrm{Dom}\Delta$, is a dense subset of $C(K)$, and such that the harmonic functions are exactly those for which $\Delta f = 0$. The Laplacian Δf can be defined as a pointwise limit of difference operators $\Delta_n f|_{V_n}$ but also by means of a Green's operator G (see [2, 3, 9]). We will say that $\Delta f = u$ if f and u are continuous and

$$f = -Gu + Hf$$

where Hf is the unique harmonic function that coincides with f on the boundary and

$$Gu(x) = \int_K u(y)g(x,y)dm(y). \tag{1.2}$$

Here $g(x,y)$ is a Green's function, which is nonnegative, symmetric and $g(x,y) = 0$ if x or y is a boundary point. Since the Sierpiński gasket is a *regular* harmonic structure, $g(x,y)$ is continuous on $K \times K$ [2, Proposition 5.4]. The relation between the Laplacian and the energy form is given by the Gauss–Green's formula

$$\mathcal{E}(u,v) = -\int_K u\Delta v dm + \sum_{p\in V_0} u(p)dv(p),$$

where $dv(p)$ is a certain normal (Neumann) derivative of v at p (see [2, Proposition 7.3]). The Laplacian satisfies the following scaling identity

$$\Delta(f_w) = 5^{-|w|}(\Delta f)_w.$$

The functions we will consider in this paper are those for which Δf is Hölder continuous. We will call such functions *smooth.*

It is proved in [8] that any function in the domain of the Laplacian is Hölder continuous with Hölder exponent $\alpha = -\frac{\log\frac{3}{5}}{\log 2}$. Thus, the important eigenfunctions of Δ and multiharmonic functions, i.e. functions for which $\Delta^n f = 0$ for some n, are smooth.

A central notion in this paper is the concept of *eccentricity* of a function defined on the Sierpiński gasket

Definition 1. For a function f defined on the Sierpiński gasket K with boundary points q_0, q_1, q_2, ordered so that $f(q_0) \le f(q_1) \le f(q_2)$, we define the *eccentricity* $e(f)$ by

$$e(f) = \begin{cases} \frac{f(q_1)-f(q_0)}{f(q_2)-f(q_0)} & \text{provided } f(q_0) < f(q_2), \\ -1 & \text{if } f(q_0) = f(q_1) = f(q_2). \end{cases}$$

For every n the Sierpiński gasket is naturally decomposed into 3^n copies K_w, $w \in W_n$ of itself. Our objective is to study how eccentricities are distributed among the restrictions f_w, $w \in W_n$, of a smooth function f to these copies (cells), generalizing results obtained in [6] for harmonic functions.

Note that the eccentricity is invariant under the symmetries of the Sierpiński gasket, and also is invariant under any affine transformation $f \mapsto af + b$, $a \neq 0$. So we may assume, without loss of generality, that if f is not constant on the boundary then $f(q_0) = 0$, $f(q_1) = e$, $f(q_2) = 1$ and if f is constant on the boundary then $f(q_0) = f(q_1) = f(q_2) = 0$

The distribution of eccentricities of harmonic functions is governed by an i.f.s. $\{\psi_i\}_{i=0}^2$ acting on $(\{-1\} \cup [0,1])$ that produces the new eccentricities on each of the three smaller copies K_i, given an eccentricity on K for a harmonic function. The i.f.s. is derived from the *harmonic extension algorithm*:

$$h(x) = \frac{2}{5}h(y) + \frac{2}{5}h(z) + \frac{1}{5}h(v) \tag{1.3}$$

where $x \in V_n \setminus V_{n-1}$, where y and z are the two neighbors of x in V_n that belong to V_{n-1}, and v is the third vertex of the triangle in V_{n-1} that contains y and z.

The maps of the i.f.s. are computed by letting the maps ψ_i be defined as

$$\psi_i(e(h)) = e(h \circ F_i),$$

where $e(h)$ is the eccentricity on K for the harmonic function h. If h is constant on the boundary, then h is a constant function, thus $\psi_i(-1) = -1$ for $i = 0, 1, 2$. If h is not constant on the boundary, we let $h(q_0) = 0$, $h(q_1) = e$ and $h(q_2) = 1$. The harmonic extension algorithm gives the new values for the blow-up $h \circ F_0$:

$$e(h \circ F_0) = (2e+1)/(e+2) = \psi_0(e),$$

since $h(F_0(q_0)) = 0$, $h(F_0(q_1)) = (2e+1)/5$ and $h(F_0(q_2)) = (e+2)/5$. The other maps, ψ_1 and ψ_2 are calculated analogously and one obtains the full iterated function system for $x \in [0,1]$:

$$\begin{cases} \psi_0(x) = \dfrac{2x+1}{x+2}, \\ \psi_1(x) = \begin{cases} \dfrac{1-3x}{2-3x}, & \text{if} \quad 0 \le x \le \dfrac{1}{3} \\ 3x-1, & \text{if} \quad \dfrac{1}{3} \le x \le \dfrac{2}{3} \\ \dfrac{1}{3x-1}, & \text{if} \quad \dfrac{2}{3} \le x \le 1, \end{cases} \\ \psi_2(x) = \dfrac{x}{3-x}. \end{cases} \tag{1.4}$$

Since the only harmonic functions for which any restriction h_w is constant on V_0 actually are the constant functions, the arbitrary definition of eccentricity for functions constant on V_0 does not give any extra information in the harmonic case. However, when working in the larger class of smooth functions it may happen that some f_w are constant on V_0 even though f is not constant. To describe the distribution of eccentricities for our larger class it is therefore necessary to define the eccentricity of functions constant on V_0.

In [6] the i.f.s. $\{\psi_i\}, i = 0, 1, 2$ acting on $[0, 1]$ were studied. It was shown that, with respect to *uniform weights* $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$, there exists a unique probability measure μ_0 which is a weak limit, as $m \to \infty$, of the distribution of the eccentricities at level m, the discrete measure $3^{-m} \sum_{|w|=m} \delta(\psi_w(e))$. This limit distribution μ does not depend on the nonconstant harmonic function, that is, the starting point of the iterations.

The case when each map in the i.f.s. is given the same weight as the restriction of the function to the corresponding subcell contributes to the energy of the whole function was also considered in [6]. Let h be the harmonic function with boundary values $h(q_0) = 0$, $h(q_1) = e$ and $h(q_2) = 1$. These *energy weights* will be

$$p_i(e) = \frac{5\mathcal{E}(h_i)}{3\mathcal{E}(h)},$$

which equals

$$\begin{cases} p_0(e) = \frac{1}{5} \frac{e^2+e+1}{e^2-e+1}, \\ p_1(e) = \frac{1}{5} \frac{3e^2-3e+1}{e^2-e+1}, \\ p_2(e) = \frac{1}{5} \frac{e^2-3e+3}{e^2-e+1}. \end{cases} \tag{1.5}$$

The same type of convergence result as for uniform weights holds in the energy case. There exists a unique probability measure $\mu_{\mathcal{E}}$, different from μ_0, that is the weak limit of the discrete measures $\sum_{|w|=m} p_w(e)\delta(\psi_w(e))$. Here $p_w(e) = \prod_{i=1}^m p_{w_i}(\psi_{w_{i-1}\dots w_1}(e))$.

In Section 2 we show that for a certain class of *nearly harmonic functions*, eccentricities are in $[0,1]$ on all scales. Using the gradient defined in [10], we identify the part of the Sierpiński gasket where a smooth function is nearly harmonic locally.

In Section 3 we define an i.f.s. $\{\Psi_i\}_{i=0}^2$ that governs the distribution of eccentricities of smooth functions. This i.f.s. will be a perturbed version of the original i.f.s. (1.4), and it will act on an infinite dimensional space, since the space of smooth functions is not finite dimensional. We prove convergence of the perturbed i.f.s. to the same measures μ_0 resp. $\mu_{\mathcal{E}}$, as in [6] with uniform weights (Theorem 4) and energy weights (Theorem 5) respectively. But with uniform weights we have the restriction that the starting point must correspond to a nearly harmonic function. This restriction is not necessary in the energy case since the subset of the Sierpiński gasket where a smooth function is nearly harmonic locally has full energy measure.

The same measures μ_0 and $\mu_{\mathcal{E}}$ occurs as limit distribution of eccentricities, because the perturbation of the original i.f.s. collapses fast enough on smaller scales. This could be interpreted that every function with Hölder continuous Laplacian in the limit satisfies the $\frac{1}{5} - \frac{2}{5}$ extension algorithm.

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2. Gradient and local eccentricities

2.1. **Nearly harmonic functions.** In this section we define a class of functions for which the local eccentricities are in $[0,1]$ on all levels. These are functions for which most of the energy comes from the harmonic part, i.e, the harmonic function with the same boundary values. We rely to a great extent on the theory of gradients developed in [10], in particular on Theorem 3 of that paper.

Let $\|f\|_\alpha$ be the Hölder norm with Hölder exponent α (with respect to the Euclidean norm in $\mathbb{R}^2$) of a function f on K. This norm is equivalent to an intrinsic norm

$$\|f\|_\rho = \|f\|_\infty + \sup_{n\geq 0} \sup_{w\in W_n} \sup_{x,y\in K_w} \rho^{-n}|f(x)-f(y)| \tag{2.1}$$

where $\alpha = -\frac{\log \rho}{\log 2}$. We will be using this intrinsic norm on the space H^α of Hölder continuous functions on K in the rest of the paper.

Following the notation in [10] we equip the space of harmonic functions $\mathcal{H}$ with the norm $\|h\|^2_{\mathcal{H}} = \mathcal{E}(h,h) + (\sum_{x\in V_0} h(x))^2$. Let $\tilde{\mathcal{H}}$ be the orthogonal complement to constant functions and $\tilde{P}$ the orthogonal projection from $\mathcal{H}$ onto $\tilde{\mathcal{H}}$. On $\tilde{\mathcal{H}}$, as well as on Dom$\mathcal{E}$ modulo constants, we will use the norm $\|f\|^2 = \mathcal{E}(f,f)$.

If $\{h_1, h_2\}$ is an orthonormal basis of $\tilde{\mathcal{H}}$ then the *Kusuoka measure*, $\nu = \nu_{h_1} + \nu_{h_1}$, is independent of the choice of orthonormal basis. The Kusuoka measure is nonatomic and ν_f is absolutely continuous with respect to ν for any $f \in$ Dom$\mathcal{E}$, see [1, 5, 10]. Again, we denote by ν its pullback on Ω under π.

For $i = 0,1,2$ let the linear map $M_i : \mathcal{H} \to \mathcal{H}$ be defined by $M_i h = h \circ F_i$ and define $\tilde{M}_i : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$, by $\tilde{M}_i = \tilde{P} M_i \tilde{P}^*$. The Sierpiński gasket is a *nondegenerate* harmonic structure, i.e., the restriction of any non-constant harmonic function to any K_w, $w \in W_*$ is non-constant, since the matrices $\tilde{M}_i$, $i = 0,1,2$ are invertible. For any continuous f we denote by Hf the unique harmonic function that coincides with f on V_0 and let $\tilde{H} = \tilde{P}H$. In [10] $\mathrm{Grad}_w f$ for $w \in W_n$ is defined as

$$\mathrm{Grad}_w f = \tilde{M}_w^{-1} \tilde{H}(f_w),$$

where $\tilde{M}_w = \tilde{M}_{w_n} \dots \tilde{M}_{w_1}$ and

$$\mathrm{Grad}_\omega f = \lim_{n\to\infty} \mathrm{Grad}_{[\omega]_n} f,$$

for $\omega \in \Omega$ whenever the limit exists.

Hölder continuity of Δf gives the following estimate of $\|\mathrm{Grad}_\omega f\|$, which is a refinement of Theorem 3 in [10].

Theorem 1 ([10] Theorem 3)**.** *Suppose that Δf is Hölder continuous on the Sierpiński gasket, that is $|\Delta f(x) - \Delta f(y)| \leqslant c\rho^n$ if $x, y \in K_w$, $w \in W_n$. Then* $\mathrm{Grad}_\omega f$ *is defined for every $\omega \in \Omega$ and*

$$\|\mathrm{Grad}_\omega f - \tilde{H} f\| \leqslant 8 \left(\frac{c}{1-\rho} + \|\Delta f(x)\|_\infty \right). \tag{2.2}$$

The estimate (2.2) *also holds for* $\mathrm{Grad}_w f$*, $w \in W_*$.*

The map $\omega \mapsto$ $\mathrm{Grad}_\omega f$ *is continuous at ω in the standard topology of Ω if ω is not constant after a finite segment or $\Delta f(\pi(\omega)) = 0$.*

For the convenience of the reader we mention that the proof in [10] consists of writing $\mathrm{Grad}_w f$ as a telescoping sum of terms $\mathrm{Grad}_{[w]_{n+1}} f - \mathrm{Grad}_{[w]_n} f$, and carefully estimating these terms using the Green's formula (1.2), and properties of the matrix $\tilde{M}_{[w]_n}^{-1}$. The constant 8 in (2.2) is not explicitly found in [10], however it follows from the argument there by inserting elementary estimates of the terms h_a and h_s into the proof. The estimates we use are

$$\|h_a\| \leq \frac{3\sqrt{6}}{5} c\rho^n$$

and

$$\|h_s\| \leq \frac{1}{\sqrt{2}} \|\Delta f\|_\infty.$$

Remark 1. If x is a point in K, then the definition of the gradient of f at x is more delicate. If $x \in K$ is not a junction point, then there is a unique $\omega \in \Omega$ such that $x = \pi(\omega)$. Then one can see that the map $x = \pi(\omega) \mapsto \mathrm{Grad}_\omega f$ is well defined, and is continuous at x in the topology of K.

However, $x = \pi(\omega)$ is a boundary point if and only if ω is constant, and $x = \pi(\omega)$ is a junction point if and only if ω is constant after a finite segment. If $x \in K$ is a junction point, then there are two different $\omega_1, \omega_2 \in \Omega$ such that $x = \pi(\omega_1) = \pi(\omega_2)$. Then there can be two different gradients, $\mathrm{Grad}_{\omega_1} f$ and $\mathrm{Grad}_{\omega_2} f$, of f at x. It is easy to construct examples of such a situation, for example, every localized eigenfunction of the Laplacian has points with this property.

Remark 2. In [10] there was an obvious typo that $-\tilde{H}f$ was omitted in (2.2)

The following theorem gives a criterion to have all local eccentricities in $[0, 1]$, i.e. for the function to have a non-constant harmonic part on all cells. It will also be a key for uniqueness of the distribution of eccentricities of such functions.

Theorem 2. *There exists a real number $\epsilon_0 > 0$, such that if f is smooth and*

$$\frac{\|\Delta f\|_\rho}{\|f\|} < \epsilon_0, \tag{2.3}$$

then $f|_{V_0}$ is not constant and

$$\|Hf_w\| \geqslant \frac{1}{2} \cdot \frac{\|f\|}{\|\tilde{M}_w^{-1}\|} > 0, \tag{2.4}$$

for any finite word w.

Proof. We write $f = Hf - Gu$, where $u = \Delta f$. If $Hf = 0$ then since

$$\begin{aligned}\mathcal{E}(Gu) &= -\int_K Gu\cdot\Delta Gudm = \int_K Gu\cdot udm\\ &= \int_{K\times K} g(x,y)u(x)u(y)d(m\times m)(x,y)\\ &\le \|g\|_\infty\|u\|_\infty^2,\end{aligned}$$

we have

$$\frac{\|\Delta f\|_\rho}{\|f\|} \ge \frac{1}{\sqrt{\|g\|_\infty}},$$

and $f|_{V_0}$ is not constant for appropriate ϵ_o in (2.3). In the sense of the energy norm, f is a slightly perturbed harmonic function, since $\mathcal{E}(f)=\mathcal{E}(Hf)+\mathcal{E}(Gu)$ and $\mathcal{E}(Gu)\le\|g\|_\infty\epsilon_0^2\mathcal{E}(f)$ implies

$$\|Hf\| \ge \sqrt{(1-\|g\|_\infty\epsilon_0^2)}\|f\|. \tag{2.5}$$

Then for $\epsilon_0>0$ small enough we have

$$\|Hf_w\| = \|\tilde{H}f_w\| = \left\|\tilde{H}(Hf)_w + \tilde{H}(Gu)_w\right\| = \left\|\tilde{M}_w\tilde{P}Hf + \tilde{M}_w\tilde{M}_w^{-1}\tilde{H}(Gu)_w\right\|$$

$$= \left\|\tilde{M}_w\left(\tilde{P}Hf + \tilde{M}_w^{-1}\tilde{H}(Gu)_w\right)\right\|\frac{\|\tilde{M}_w^{-1}\|}{\|\tilde{M}_w^{-1}\|} \ge \frac{1}{\|\tilde{M}_w^{-1}\|}\left\|\tilde{P}Hf + \tilde{M}_w^{-1}\tilde{H}(Gu)_w\right\|$$

$$= \frac{1}{\|\tilde{M}_w^{-1}\|}\left\|\tilde{P}Hf + \mathrm{Grad}_w Gu\right\| \ge \frac{\|Hf\| - \|\mathrm{Grad}_w Gu\|}{\|\tilde{M}_w^{-1}\|} \ge \frac{1}{2}\cdot\frac{\|f\|}{\|\tilde{M}_w^{-1}\|}.$$

The last inequality follows from Theorem 1. □

Definition 2. A smooth function f defined on K is *nearly harmonic* if f satisfies (2.3) with ϵ_0 small enough that the conclusions of Theorem 2 hold.

The term nearly harmonic stems from inequality (2.5). Note that if h is a nonconstant harmonic function and u is any Hölder continuous function on K with $\|u\|_\rho = 1$, then $h + tGu$ is nearly harmonic whenever $0\le|t|\le\epsilon_0\|h\|$.

Proposition 3. *If* $\rho \le 1-\frac{3}{20}\sqrt{\frac{3}{2}} \approx 0.816288\ldots$ *then* ϵ_0 *is independent of* ρ, *and can be put to* $\epsilon_0 = 0.06$

Proof. To give a numerical value of ϵ_0 it is necessary to estimate the supremum norm of the Green's function g, which is defined by (see [2] and [3]),

$$g(x,y) = \sum_{w \in W_* \cup \varnothing} r_w \Psi_w(x,y), \tag{2.6}$$

where

$$\Psi_w(x,y) = \begin{cases} \Psi((F_w)^{-1}(x)), (F_w)^{-1}(y)) & \text{if } x,y \in K_w \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Psi(x,y) = \sum_{p,q \in V_1 \setminus V_0} X_{p,q} \psi_p(x) \psi_q(y). \tag{2.7}$$

Since the functions ψ_p are 1-harmonic the maximum of Ψ will be obtained for x and y in V_1, which gives $\|\Psi\|_\infty = \frac{9}{50}$.

For any pair of points x and y, it is clear that $\Psi_w(x,y)$ can be non-zero for more than one $w \in W_k$, only if x and y lie in V_k, but for such points $\Psi_w(x,y) = 0$. Thus, for every k, there can only be at most one non-zero term $\Psi_w(x,y)$, $w \in W_k$, and

$$\|g\|_\infty \leq \sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k \|\Psi\|_\infty = \frac{9}{20}.$$

From the proof of Theorem 1 it follows that if $\rho \leq 1 - \frac{3}{20}\sqrt{\frac{3}{2}}$ the sum of the asymmetric parts are bounded by $8c$ and thus the right hand side of (2.2) can be replaced by $8\|\Delta f\|_\rho$. In the last step of the proof of Theorem 2 we choose ϵ_0 small enough that

$$\|Hf\| - \|\mathrm{Grad}_w Gu\| \geq \left(\sqrt{(1 - \|g\|_\infty \epsilon_0^2)} - 8\epsilon_0\right) \|f\| \geq \frac{1}{2}\|f\|.$$

which holds for $\epsilon_0 = 0.06$. This value is also small enough to assure that $f|_{V_0}$ is not constant. □

Remark 3. In [4] it is conjectured that $\|g\|_\infty = 178839/902500$.

Remark 4. Note that in the important case $\rho = \frac{3}{5}$, which includes all functions whose Laplacian is itself in DomΔ, the hypothesis of Proposition 3 is satisfied.

Remark 5. The value $\frac{1}{2}$ in (2.5) is of course arbitrarily chosen from $(0,1)$. Replacing it with a number close to 0, it is possible to obtain a value of ϵ_0 arbitrarily close to $\frac{1}{\sqrt{64+\|g\|_\infty}}$ in Proposition 3. Also note that we can change $\frac{1}{2}$ to a factor $b(\rho) \in (0,1)$ depending on ρ to have Proposition 3 valid for more values of ρ. For $\rho < 1 - \frac{3\sqrt{6}\epsilon_0}{5\sqrt{1-\|g\|_\infty \epsilon_o^2}}$ it is possible to choose $b(\rho)$ so that Proposition 3 is valid but it seems impossible to have a value ϵ_0 valid for all ρ.

2.2. **Eccentricities of restrictions of smooth functions.** In this section we show that the value of eccentricities of restrictions of smooth functions depend on whether the gradient vanishes or not. In particular we prove that restrictions of smooth functions are nearly harmonic on small enough cells where the gradient does not vanish.

Proposition 4. *Suppose f is a smooth function. Let O be the subset of Ω where $\mathrm{Grad}_\omega f \neq 0$. Then for any $\epsilon > 0$ there exists an open set $O_\epsilon \subseteq O$ with the following property. For any $\omega \in O_\epsilon$ there is n such that*

$$\frac{\|\Delta f_{[\omega]_m}\|_\rho}{\|f_{[\omega]_m}\|} < \epsilon \tag{2.8}$$

for all $m \geqslant n$. Moreover, $O \backslash O_\epsilon$ consists only of sequences which are constant after a finite segment. In particular, $O \backslash O_\epsilon$ is at most countable.

Proof. We have,

$$\|\Delta f_w\|_\rho \leqslant 5^{-|w|} \|\Delta f\|_\rho \tag{2.9}$$

and

$$\|f_{[\omega]_m}\| \geqslant \|\tilde{H} f_{[\omega]_m}\| = \|\tilde{M}_{[\omega]_m} \mathrm{Grad}_{[\omega]_m} f\| \geqslant \frac{1}{\|\tilde{M}^{-1}_{[\omega]_m}\|} \|\mathrm{Grad}_{[\omega]_m} f\|. \tag{2.10}$$

Suppose $\mathrm{Grad}_{\omega_0} f \neq 0$ and ω_0 is not constant after a finite segment. Then $\liminf_{m\to\infty} \|\mathrm{Grad}_{[\omega]_m} f\| > 0$ uniformly in a neighborhood of ω_0, we even have for some n that $\|\mathrm{Grad}_{[\omega_0]_n w} f\| \geq c_{\omega_0} \|\mathrm{Grad}_{\omega_0} f\|$ for every $w \in W_*$. In addition, $\|\tilde{M}_j^{-1}\| = 5$ and $\lim_{n\to\infty} 5^{-n} \|\tilde{M}^{-1}_{[\omega_0]_n}\| = 0$ by the estimate in Theorem 2 in [10] (see also Lemma 8 below). We thus have

$$\frac{\|\Delta f_{[\omega_0]_n w}\|_\rho}{\|f_{[\omega_0]_n w}\|} \leq \frac{5^{-n} 5^{-|w|} \|\tilde{M}^{-1}_{[\omega_0]_n w}\| \|\Delta f\|_\rho}{\|\mathrm{Grad}_{[\omega_0]_n w} f\|} \leq \frac{5^{-n} \|\tilde{M}^{-1}_{[\omega_0]_n}\| \|\Delta f\|_\rho}{c_{\omega_0} \|\mathrm{Grad}_{\omega_0} f\|},$$

for every $w \in W_*$. This completes the proof.

□

Corollary 5. *Suppose f is a non-constant smooth function. Then for any $\epsilon > 0$ there exists $W'_\epsilon \subseteq W_*$ such that*

$$\frac{\|\Delta f_w\|_\rho}{\|f_w\|} < \epsilon \tag{2.11}$$

for all w that can be written as $w = w'w_*$ where $w' \in W'_\epsilon$ and $w_* \in W_*$. Moreover, if O is the subset of Ω where $\mathrm{Grad}_\omega f \neq 0$, then $\pi(O)\backslash(\bigcup_{w\in W'_\epsilon} K_w)$ consists only of boundary and junction points. In particular, this set is at most countable.

Proof. As W'_ϵ take the set of all $[\omega]_n$ with $\omega \in O_\epsilon$ not constant after a finite segment, where n is the least possible value for which (2.8) holds. Then apply the projection π to the objects in the previous corollary. □

This corollary tells us that any restriction f_w, $w \in W'_{\epsilon_0}$ is nearly harmonic. We want to show that f is constant on cells whose intersection with $\cup_{w\in W'_{\epsilon_0}} K_w$ is at most finite. This does not follow directly from Theorem 1 since the set $\pi(O)\backslash(\bigcup_{w\in W'_\epsilon} K_w)$ might intersect such cells. We will need the following result.

Proposition 6. *Suppose f is a smooth function and that*

$$\nu\left(\{\omega \in \Omega \mid \mathrm{Grad}_\omega f = 0\}\right) = 1,$$

where ν is the Kusuoka measure. Then f is constant.

Proof. We prove that $\mathcal{E}(f) = 0$. Let f_n be the n-harmonic function that coincides with f on V_n. Then $\mathcal{E}(f) = \lim_n \mathcal{E}(f_n)$. Let

$$g_n = \sum_{w\in W_n} < \mathrm{Grad}_w f, Z_n(w)\mathrm{Grad}_w f > 1_{K_w}$$

where 1_{K_w} denotes the characteristic function of K_w and

$$Z_n(w) = \frac{\tilde{M}_w^* \tilde{M}_w}{Tr \tilde{M}_w^* \tilde{M}_w}$$

It is noted in [10, section 4] that

$$\mathcal{E}(f_n) = \int_K g_n d\nu.$$

Theorem 1 implies g_n is uniformly bounded and $\mathrm{Grad}_\omega f = 0$ for ν a.e. ω gives $\lim_{n\to\infty} g_n(x) = 0$ for ν a.e. x. Dominated convergence completes the proof. □

Remark 6. If the set

$$K_z \cap \left(\cup_{w\in W'_{\epsilon_0}} K_w\right), \; z \in W_*$$

is finite or empty, then f_z is constant, since by Corollary 5, $\mathrm{Grad}_\omega f_z \neq 0$ for at most a countable number of ω, and ν has no atoms, so Proposition 6 applies. The converse is trivially true.

For smooth functions, depending on where in K a point x lies, restrictions to small enough neighborhoods of x will exhibit one of three possible behaviors. Either they will be constant, nearly harmonic or exhibit what we will call exceptional behavior.

Theorem 3. *Let f be a smooth function on K. Then there are sets K_f^H, K_f^C and K_f^E such that*

$$K = K_f^H \cup K_f^C \cup K_f^E$$

where pairwise intersections between the sets in the union are at most countable and such that f is nearly harmonic locally on K_f^H, in the sense that the restriction to any cell contained in K_f^H is nearly harmonic. Also f is constant locally on K_f^C in the same sense. The set K_f^E is closed and nowhere dense.

Proof. The different parts of K can be constructed as follows. Partition W_n into

$$W_{n,f}^H = \{w \mid [w]_k \in W'_{\epsilon_0} \text{ for some } k \leq n\}$$
$$W_{n,f}^C = \{w \mid f|_{K_w} = \text{const}\}$$

and $W_{n,f}^E$ what is left. Then define three sequences of subsets of K

$$K_{n,f}^H = \cup_{w \in W_{n,f}^H} K_w,\ K_{n,f}^C = \cup_{w \in W_{n,f}^C} K_w, \text{ and } K_{n,f}^E = \cup_{w \in W_{n,f}^E} K_w,$$

with the property that

$$K = K_{n,f}^H \cup K_{n,f}^C \cup K_{n,f}^E.$$

Note that $K_{n,f}^H$ and $K_{n,f}^C$ are increasing and $K_{n,f}^E$ decreasing. Define

$$K_f^H = \cup_{n \geq 1} K_{n,f}^H,\ K_f^C = \cup_{n \geq 1} K_{n,f}^C \text{ and } K_f^E = \cap_{n \geq 1} K_{n,f}^E.$$

Then $K = K_f^H \cup K_f^C \cup K_f^E$ with pairwise intersections at most countable, f is nearly harmonic (constant) locally in K_f^H (K_f^C) and the closed set K_f^E has empty interior (Remark 6). □

On the exceptional set K_f^E we can not say anything about the local behavior of f. If $x = \pi(\omega) \in K_f^E$ is not a junction or boundary point then $\text{Grad}_\omega f = 0$ but we don't have $\text{Grad}_{[\omega]_n} f = 0$ for n big enough. Thus the eccentricity of $f_{[\omega]_n}$ might very well jump between -1 and $[0, 1]$.

This partition shows that in the case of uniform weights we can not hope for convergence of the perturbed i.f.s. for arbitrary starting points since possibly $m(K_f^E) > 0$. But in the energy case this is true because of the following fact.

Proposition 7. *If f is a function with Hölder continuous Laplacian then $\nu_f(K_f^C) = \nu_f(K_f^E) = 0$.*

Proof. It is trivial that $\nu_f(K_f^C)=0$, so we can suppose that f is not constant on any subcell of K.

Let f_n be the n-harmonic function that coincides with f on V_n. From [10, section 4], we know that

$$\nu_{f_n}(K^E_{m,f}) = \sum_{w\in W_m^E}\sum_{w'\in W_{n-m}} < \mathrm{Grad}_{ww'}f, Z_n(ww')\mathrm{Grad}_{ww'}f > \nu(K_{ww'}),$$

for $n\geq m$.

Then, because $K^E_{m,f}$ is a finite union of cells, we have

$$\nu_f(K^E_{m,f}) = \lim_{n\to\infty}\nu_{f_n}(K^E_{m,f})$$

$$= \lim_{n\to\infty}\sum_{w\in W_m^E}\sum_{w'\in W_{n-m}} < \mathrm{Grad}_{ww'}f, Z_n(ww')\mathrm{Grad}_{ww'}f > \nu(K_{ww'})$$

$$= \int_{\pi^{-1}(K^E_{m,f})\setminus\pi^{-1}(K^E_f)} < \mathrm{Grad}_\omega f, Z(\omega)\mathrm{Grad}_\omega f > d\nu(\omega),$$

where we have used that $\mathrm{Grad}_\omega f=0$, ν a.e. on $\pi^{-1}(K_f^E)$. Since Hölder continuity of Δf implies that $< \mathrm{Grad}_\omega f, Z(\omega)\mathrm{Grad}_\omega f >$ is uniformly bounded, we see that

$$\nu_f(K_f^E) = \lim_{m\to\infty}\nu_f(K^E_{m,f}) = 0.$$

□

3. Distribution of eccentricities

3.1. **Perturbation of the iterated function system.** To study the limit distribution of eccentricities of smooth functions it is necessary to extend the original i.f.s. on $\{-1\}\cup[0,1]$ describing the harmonic case to $(\{-1\}\cup[0,1])\times H^\alpha$, where H^α is the space of Hölder continuous functions on K. For this purpose we make the following identification, the notation for which will be used throughout this section. Let $(e,u)\in(\{-1\}\cup[0,1])\times H^\alpha$ correspond to a function with Hölder continuous Laplacian through the following identification. If $e\in[0,1]$ let $f=h-Gu$ where h is the unique harmonic function such that $h(q_0)=0$, $h(q_1)=e$ and $h(q_2)=1$, and if $e=-1$ let $f=-Gu$. After composition with a symmetry of the Sierpiński gasket and an affine transformation any function with Hölder continuous Laplacian is of this form so it is sufficient to study such functions.

The i.f.s. that describes the distribution of eccentricities on this larger class of functions is of course the same as the original i.f.s. on $(\{-1\}\cup[0,1])\times\{0\}$

but for non-zero second coordinate the maps are perturbed to

$$\Psi_j(e,u) = \begin{cases} \left(e(f_j), \frac{u_j'}{5(\max_{V_0} f_j - \min_{V_0} f_j)}\right) & \text{if } f_j|_{V_0} \text{ is not constant} \\ \\ \left(-1, \frac{u_j}{5}\right) & \text{if } f_j|_{V_0} \text{ is constant} \end{cases} \tag{3.1}$$

with $u_j' = u_j \circ R$ where R is a symmetry of K such that $f_j' = f_j \circ R$ has the property that $\max_{V_0} f_j'$ is achieved at the vertex q_2 of K and $\min_{V_0} f_j'$ is achieved at the vertex q_0 of K. Thus, in the above identification $\Psi_j(e,u)$ corresponds to f_j if $f_j|_{V_0}$ is constant and to $\frac{f_j'}{(\max_{V_0} f_j - \min_{V_0} f_j)}$ if $f_j|_{V_0}$ is not constant.

In the case of energy weights there will also be new weights $p_i(e,u)$ that depend on the second coordinate.

For ease of notation we will let $\Psi_w = \Psi_{w_1'} \circ \cdots \circ \Psi_{w_n'}$ where $w \mapsto w'$ is the permutation of W_n such that

$$\Psi_{w'}(e,u) = \begin{cases} \left(e(f_w), \frac{u_w'}{5^n(\max_{V_0} f_w - \min_{V_0} f_w)}\right) & \text{if } f_w|_{V_0} \text{ is not constant} \\ \\ \left(-1, \frac{u_w}{5^n}\right) & \text{if } f_w|_{V_0} \text{ is constant.} \end{cases}$$

Since we will only be interested in estimating the norm of the second coordinate we will skip the prime notation.

Lemma 8. *The second component in the perturbed i.f.s.* Ψ_w *tends to* 0 *for every orbit* $\omega \in \Omega$ *that is not constant after a finite segment, from any starting point* $(e,u) \in (\{-1\} \cup [0,1]) \times H^\alpha$ *corresponding to a function* f *such that* $\mathrm{Grad}_\omega f \neq 0$.

Proof. We know from Corollary 5 that $[\omega]_m = w$ for some $w \in W_{\epsilon_0}'$. Then according to Theorem 2

$$(\max_{V_0} f_{[\omega]_n} - \min_{V_0} f_{[\omega]_n})^2 \geqslant \frac{1}{3}\mathcal{E}(H f_{[\omega]_n}) \geqslant \mathrm{Const}\frac{\mathcal{E}(f_{[\omega]_m})}{\|\tilde{M}^{-1}_{[\sigma^m(\omega)]_{n-m}}\|^2}.$$

With the estimate

$$\|\tilde{M}^{-1}_{[\omega]_n}\| \leq 5^n \beta^{C(\omega,n)}, \tag{3.2}$$

where $\beta < 1$ and $C(\omega,n)$ is the number of changes in $[\omega]_n$, from the proof of Theorem 2 in [10], it follows that for any $\omega \in \Omega$ we have

$$5^n(\max_{V_0} f_{[\omega]_n} - \min_{V_0} f_{[\omega]_n}) \geqslant \mathrm{Const}\frac{\|f_{[\omega]_m}\|}{\beta^{C(\sigma^m(\omega),n-m)}} \to \infty. \tag{3.3}$$

We conclude that the second term of the iterates

$$\frac{u_{[\omega]_n}}{5^n(\max_{V_0} f_{[\omega]_n} - \min_{V_0} f_{[\omega]_n})} \to 0$$

in Hölder norm. □

Remark 7. Note that if (e, u) corresponds to a nearly harmonic function f then Lemma 8 is true without any assumption on $\text{Grad}_\omega f$. However, for nearly harmonic functions $\text{Grad}_\omega f \neq 0$ for every $\omega \in \Omega$ anyway, because of (2.2).

3.2. **Limit distribution with uniform weights.** It was shown in [6] that the i.f.s. (1.4) on $[0, 1]$ with uniform weights has a unique invariant measure μ_0 in the sense that

$$\mu_0 = \sum_{j=1}^{3} \frac{1}{3} \mu_0 \circ \psi_j^{-1}. \tag{3.4}$$

Our extension of this i.f.s. to $\{-1\} \cup [0, 1]$ trivially gives rise to some new invariant measures that satisfy (3.4), namely

$$\mu_t = t\delta_{-1} + (1 - t)\mu_0, \; t \in [0, 1].$$

Since $\Psi_j(x, 0) = (\psi_j(x), 0)$, it is obvious that $\mu_t \times \delta_0$ are invariant measures of the perturbed i.f.s. (3.1) in the sense that

$$\mu_t \times \delta_0 = \sum_{j=1}^{3} \frac{1}{3} (\mu_t \times \delta_0) \circ \Psi_j^{-1}.$$

We define the action of an operator A on a probability measure λ on $(\{-1\} \cup [0, 1]) \times H^\alpha$ by

$$A\lambda(B) = \sum_{j=1}^{3} \frac{1}{3} \lambda(\Psi_j^{-1}(B)) = \int_{(\{-1\} \cup [0,1]) \times H^\alpha} P((e, u), B) d\lambda(e, u),$$

where B is any Borel subset of $(\{-1\} \cup [0, 1]) \times H^\alpha$ and

$$P((e, u), B) = \sum_{j=1}^{3} \frac{1}{3} \delta_{\Psi_j(e,u)}(B) = m(\omega \mid \psi_{[\omega]_1}(e, u) \in B)$$

is the probability, with respect to uniform weights, of ending up in B when starting from (e, u). Then the invariant measures $\mu_t \times \delta_0$ are exactly the fixed points of A.

To state our main result we need the following definition.

Definition 9. The *Wasserstein metric* for probability measures μ and ν on a measurable set X is defined as

$$d_W(\mu,\nu) = \sup_{\|f\|_{Lip}\le 1} \left| \int_X f d\mu - \int_X f d\nu \right|.$$

In [6] it was proven that $A^n\delta_e \to \mu_0$ in the Wasserstein metric, regardless of the starting point e. Next, we prove that the limit distribution of eccentricities for nearly harmonic functions is the same as for harmonic functions.

Theorem 4. *For any $(e,u) \in (\{-1\}\cup[0,1])\times H^\alpha$ corresponding to a nearly harmonic function f,*

$$A^n\delta_{(e,u)} \to \mu_0 \times \delta_0$$

in the Wasserstein metric.

Theorem 4 does not follow immediately from Lemma 8. That Lemma only tells us that if $A^n\delta_{(e,u)}$ converges in the Wasserstein metric it must converge to a measure with support in $(\{-1\}\cup[0,1])\times\{0\}$. However, to prove Theorem 4 it is necessary to show that the perturbation of the original i.f.s. is, in some sense, continuous in the second coordinate; if a function is close enough to harmonic, eccentricities distribute almost like in the harmonic case.

Lemma 10. *Suppose $f|_{V_0}$ and $f_i|_{V_0}$, $i = 0,1,2$ are not constant. If $\|u\|_\infty \le \frac{1}{20\|g\|_\infty}$ then*

$$|e(f_i) - \psi_i(e)| \le \text{Const}\|u\|_\infty \quad i = 0,1,2. \tag{3.5}$$

Proof. Let $V_1 \setminus V_0 = \{p_0,p_1,p_2\}$ where $p_0 = F_1(q_2)$, $p_1 = F_2(q_0)$, $p_2 = F_0(q_1)$ and $f = Hf - Gu$ with $f(q_0) = 0 \le f(q_1) = e \le f(q_2) = 1$. The harmonic extension algorithm (1.1) gives that $Hf(p_0) = \frac{2}{5} + \frac{2e}{5}$, $Hf(p_1) = \frac{2}{5} + \frac{e}{5}$, and $Hf(p_2) = \frac{1}{5} + \frac{2e}{5}$.

Under the hypothesis of the lemma it is clear from (1.2) that $\|Gu\|_\infty \le \frac{1}{20}$ and this is enough to control in what point of $F_i(V_0)$ either $\max_{V_0} f_i$ or $\min_{V_0} f_i$ will occur. In the case $i = 0$ it is clear that $\min_{V_0} f_0 = f(q_0) = 0$ and independently of e we have

$$e(f_1) = \min\left(\frac{\frac{2}{5} + \frac{e}{5} + Gu(p_1)}{\frac{1}{5} + \frac{2e}{5} + Gu(p_2)}, \frac{\frac{1}{5} + \frac{2e}{5} + Gu(p_2)}{\frac{2}{5} + \frac{e}{5} + Gu(p_1)} \right).$$

Define

$$\text{ecc}_0 : I \times [-\frac{1}{20}, \frac{1}{20}] \times [-\frac{1}{20}, \frac{1}{20}] \to R$$

$$(e,x,y) \mapsto \min\left(\frac{\frac{2}{5}+\frac{e}{5}+x}{\frac{1}{5}+\frac{2e}{5}+y}, \frac{\frac{1}{5}+\frac{2e}{5}+y}{\frac{2}{5}+\frac{e}{5}+x}\right).$$

Note that ecc_0 is Lipschitz continuous and that $ecc_0(e,0,0) = \psi_0(e)$ and $ecc_0(e, Gu(p_1), Gu(p_2)) = e(f_0)$, hence

$$|e(f_0) - \psi_0(e)| \leq \mathrm{Const}\|(e, Gu(p_1), Gu(p_2)) - (e,0,0)\|$$
$$\leq \mathrm{Const}\max(|Gu(p_1)|, |Gu(p_2)|) \leq \mathrm{Const}\|Gu\|_\infty \leq \mathrm{Const}\|u\|_\infty.$$

For $i = 2$ we know that $\max_{V_0} f_2 = f(q_2) = 1$ so

$$e(f_2) = \min\left(\frac{\left|\frac{e}{5} + Gu(p_0) - Gu(p_1)\right|}{\frac{3}{5} - \frac{e}{5} - Gu(p_1)}, \frac{\left|\frac{e}{5} + Gu(p_0) - Gu(p_1)\right|}{\frac{3}{5} - \frac{2e}{5} - Gu(p_0)}\right)$$

and a similar proof as for $i = 0$ can be done with

$$ecc_2(e,x,y) = \min\left(\frac{\left|\frac{e}{5} + x - y\right|}{\frac{3}{5} - \frac{e}{5} - y}, \frac{\left|\frac{e}{5} + x - y\right|}{\frac{3}{5} - \frac{2e}{5} - x}\right).$$

The case $i = 2$ is a mixture of the two previous cases and is treated similarly.

□

Proof of Theorem 4. We must estimate

$$d_W(A^N\delta_{(e,u)}, \mu_0 \times \delta_0)$$
$$= \sup_{\|h\|_{\mathrm{Lip}}\leq 1}\left|\frac{1}{3^N}\sum_{w\in W_N} h(\Psi_w(e,u)) - \int h(x,0)d\mu_0(x)\right|$$
$$= \sup_{\|h\|_{\mathrm{Lip}}\leq 1}\left|\frac{1}{3^N}\sum_{w\in W_N} h(e(f_w), \frac{u_w}{5^N(\max_{V_0} f_w - \min_{V_0} f_w)}) - \int h(x,0)d\mu(x)\right|.$$

For this it is necessary to first iterate a certain number of steps so that the norm of the second coordinate is small enough on most subcells of K, and then use the result obtained in [6, Thm 5.6] together with Lemma 10 on those subcells.

Inequality (3.3) from the proof of Lemma 8 tells us that

$$\left\|\frac{u_w}{5^{|w|}(\max_{V_0} f_w - \min_{V_0} f_w)}\right\|_\infty \leq \mathrm{Const}\frac{\|u\|_\infty \beta^{C(w,n)}}{\|f\|} \tag{3.6}$$

for any $w \in W_n$. In the rest of the proof we will always assume that M is big enough that

$$\mathrm{Const}\frac{\|u\|_\infty\beta^M}{\|f\|} < \frac{1}{20\|g\|_\infty},$$

so that Lemma 10 applies whenever $C(w,n) \geqslant M$.

Let m, m' and M be such that $m + m' = N$ and $M \leq m$. Then for any $\|h\|_{\mathrm{Lip}} \leq 1$ we have

$$\left| \frac{1}{3^N} \sum_{w \in W_N} h(e(f_w), \frac{u_w}{5^{|w|}(\max_{V_0} f_w - \min_{V_0} f_w)}) - \int h(x,0) d\mu_0(x) \right|$$

$$\leq \frac{1}{3^N} \sum_{w \in W_N, C(w,m) < M} \left| h(e(f_w), \frac{u_w}{5^{|w|}(\max_{V_0} f_w - \min_{V_0} f_w)}) - \int h(x,0) d\mu_0(x) \right|$$

$$+ \frac{1}{3^N} \sum_{w \in W_N, C(w,m) \geq M} \left| h(e(f_w), \frac{u_w}{5^{|w|}(\max_{V_0} f_w - \min_{V_0} f_w)}) - h(e(f_w),0) \right|$$

$$+ \frac{1}{3^m} \sum_{w_0 \in W_m, C(w_0,m) \geqslant M} \frac{1}{3^{m'}} \sum_{z \in W_{m'}} |h(e(f_{w_0 z}),0) - h(\psi_z(e(f_{w_0})),0)|$$

$$+ \frac{1}{3^m} \sum_{w_0 \in W_m, C(w_0,m) \geqslant M} \left| \frac{1}{3^{m'}} \sum_{z \in W_{m'}} h(\psi_z(e(f_{w_0}),0) - \int h(x,0) d\mu_0(x) \right|.$$

The last term in the previous inequality is in [6, Thm 5.6] shown to be bounded by $\mathrm{Const} a^{m'}$, with $a < 1$. To estimate the third term, let $B = \max_{i=1,2,3} \|\psi_i\|_{\mathrm{Lip}}$ and use that if $C(w_0,m) \geqslant M$ we obtain by using (3.2) and Lemma 10 that for every $z \in W_{m'}$

$$\begin{aligned} &|e(f_{w_0 z}) - \psi_z(e(f_{w_0}))| \\ &\leq |e(f_{w_0 z}) - \psi_{z_{m'}}(e(f_{w_0 z_1 \ldots z_{m'-1}}))| \\ &+ |\psi_{z_{m'}}(e(f_{w_0 z_1 \ldots z_{m'-1}})) - \psi_{z_{m'} z_{m'-1}}(e(f_{w_0 z_1 \ldots z_{m'-2}}))| \\ &+ \ldots \\ &+ |\psi_{z_{m'} \ldots z_2}(e(f_{w_0 z_1})) - \psi_z(e(f_{w_0}))| \\ &\leq \mathrm{Const} \sum_{k=0}^{m'-1} B^k \frac{\|u\|_\infty \beta^M}{\|f\|}. \end{aligned}$$

We can conclude that

$$\left| \frac{1}{3^N} \sum_{w \in W_N} h(e(f_w), \frac{u_w}{5^{|w|}(\max_{V_0} f_w - \min_{V_0} f_w)}) - \int h(x,0) d\mu_0(x) \right|$$

$$\leq 2m[C(\omega,m)<M]+\frac{1}{3^N}\sum_{w\in W_N, C(w,m)\geqslant M}\left\|\frac{u_w}{5^{|w|}(\max_{V_0} f_w-\min_{V_0} f_w)}\right\|$$

$$+\mathrm{Const}\frac{B^{m'}-1}{B-1}\beta^M+\mathrm{Const}a^{m'}$$

$$\leq 2m[C(\omega,m)<M]+\beta^M(\mathrm{Const}+\mathrm{Const}\frac{B^{m'}-1}{B-1})+\mathrm{Const}a^{m'}$$

where $a<1$, $\beta<1$ and $B>1$. This completes the proof.

□

With Theorem 4 we know that eccentricities of smooth functions locally has the same limit distribution of eccentricities as harmonic functions in the set K_f^H. In particular Theorem 4 remains true if $\mathrm{Grad}_\omega f\neq 0$ for every $\omega\in\Omega$. Without control on the behavior of the perturbed i.f.s. on $\pi^{-1}(K_f^E)$ it is not possible to have convergence for arbitrary starting points. But in case K_f^E is negligible we still have convergence.

Corollary 11. *If $(e,u)\in(\{-1\}\cup[0,1])\times H^\alpha$ corresponds to a function f for which $m(K_f^E)=0$. Then*

$$A^n\delta_{(e,u)}\to\mu_t\times\delta_0,$$

with $t=m(K_f^C)$, in the Wasserstein metric.

Proof. Partition W_n into $W_{n,f}^H$, $W_{n,f}^C$ and $W_{n,f}^E$ as in section 2. We estimate

$$d_W(A^N\delta_{(e,u)},\mu_t\times\delta_0)$$

$$=\sup_{\|h\|_{\mathrm{Lip}}\leq 1}\left|\frac{1}{3^N}\sum_{w\in W_N}h(\Psi_w(e,u))-(1-t)\int h(x,0)d\mu_0(x)-th(-1,0)\right|$$

$$\leq\sup_{\|h\|_{\mathrm{Lip}}\leq 1}\left|\frac{1}{3^N}\sum_{w\in W_{N,f}^C}h(\Psi_w(e,u))-th(-1,0)\right|$$

$$+\sup_{\|h\|_{\mathrm{Lip}}\leq 1}\left|\frac{1}{3^N}\sum_{w\in W_{N,f}^H}h(\Psi_w(e,u))-(1-t)\int h(x,0)d\mu_0(x)\right|$$

$$+\sup_{\|h\|_{\mathrm{Lip}}\leq 1}\left|\frac{1}{3^N}\sum_{w\in W_{N,f}^E}h(\Psi_w(e,u))\right|.$$

Since $\Psi_w(e,u)=(-1,0)$ if $w\in W_{N,f}^C$, we have

$$\sup_{\|h\|_{\mathrm{Lip}}\le 1}\left|\frac{1}{3^N}\sum_{w\in W^C_{N,f}} h(\Psi_w(e,u)) - th(-1,0)\right| \le \left|m\left(K^C_{N,f}\right)-t\right| \to 0$$

and for the last term note that

$$\sup_{\|h\|_{\mathrm{Lip}}\le 1}\left|\frac{1}{3^N}\sum_{w\in W^E_{N,f}} h(\Psi_w(e,u))\right| \le m(K^E_{N,f}) \to 0.$$

Given $\epsilon > 0$, take n such that $m(K^H_f \setminus K^H_{n,f}) < \epsilon$. Then for any $N \ge n$ we can estimate the mid term by,

$$\sup_{\|h\|_{\mathrm{Lip}}\le 1}\left|\frac{1}{3^N}\sum_{w\in W^H_{N,f}} h(\Psi_w(e,u)) - (1-t)\int h(x,0)d\mu_0(x)\right|$$

$$\le \sup_{\|h\|_{\mathrm{Lip}}\le 1}\left|\frac{1}{3^N}\sum_{w\in W^H_{N,f}} h(\Psi_w(e,u)) - m(K^H_{n,f})\int h(x,0)d\mu_0(x)\right|$$

$$+ \sup_{\|h\|_{\mathrm{Lip}}\le 1}\left|\left(m(K^H_{n,f})-(1-t)\right)\int h(x,0)d\mu_0\right|$$

$$\le \sum_{w'\in W'_{\epsilon_0}\ |w'|\le n}\frac{1}{3^{|w'|}}\sup_{\|h\|_{\mathrm{Lip}}\le 1}\left|\frac{1}{3^{N-|w'|}}\sum_{w\in W_{N-|w'|}} h(\Psi_{w'w}(e,u)) - \int h(x,0)d\mu_0(x)\right|$$

$$+2\epsilon.$$

and for each $w' \in W'_{\epsilon_0}$ with $|w'| \le n$ this supremum goes to 0 by Theorem 4. □

3.3. **Limit distribution with energy weights.** Energy weights are naturally expressed as normalized energy measure of the subcells of level one. If f is the function corresponding to the point $(e,u) \ne (-1,0)$, then $p_i(e,u) = \bar\nu_f(K_i)$ where

$$\bar\nu_f = \frac{\nu_f}{\mathcal{E}(f)}.$$

Cells on which a function is constant do not matter since they give no contribution to the energy. Thus we can arbitrarily define $p_i(-1,0) = \frac{1}{3}$.

It was shown in [6] that there is a unique invariant measure $\mu_{\mathcal{E}}$ to the i.f.s. (1.4) on $[0,1]$ with energy weights $p_i(e) = p_i(e,0)$ satisfying

$$\mu_{\mathcal{E}} = \sum_{j=1}^{3} p_i(e)\mu_{\mathcal{E}} \circ \psi_j^{-1}. \tag{3.7}$$

The extension of the original i.f.s. with energy weights to $(\{-1\} \cup [0,1])$ will then have invariant measures

$$\mu_{\mathcal{E},t} = t\delta_{-1} + (1-t)\mu_{\mathcal{E}},$$

and clearly $\mu_{\mathcal{E},t} \times \delta_0$ are invariant measures to the perturbed i.f.s. (3.1) with energy weights, and in fact there are no others.

Proposition 12. *$\mu_{\mathcal{E},t} \times \delta_0$ are the only invariant measure for the perturbed i.f.s.* (3.1) *with energy weights.*

Proof. The result follows from Lemma 8 and Proposition 7. Suppose λ is an invariant measure. Then λ is a fixed point of the operator

$$A_{\mathcal{E}}\lambda(B) = \sum_{j=1}^{3} \int_{\Psi_j^{-1}(B)} p_j(e,u)d\lambda(e,u) = \int_{(\{-1\}\cup[0,1])\times H^\alpha} P_{\mathcal{E}}[(e,u),B]d\lambda(e,u)$$

acting on the probability measures on $(\{-1\}\cup[0,1])\times H^\alpha$. Here $P_{\mathcal{E}}[(e,u),B] = \bar{\nu}_f(\omega \mid \Psi_{[\omega]_1}(e,u) \in B)$ is the probability, with respect to energy weights, of ending up in the Borel set B starting from (e,u). So

$$\lambda(B) = A_{\mathcal{E}}^n\lambda(B) = \int P_{\mathcal{E}}^{(n)}((e,u),B)d\lambda(e,u), \tag{3.8}$$

where

$$P_{\mathcal{E}}^{(n)}((e,u),B) = \bar{\nu}_f(\omega \mid \Psi_{[\omega]_n}(e,u) \in B).$$

Let $B = (\{-1\}\cup[0,1])\times B_m$ in equality (3.8), with $B_m = \{u \in H^\alpha \mid \|u\|_\rho > \frac{1}{m}\}$. The second coordinate of $\Psi_{[\omega]_n}(e,u)$ tends to zero in Hölder norm for every $\omega \in O_\epsilon$ that is not constant after a finite segment. This is a set of full $\bar{\nu}_f$ measure thus $P_{\mathcal{E}}^{(n)}((e,u),(\{-1\}\cup[0,1])\times B_m) \to 0$ for every (e,u).

Dominated convergence gives that $\lambda((\{-1\}\cup[0,1])\times B_m) = 0$ and thus $\lambda((\{-1\}\cup[0,1])\times\{0\}^c) = 0$ and the support of λ must be included in $(\{-1\}\cup[0,1])\times\{0\}$. The only possibilities are $\lambda = \mu_{\mathcal{E},t}\times\delta_0$. □

With energy weights we have a nicer convergence result than for uniform weights since we have convergence, to the same measure, no matter what starting point. With respect to energy, the limit distribution of eccentricities is the same for all non-constant smooth functions. This is a consequence of the fact that the set K_f^H where a non-constant smooth function f is nearly harmonic locally has full energy measure.

Theorem 5. *For any* $(e,u) \in (\{-1\} \cup [0,1]) \times H^\alpha$*, with* $(e,u) \neq (-1,0)$

$$A_{\mathcal{E}}^n \delta_{(e,u)} \to \mu_{\mathcal{E}} \times \delta_0$$

in the Wasserstein metric.

Proof. The proof follows the same path as the proofs of Theorem 4 and Corollary 11, only some more attention to the weights has to be paid. In view of Proposition 7, one can mimic the proof of Corollary 11 to see that it is enough to consider starting points (e,u) corresponding to a nearly harmonic function f.

We must show

$$d_W(A_{\mathcal{E}}^N \delta_{(e,u)}, \mu_{\mathcal{E}} \times \delta_0)$$

$$= \sup_{\|h\|_{\mathrm{Lip}} \leq 1} \left| \sum_{w \in W_n} \bar{\nu}_f(K_w) h(\Psi_w(e,u)) - \int h(x,0) d\mu_{\mathcal{E}}(x) \right|$$

$$= \sup_{\|h\|_{\mathrm{Lip}} \leq 1} \left| \sum_{w \in W_n} \bar{\nu}_f(K_w) h\left(\left(e(f_w), \frac{u_w}{5^{|w|}(\max_{V_0} f_w - \min_{V_0} f_w)}\right)\right) - \int h(x,0) d\mu_{\mathcal{E}}(x) \right| \to 0.$$

As in the proof of Theorem 4 we will always assume that M is big enough that

$$\mathrm{Const} \frac{\|u\|_\infty \beta^M}{\|f\|} < \frac{1}{20\|g\|_\infty},$$

so that Lemma 10 applies whenever $C(w,n) \geqslant M$.

Let m, m' and M be such that $m + m' = n$ and $M \leq m$. Then for any $\|h\|_{\mathrm{Lip}} \leq 1$ we have

$$\left| \sum_{w \in W_n} \bar{\nu}_f(K_w) h(e(f_w), \frac{u_w}{5^{|w|}(\max f_w - \min f_w)}) - \int h(x,0) d\mu_{\mathcal{E}}(x) \right| \tag{3.9}$$

$$\leq \sum_{w \in W_n, C(w,m) < M} \bar{\nu}_f(K_w) \left| h(e(f_w), \frac{u_w}{5^{|w|}(\max f_w - \min f_w)}) - \int h(x,0) d\mu_{\mathcal{E}}(x) \right|$$

$$+ \sum_{w \in W_n, C(w,m) \geq M} \bar{\nu}_f(K_w) \left| h(e(f_w), \frac{u_w}{5^{|w|}(\max f_w - \min f_w)}) - h(e(f_w), 0) \right|$$

$$+ \sum_{w_0 \in W_m, C(w_0,m) \geqslant M} \bar{\nu}_f(K_{w_0}) \sum_{z \in W_{m'}} \bar{\nu}_{f_{w_0}}(K_z) \left| h(e(f_{w_0 z}), 0) - h(\psi_z(e(f_{w_0})), 0) \right|$$

$$+ \sum_{w_0 \in W_m, C(w_0,m) \geqslant M} \bar{\nu}_f(K_{w_0}) \sum_{z \in W_{m'}} |\bar{\nu}_{f_{w_0}}(K_z) - \bar{\nu}_{Hf_{w_0}}(K_z)| |h(\psi_z(e(f_{w_0})), 0)|$$

$$+ \sum_{w_0 \in W_m, C(w_0,m) \geqslant M} \bar{\nu}_f(K_{w_0}) \left| \sum_{z \in W_{m'}} \bar{\nu}_{Hf_{w_0}}(K_z) h(\psi_z(e(f_{w_0}), 0) - \int h(x, 0) d\mu_{\mathcal{E}}(x) \right|.$$

The three first terms can be handled as in the proof of Theorem 4. The last term in the previous inequality is in [6, Thm 5.9] shown to be bounded by $\mathrm{Const} a^{m'}$, with $a < 1$. So what is new in this proof is the fourth term.

To estimate it note that

$$\bar{\nu}_{f_{w_0}}(K_z) = \prod_{j=1}^{m'} \bar{\nu}_{f_{w_0 z_1 \ldots z_{j-1}}}(K_{z_j})$$

and

$$\bar{\nu}_{Hf_{w_0}}(K_z) = \prod_{j=1}^{m'} \bar{\nu}_{(Hf_{w_0})_{z_1 \ldots z_{j-1}}}(K_{z_j})$$

so using the fact that all terms in the product are bounded by 1

$$|\bar{\nu}_{f_{w_0}}(K_z) - \bar{\nu}_{Hf_{w_0}}(K_z)| \leq \sum_{j=1}^{m'} |\bar{\nu}_{f_{w_0 z_1 \ldots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{(Hf_{w_0})_{z_1 \ldots z_{j-1}}}(K_{z_j})|, \tag{3.10}$$

and each term can be estimated by

$$\begin{aligned} &|\bar{\nu}_{f_{w_0 z_1 \ldots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{(Hf_{w_0})_{z_1 \ldots z_{j-1}}}(K_{z_j})| \\ &\leq |\bar{\nu}_{f_{w_0 z_1 \ldots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{Hf_{w_0 z_1 \ldots z_{j-1}}}(K_{z_j})| \\ &+ |\bar{\nu}_{Hf_{w_0 z_1 \ldots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{(Hf_{w_0})_{z_1 \ldots z_{j-1}}}(K_{z_j})|. \end{aligned} \tag{3.11}$$

To bound the first term of (3.11) we show that if $f = Hf - Gu$ with $\max_{V_0} f = 1$ and $\min_{V_0} f = 0$, then for $\|u\|_\infty$ small enough

$$|\bar{\nu}_f(K_i) - \bar{\nu}_{Hf}(K_i)| \leq \mathrm{Const} \|u\|_\infty. \tag{3.12}$$

Since the difference in the first term does not change if we rescale $f_{w_0 z_1 \ldots z_{j-1}}$ as in the i.f.s. (3.1) and that u for this function is bounded by (3.6) inequality (3.12) will hold for large enough M.

Note the estimates $\mathcal{E}(Gu) \leq \text{Const}\|u\|_\infty^2$ and $\mathcal{E}((Gu)_i) \leq \text{Const}\|u\|_\infty^2$ that follows by the same reasoning as in the proof of Theorem 2 and $\mathcal{E}((Hf)_i, (Gu)_i) = \mathcal{E}_0((Hf)_i, (Gu)_i) \leq \text{Const}\|u\|_\infty$, where the equality holds since Hf is harmonic. Thus (3.12) for small enough $\|u\|_\infty$ is a consequence of the equalities

$$\frac{3}{5}\bar{\nu}_f(K_i) = \frac{\mathcal{E}(f_i)}{\mathcal{E}(f)} = \frac{\mathcal{E}((Hf)_i) + \mathcal{E}((Gu)_i) + 2\mathcal{E}((Hf)_i, (Gu)_i)}{\mathcal{E}(Hf) + \mathcal{E}(Gu)}$$

and

$$\frac{3}{5}\bar{\nu}_{Hf}(K_i) = \frac{\mathcal{E}((Hf)_i)}{\mathcal{E}(Hf)}$$

together with $\mathcal{E}(Hf) \geq \frac{3}{2}$. Using (3.6) once more gives

$$\left|\bar{\nu}_{f_{w_0 z_1 \ldots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{Hf_{w_0 z_1 \ldots z_{j-1}}}(K_{z_j})\right| \leq \text{Const}\beta^M.$$

Using Lemma 10 and once again that the second coordinate of the iterates satisfies (3.6) we estimate the second term of inequality (3.11) by

$$\left|\bar{\nu}_{Hf_{w_0 z_1 \ldots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{(Hf_{w_0})_{z_1 \ldots z_{j-1}}}(K_{z_j})\right|$$

$$\leq C|e(f_{w_0 z_1 \ldots z_{j-1}}) - \psi_{z_1 \ldots z_{j-1}}(e(f_{w_0}))|$$

$$\leq C \sum_{k=0}^{j-1} \text{Const} B^k \beta^M \leq \text{Const}\frac{B^j}{B-1}\beta^M,$$

where $B = \max_{i=1,2,3} \|\psi_i\|_{\text{Lip}}$ and $C = \max_{i=1,2,3} \|p_i\|_{\text{Lip}}$, where p_i are the energy weights (1.5) for the original i.f.s.

Summing up all terms on the right hand side of (3.10) we have

$$\left|\bar{\nu}_{f_{w_0}}(K_z) - \bar{\nu}_{Hf_{w_0}}(K_z)\right|$$

$$\leq m'\beta^M(1 + \frac{B^{m'}}{B-1}).$$

It follows from (3.9) that

$$\left|\sum_{w \in W_n} \bar{\nu}_f(K_w) h(e(f_w), \frac{u_w}{5^{|w|}(\max_{V_0} f_w - \min_{V_0} f_w)}) - \int h(x,0) d\mu_{\mathcal{E}}(x)\right|$$

$$\leq 2\bar{\nu}_f[C(\omega, m) < M]$$

$$+\beta^M \left(\text{Const} + \text{Const}\frac{B^{m'}-1}{B-1} + m'(1+\frac{B^{m'}}{B-1}) \right)$$

$$+\text{Const}a^{m'}$$

where $a < 1$, $\beta < 1$ and $B > 1$. Note that $\bar{\nu}_f[C(\omega, m) < M] \to 0$ since $\bar{\nu}_f$ does not have atoms. This completes the proof. □

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