PRODUCTS OF RANDOM MATRICES AND DERIVATIVES ON P.C.F. FRACTALS

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ABSTRACT. We define and study intrinsic first order derivatives on post critically finite fractals and prove differentiability almost everywhere with respect to self-similar measures for certain classes of fractals and functions. We apply our results to extend the geography is destiny principle to these cases, and also obtain results on the pointwise behavior of local eccentricities on the Sierpiński gasket, previously studied by Öberg, Strichartz and Yingst, and the authors. We also establish the relation of the derivatives to the tangents and gradients previously studied by Strichartz and the authors. Our main tool is the Furstenberg-Kesten theory of products of random matrices.

CONTENTS

1. INTRODUCTION

For the last twenty years a theory of analysis on fractals has evolved, with the construction of Laplacians and Dirichlet forms as cornerstones. There is both a probabilistic approach, where the Laplacian is constructed as an infinitesimal generator of a diffusion process, and an analytic approach where the Laplacian can be defined as a limit of difference operators. In this article we will work in the context of post critically finite (p.c.f.) fractals, for which Kigami laid the foundations of an analytic theory [8, 9, 10, 11].

We consider one of the most fundamental topics in analysis; the local structure of smooth functions. This is not only an interesting matter as such, it also shed light on an important phenomenon that does not occur when the underlying set is smooth.

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In classical analysis any two points in the interior of the considered set have homeomorphic neighborhoods. This is not the case in analysis on fractals. Some points, called junction points, are boundary points of several copies of the self-similar set and neighborhoods of such points are different from those at nonjunction points that have a canonical basis of neighborhoods consisting of copies of the self-similar set. However, although two nonjunction points x, x' have bases of homeomorphic neighborhoods, the homeomorphisms do not in general map x onto x'.

It turns out that, as a consequence of the above, the local behavior of functions depend on the point under consideration. This *geography is destiny* principle, that has no analog whatsoever in analysis on smooth sets, were proven for harmonic functions on the Sierpiński gasket by Öberg, Strichartz and Yingst in [15]. Restrictions to the canonical neighborhoods will, for most harmonic functions, line up in the same direction, a direction that depends on the point, or rather the neighborhood. This property follows from theorems on products of random matrices since the restrictions to the canonical neighborhoods are given by linear mappings.

We will show that the geography is destiny principle extends to other fractals and to larger classes of functions with certain smoothness properties.

Generally speaking, the notion of smoothness of functions addresses the degree of differentiability of the function and its derivatives. Since the basic differential operator in analysis on fractals is the Laplacian, the term smooth has mostly been used for a function f in the domain of the Laplacian. It has also been used to refer to those f for which $\Delta^k f$ is continuous for some or all k.

On the other hand, in the classical calculus a differentiable function locally behaves like an affine linear mapping. In fractal analysis the analogs of such mappings are the harmonic functions, and from this point of view we make a natural definition of a derivative, and thus a concept of differentiability, of a function with respect to a harmonic function. This gives us wider classes of functions with some degree of smoothness for which we can prove geography is destiny. We also relate this derivative to the gradient defined by the second author [21].

Our results concerns generic, with respect to a self-similar measure, properties of the local behavior of smooth functions at nonjunction points. It would be interesting to know if the same properties hold generically with respect to the Kusuoka energy measure [13, 21]. Local behavior at junction points were studied in [19].

It is likely that our results can be extended to the category of self-similar finitely ramified fractals defined in [22].

We need to fix some notation, and at the same time recall some of the basic results of the theory. We refer to the books by Kigami [12] and Strichartz [20] for the whole story.

Positive constants in estimates will be denoted by C. The value of C might thus change from line to line.

Throughout this paper, F will denote a p.c.f. self-similar fractal, or post critically finite self-similar set, as defined in [12]. By [7], F is a compact connected metric space and there are contractions $\psi_1, ..., \psi_m : F \to F$ such that

$$F = \bigcup_{i=1}^{m} \psi_i(F), \tag{1.1}$$

and a finite set $V_0 \subset F$ such that for any n and for any two distinct words $w, w' \in W_n = \{1, ..., m\}^n$ we have

$$F_w \cap F_{w'} = V_w \cap V_{w'}, \tag{1.2}$$

where $F_w = \psi_w(F)$ and $V_w = \psi_w(V_0)$. Here for a finite word $w = w_1...w_n \in W_n$ we denote

$$\psi_w = \psi_{w_1} \circ ... \circ \psi_{w_n}. \tag{1.3}$$

We call F_w, $w \in W_n$ a *cell* of level n. If f is any function defined on F we use notation $f_w = f \circ \psi_w$ for its restriction to F_w.

The set V_0 is called the *boundary* of F and consequently points in V_0 are referred to as *boundary points*. The fractal F is p.c.f. self-similar fractal if every boundary point is contained in only one 1-cell. We denote the number of boundary points by N_0 and will assume that $N_0 \geqslant 2$. A point $x \in F$ is called a *junction point* if $x \in F_w \cap F_{w'}$, for two distinct w, $w' \in W_n$.

Define $V_n = \bigcup_{w \in W_n} V_w$, $V_* = \bigcup_{n \geqslant 1} V_n$ and $W_* = \bigcup_{n \geqslant 1} W_n$. If $w = w_1 \dots w_k \in W_*$, we say that $|w| = k$ is the *length* of w. It is easy to see that V_* is dense in F. Note that, by definition, each ψ_i maps V_* into itself injectively.

Let $\Omega = \{1, \dots, m\}^{\mathbb{N}}$ be the space of infinite sequences $\omega = w_1 w_2 \dots$, and $W_n = \{1, \dots, m\}^n$ the set of finite words in letters $w_j \in W_1 = \{1, \dots, m\}$. For any $\omega \in \Omega$ let $[\omega]_n = w_1 \cdots w_n \in W_n$ and $[\omega]_{n,k} = w_{n+1} \cdots w_k \in W_{k-n}$, $k > n$. These notations will be used also for $w \in W_*$ and $k < n \leqslant |w|$.

There is a natural continuous projection $\pi : \Omega \to F$ defined by

$$\pi(\omega) = \bigcap_{n \geqslant 0} F_{[\omega]_n}, \tag{1.4}$$

and $\pi^{-1}\{x\}$ is finite for any x by the p.c.f. assumption. Moreover, $\pi^{-1}\{x\}$ consists of more than one element if and only if x is a junction point. In case x is not a junction point we can therefore define $[x]_n = [\omega]_n$ and $[x]_{n,k} = [\omega]_{n,k}$ if $x = \pi(\omega)$. In particular, $[x]_n$ is well defined for any $x \notin V_*$.

We assume that a harmonic structure, as defined in [12], is fixed on the p.c.f. self-similar structure. This will give rise to a self-similar Dirichlet (resistance, energy) form

$$\mathcal{E}(f, f) = \sum_{i=1}^{m} \rho_i \mathcal{E}(f_i, f_i) = \sum_{w \in W_n} \rho_w \mathcal{E}(f_w, f_w). \tag{1.5}$$

Here $\rho_w = \rho_{w_1} \dots \rho_{w_n}$, where $\rho = (\rho_1, ..., \rho_m)$ are the energy renormalization factors. The energy renormalization factors, or weights, are often called conductance scaling factors because of the relation of resistance forms and electrical networks. They are reciprocals of the resistance scaling factors $r_j = 1/\rho_j$. We will always assume that the resistance form is *regular*, i.e. $\rho_j > 1$, $j = 1, \dots, m$.

The domain, $\operatorname{Dom} \mathcal{E}$, of $\mathcal{E}$ consists of continuous functions such that the *energy*, $\mathcal{E}(f) = \mathcal{E}(f, f) < \infty$. A function on F is *harmonic* if it minimizes the energy for the given set of boundary values.

Harmonic functions are uniquely defined by their restrictions to V_0 and we often, for convenience, identify the space of harmonic functions with the N_0-dimensional space $l(V_0)$ of functions on V_0.

The restrictions of a harmonic function to cells of level 1 give rise to linear mappings A_i, $i = 1, \dots, m$ on $l(V_0)$ through $A_i h = h_i = h \circ \psi_i$. The restrictions to smaller cells are given by products of these matrices since $h_w = h \circ \psi_w = A_w h$, where $A_w = A_{w_n} \dots A_{w_1}$ for $w \in W_n$.

Constant functions are harmonic so constant functions on $l(V_0)$ will be eigenvectors of all the mappings A_i, $i = 1, \dots, m$ with the corresponding eigenvalue equal to 1. To study the local behavior of harmonic functions it is therefore useful to factor out the constant functions. Denote by $\mathcal{H}$ the space of harmonic functions such that $\sum_{q \in V_0} h(q) = 0$ and define operators A_i', $i = 1, \dots, m$ on $\mathcal{H}$ by $A_i' = P_{\mathcal{H}} A_i P_{\mathcal{H}}^*$, where $P_{\mathcal{H}}$ is the projection of $l(V_0)$ onto $\mathcal{H}$ given by $P_{\mathcal{H}} h = h - \sum_{q \in V_0} h(q)$. Note that each A_j commutes with $P_{\mathcal{H}}$.

we will from now on assume that the matrices A_i are invertible, which implies that A_i' are invertible. This is an underlying assumption in the theory of product of random matrices that we will use. It is equivalent to that the restriction of a nonconstant harmonic function to any cell is itself nonconstant. Harmonic structures with this property are called *nondegenerate*. To see what the local behavior of harmonic functions on a degenerate harmonic structure might be like, there is an interesting study in [15, Section 7] on the case of the hexagasket.

For any function f defined on F we will denote by Hf the unique harmonic function that coincides with f on the boundary.

Given a finite nonatomic measure μ on F with the property that $\mu(O) > 0$ for any nonempty open set O there is a Laplacian Δ_μ that is an unbounded operator defined on a dense set of continuous functions by

$$\mathcal{E}(u, v) = -\int_F u \Delta_\mu v \, d\mu \tag{1.6}$$

for any $u \in \operatorname{Dom} \mathcal{E}$ with $u|_{V_0} = 0$. In this paper we will always assume that $\Delta_\mu v \in L^\infty(F)$. Functions with this property is denoted $\operatorname{Dom}_{L^\infty} \Delta_\mu$ but we will in what follows omit the index L^∞. We will also always assume that μ is self-similar, i.e. that there are real numbers μ_i, $i = 1, \dots, m$ such that $\mu(F_w) = \mu_w$ for any $w \in W_*$. For convenience we will assume that $\mu(F) = 1$.

Harmonic functions are exactly those for which $\Delta_\mu h = 0$. It should be noted that even though the Laplacian depends on the measure μ, the set of harmonic functions only depend on the harmonic structure.

There is a *Green's operator*

$$Gu(x) = \int_F g(x, y) u(y) \, d\mu(y) \tag{1.7}$$

acting on $L^\infty(F)$ such that $-\Delta G u = u$, and $Gu|_{V_0} = 0$. Thus, any function $f \in \operatorname{Dom} \Delta_\mu$ can be written $f = Hf - Gu$. The *Green's function* $g(x, y)$ is continuous for regular harmonic structures.

We next define some regularity classes of functions on F.

Definition 1.1. We say that $f \in C^k(\mathcal{H})$ if there are harmonic functions $h_1, ..., h_l \in \mathcal{H}$ and $u \in C^k(\mathbb{R}^l)$ such that $f = u(h_1, ..., h_l)$. We say that $f \in C^k(\operatorname{Dom} \Delta_\mu)$, if there are $g_1, ..., g_l \in \operatorname{Dom} \Delta_\mu$ and $u \in C^k(\mathbb{R}^l)$ such that $f = u(g_1, ..., g_l)$.

Note that whereas $C^k(\operatorname{Dom} \Delta_\mu)$ and $C^k(\mathcal{H})$ are multiplication domains, in general $\operatorname{Dom} \Delta_\mu$ is not by [2, 5, 6]. Also note that by definition $C^k(\mathcal{H}) \bigcup \operatorname{Dom} \Delta_\mu \subset C^k(\operatorname{Dom} \Delta_\mu)$.

There are several approaches to define derivatives on a p.c.f. fractal F. A weak gradient was studied by Kusuoka in [13, 14]. A stronger notion of gradients and tangents was considered in [19, 21] by Strichartz and the second author. In this paper we introduce the following definition.

Definition 1.2. Let f and h be real valued functions on a p.c.f. fractal F, and suppose h is continuous at $x \in F$. For $S \subseteq F$ let $Osc_S h = \sup_{x,y \in S} |h(y) - h(x)|$. Then we say that f is *differentiable* with respect to h at a nonjunction point x if there is a real number $\frac{df}{dh}(x)$ such that

$$f(y) = f(x) + \frac{df}{dh}(x)(h(y) - h(x)) + o\big(Osc_{F_{[x]_n}} h\big)_{y \to x}, \tag{1.8}$$

where n is such that $y \in F_{[x]_n}$, and at a junction point x if

$$f(y) = f(x) + \frac{df}{dh}(x)(h(y) - h(x)) + o\big(Osc_{U_n(x)} h\big)_{y \to x}, \tag{1.9}$$

where $U_n(x)$ is a canonical basis of neighborhoods and n is such that $y \in U_n(x)$. Naturally, $\frac{df}{dh}(x)$ is called the *derivative* of f at x with respect to h.

It is easy to show usual properties of the derivative $\frac{df}{dh}(x)$, such as sum, product, ratio and chain rules. Also if f is differentiable with respect to h at x, then f is continuous at x. For later use we formulate the following version of the chain rule.

Proposition 1.3. *Suppose $f_j : F \to \mathbb{R}$, $j = 1, \dots, l$ are differentiable with respect to h at x and that $g : \mathbb{R}^l \to \mathbb{R}$ is in $C^1(\mathbb{R}^l)$. Then $g(f_1, \dots, f_l)$ is differentiable with respect to h at x and*

$$\frac{d(g(f_1, \dots, f_l))}{dh}(x) = \sum_{j=1}^{l} \frac{\partial g}{\partial f_j}(f_1, \dots, f_l) \frac{df_j}{dh}(x). \tag{1.10}$$

We will only use Definition 1.2 for h harmonic. Harmonic functions are the natural choice with respect to which one should differentiate since they are, in a sense, the analogues of linear functions on the interval. In fact, we will only differentiate with respect to $h \in \mathcal{H}$ since $\frac{df}{d(h+c)} = \frac{df}{dh}$ for any constant c. The maximum and minimum of a harmonic function is always attained on the boundary and we can therefore replace $Osc_{F_{[x]_n}} h_{[x]_n}$ by $\|A'_{[x]_n} h\|$ in (1.8).

In section 2 we state the results on products of random matrices that will be used subsequently and in section 3 we formulate a condition on the harmonic structure that is necessary to apply most of these results. We also state two main assumptions, a weak and a strong, on the self-similar measure. Each of these is precisely the condition, the weak one for the derivative and the strong one for the gradient, that allows one to say that on sufficiently small cells the influence of $Hf_{[x]_n}$ dominates the term from the Green's function μ a.e. . This is the basis of essentially all of the results that do not follow directly of the theory on products of random matrices.

In section 4 we prove that a function $f \in C^1(\mathcal{H})$ is differentiable with respect to arbitrary nonconstant harmonic functions μ a.e. (see Theorem 4.7). Then, according to Definition 1.2, the function f behaves as a function of one variable up to smaller order terms. This means, in a sense, that the space F is essentially one dimensional. We then prove, under the weak main assumption, the same result for any function $f \in C^1(\mathrm{Dom}\,\Delta_\mu)$ in Theorem 4.8. We also prove an analog of Fermat's theorem on stationary points and discuss the relationship between our derivative and the local derivatives defined at periodic points in [1, 3].

In section 5 we prove the "geography is destiny" principle for smooth functions on the set where the derivative is different from zero and then use this to prove a result on the local behavior of the eccentricity for functions defined on fractals with three boundary points. The concept of eccentricity was introduced and studied for harmonic functions on the Sierpiński gasket in [15] and were studied for larger classes of functions in [16].

In section 6 we relate the derivative to the gradient defined in [19, 21] under the strong main assumption. Using this relation and technical results from the theory of products of random matrices we are also able to show geography is destiny on the set where the gradient is different from zero.

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2. Products of random matrices

Since our aim is to describe the local behavior of functions with certain smoothness properties with that of harmonic functions it is essential to understand their local structure.

If $x \in F$ is a nonjunction point it is contained in a unique sequence of cells $F_{[x]_n}$, and the local behavior of harmonic functions at x is given by the properties of the products $A'_{[x]_n}$. The generic local behavior of harmonic functions with respect to a self-similar measure μ will thus be governed by the product of i.i.d. random matrices. We define random matrices

$$M_n(x) = A'_{[x]_n}$$

on the probability space (F, μ) with the Borel sigma-field. Note that we have

$$\mathbb{P}[M_n = A'_w] = \mu_w,$$

and the random matrices M_n are products of i.i.d. random matrices with a common Bernoulli distribution given by

$$\mathbb{P}[M_1 = A'_i] = \mu_i, \ i = 1, \dots, m. \tag{2.1}$$

In the 60s and 70s a theory of products of random matrices, as a natural generalization of the classical limit theorems to products of i.i.d. invertible matrices, was developed by Furstenberg, Kesten, Guivarch, Le Page, Raugi, Osseledec et al. In this section results and concepts from this theory that we will rely upon are summarized. They can all be found in [4], where the reader will find references to the original sources. However, we start by introducing the following notation.

Notation 2.1. We use notation $c_n = \varnothing(a^n)$ if $\lim_{n\to\infty} \frac{1}{n} \log c_n = \log a$.

The next Lemma collects some properties of the notion $\varnothing(a^n)$. As the proof is elementary we omit it.

Lemma 2.2. *Suppose $c_n = \varnothing(a^n)$ and $d_n = \varnothing(b^n)$. Then the following properties hold.*

i) $1/c_n = \varnothing((1/a)^n)$
ii) $c_n d_n = \varnothing((ab)^n)$
iii) $\sum_{n\leqslant N} c_n$ *is* $\varnothing(a^N)$ *if* $a > 1$, $O(1)$ *if* $a < 1$ *and* $\varnothing(1)$ *if* $a = 1$.
iv) $\sum_{n>N} c_n = \varnothing(a^N)$ *if* $a < 1$.

Moreover, $c_n = \text{Ø}(a^n)$ *if and only if* $c_n = o((a+\epsilon)^n)$ *and* $(a-\epsilon)^n = o(c_n)$ *for any* $\epsilon > 0$ *but* $c_n = \text{Ø}(a^n)$ *is not equivalent to* $c_n = O(a^n)$*.*

Throughout the rest of this section $Y_n \in Gl(\mathbb{R}, d)$, $n \geqslant 1$, will be any sequence of i.i.d. invertible $d \times d$ random matrices with common distribution $\mathcal{M}$ and $S_n = Y_n \dots Y_1$. We also suppose the support of $\mathcal{M}$ is finite since this obviously holds for M_n with distribution given by (2.1). It should be noted that the results we present do not depend on the particular norms chosen on $\mathbb{R}^d$ and $Gl(\mathbb{R}, d)$.

Theorem 2.3 (Theorem I.4.1 and Proposition III.5.6 [4])**.** *Let* $a_1(n) \geqslant a_2(n) \geqslant \dots \geqslant a_d(n) > 0$ *be the square roots of the eigenvalues of* $(Y_n \dots Y_1)^*(Y_n \dots Y_1)$*. Then there are numbers* $\alpha_+ = \alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant \alpha_d = \alpha_- > 0$ *such that with probability one*

$$a_p(n) = \text{Ø}(\alpha_p^n),\ p = 1, \dots, d \tag{2.2}$$

and moreover

$$\|S_n\| = \|Y_n \dots Y_1\| = \text{Ø}(\alpha_+^n) \tag{2.3}$$

Definition 2.4. Let $\alpha_+ = \alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant \alpha_d = \alpha_- > 0$ be as in Theorem 2.3. The numbers $\log \alpha_p$, $p = 1, \dots, d$ are called the Lyapunov exponents associated to Y_n. The upper, respectively lower, Lyapunov exponents are $\log \alpha_+$ respectively $\log \alpha_-$.

It is clear that the Lyapunov exponents of Y_n^{-1} are $-\log \alpha_- \geqslant -\log \alpha_{d-1} \geqslant \dots \geqslant -\log \alpha_+$. It should also be remarked that in general some Lyapunov exponents can be $-\infty$, however this possibility is excluded by the assumption that $\mathcal{M}$ has finite support.

Our interest lies in $h_{[x]_n} = M_{[x]_n} h$, i.e. in the long term behavior of $S_n v$, $v \in \mathbb{R}^d$ and to apply the results on products of random matrices it is then necessary to make additional assumptions on the distribution $\mathcal{M}$, i.e. on the matrices A_i' in the fractal setting. We need the following definitions, which are [4, Definitions III.2.1 and III.1.3].

Definition 2.5. A subset S of $Gl(d, \mathbb{R})$ is *strongly irreducible* if there does not exist a finite family $\{L_1, \dots, L_k\}$ of proper linear subspaces of $\mathbb{R}^d$ such that

$$M(L_1 \cup L_2 \cup \dots \cup L_k) = L_1 \cup L_2 \cup \dots \cup L_k, \tag{2.4}$$

for any $M \in S$.

Definition 2.6. The *index* of a subset T of $Gl(d, \mathbb{R})$ is the least integer p such that there exists a sequence M_n in T for which $\|M\|_n^{-1} M_n$ converges to a rank p matrix. T is *contracting* if its index is one.

Denote by $T_{\mathcal{M}}$ the smallest closed semigroup that contains the support of $\mathcal{M}$.

Theorem 2.7 (Corollary III.3.4 and Theorem III.6.1 [4])**.** *Suppose* $T_{\mathcal{M}}$ *is strongly irreducible, then for any* $v \in \mathbb{R}^d$*,* $v \neq 0$*, with probability one*

$$\|S_n v\| = \text{Ø}(\alpha_+^n). \tag{2.5}$$

Moreover, if $T_{\mathcal{M}}$ *also is contracting then the two first Lyapunov exponents are distinct, i.e.,*

$$\alpha_+ > \alpha_2. \tag{2.6}$$

For $v \in \mathbb{R}^d$, $v \neq 0$, denote by $\bar{v}$ the corresponding element in the real projective space $\mathbb{P}(\mathbb{R}^d)$, and let δ be the natural angular distance in $\mathbb{P}(\mathbb{R}^d)$. For $Y \in Gl(\mathbb{R}, d)$ let $Y \cdot \bar{v} = \overline{Yv} \in \mathbb{P}(\mathbb{R}^d)$.

Theorem 2.8 (Theorem III.3.1, Corollary VI.1.7 and Theorem VI.3.1 [4]). *Suppose $T_{\mathcal{M}}$ is strongly irreducible and contracting. Then, there is a random direction $\bar{Z}$ (depending on S_n), such that for any $\bar{v} \in \mathbb{P}(\mathbb{R}^d)$*

$$S_n^* \cdot \bar{v} \to \bar{Z}, \tag{2.7}$$

with probability one. If $\bar{v}$ is not orthogonal to $\bar{Z}$, then

$$\|S_n v\| = \varnothing(\alpha_+^n), \tag{2.8}$$

and if $\bar{v}$ is orthogonal to $\bar{Z}$ then

$$\limsup \frac{1}{n} \log \|S_n v\| \leqslant \log \alpha_2. \tag{2.9}$$

Moreover, for any nonzero $v \in \mathbb{R}^d$ the probability that v is orthogonal to $\bar{Z}$ is zero.

The next theorem formulates the contraction property that is the basis for the Geography is destiny principle.

Theorem 2.9 (Theorem III.4.3 and Proposition III.6.4 [4]). *Suppose $T_{\mathcal{M}}$ is strongly irreducible and contracting. Then for any $\bar{v}, \bar{w} \in \mathbb{P}(\mathbb{R}^d)$,*

$$\frac{\delta(S_n \cdot \bar{v}, S_n \cdot \bar{w})}{\delta(\bar{v}, \bar{w})} = \varnothing((\alpha_2/\alpha_+)^n), \tag{2.10}$$

with probability one.

In section 6 we will make use of the following.

Theorem 2.10 (Lemma V.5.2 and Theorem V.6.2 [4]). *Suppose $T_{\mathcal{M}}$ is strongly irreducible and contracting. For any unit vector $v \in \mathbb{R}^d$ there is $a > 0$ so that*

$$\mathbb{E}(\log \|S_n v\| - n \log \alpha_+)^2 - na \tag{2.11}$$

converges to a finite limit. Moreover, there exists $b > 0$ such that for any $\varepsilon > 0$

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}[\,|\log \|S_n\| - n \log \alpha_+| > n\varepsilon] < -b, \tag{2.12}$$

where $\mathbb{E}$ denotes expectation and $\mathbb{P}$ probability.

3. Main assumptions

Motivated by Theorems 2.7–2.10 in the previous section we make the following definition.

Definition 3.1. We say that F satisfies the *SC-assumption* if the semigroup generated by A_i', $i = 1, \ldots, m$ is strongly irreducible and contracting.

The index of a set is in general difficult to determine, however in the case of semigroups there is a useful result in [4, Corollary IV.2.2]. Recall that an eigenvalue λ of a matrix M is *simple* if $\mathrm{Ker}\,(M - \lambda Id)$ has dimension one and equals $\mathrm{Ker}\,(M - \lambda Id)^2$ and it is *dominating* if $|\lambda| > |\lambda'|$ for any other eigenvalue λ'.

Proposition 3.2. *A semigroup T in $Gl(d, \mathbb{R})$ which contains a matrix with a simple dominating eigenvalue is contracting.*

Suppose a matrix $M \in Gl(2, \mathbb{R})$ has two distinct real eigenvalues. A finite union of lines invariant under M consists of either one or both of the eigenspaces, so we have the following.

Proposition 3.3. *If the boundary V_0 consists of three points, then F satisfy the SC-assumption if there is some M_v with a simple dominating eigenvalue and there are two matrices M_w, $M_{w'}$ both with two distinct real eigenvalues and no eigenvector in common.*

It is readily verified that for instance the standard harmonic structures on the Sierpiński gasket, as noted in [15, 19], and the level 3 Sierpiński gasket satisfies the SC-assumption. In fact, any nondegenerate structure with D_3 symmetry considered in [19, Section 5] satisfies the SC-assumption if $a \neq b$ where

$$\begin{pmatrix} 1 & 0 & 0 \\ 1-a-b & a & b \\ 1-a-b & b & a \end{pmatrix} \tag{3.1}$$

is the matrix corresponding to the restriction to a level 1 cell containing one of the boundary points.

With the SC-assumption one can obtain differentiability results for $C^1(\mathcal{H})$. For the same results on $C^1(\operatorname{Dom}\Delta_\mu)$ an additional assumption on the measure μ is needed. In section 6 we will use another, stronger, assumption on μ to have a.e. existence of the gradient. To this end, we define γ by

$$\log \gamma = \sum_{j=1}^{m} \mu_j \log(r_j \mu_j). \tag{3.2}$$

Then

$$r_{[x]_n} \mu_{[x]_n} = \emptyset(\gamma^n) \tag{3.3}$$

for μ a.e. x, essentially because the probability of occurrence of the scaling factor $r_j \mu_j$ is μ_j. One can see that $\log \gamma$ is the analog of the Lyapunov exponent for the Laplacian scaling factor $r_{[x]_n} \mu_{[x]_n}$, which in turn is the product of energy and measure scaling factors.

Definition 3.4. We will say that (F, μ) satisfies the *weak main assumption* respectively the *strong main assumption* if F satisfies the SC-assumption and

$$\gamma < \alpha_+. \tag{3.4}$$

respectively

$$\gamma < \alpha_-. \tag{3.5}$$

Essentially the weak main assumption says that, μ a.e. , restrictions of harmonic functions to small cells scale to zero exponentially more slowly than the Laplacian scales, while the strong main assumption says that extensions of harmonic functions from smaller to larger cells scale to infinity exponentially faster than the Laplacian scales.

It is known that the Sierpiński gasket with the standard harmonic structure and uniform self-similar measure satisfies the weak main assumption. It also holds for the level 3 Sierpiński gasket with the uniform self-similar measure and standard harmonic structure, which is discussed in detail in [19, 20]. In this case $\gamma = 7/90$ and of the six restriction matrices three have determinant $7/15^2$ and three have determinant $8/15^2$. It is known that if all determinants equal one, then $\alpha_+ > 1$. It follows that for the level 3 Sierpiński gasket $\alpha_+ > \sqrt{7}/15 > \gamma$.

It has been shown [23, 19] that the Sierpiński gasket with standard harmonic structure and uniform self-similar measure satisfies the inequality,

$$\gamma\alpha_+ < \alpha_-^2 \tag{3.6}$$

which is even stronger than (3.5).

For the standard harmonic structure on the Sierpiński gasket the resistance scaling factors are all 3/5. Sabot showed in [17] that for small perturbations of these factors there is a unique harmonic structure on the Sierpiński gasket, see also [18]. Since the harmonic restriction mappings depend continuously on the resistances, (3.6) implies that for small enough perturbations of the harmonic structure the Sierpiński gasket, with a self-similar measure not far from being uniform, will still satisfy the strong main assumption.

4. Derivatives on p.c.f. fractals

We start this section by translating some of the theorems in section 2 to properties of the local behavior of harmonic function and then go on to prove a.e. differentiability in $C^1(\mathcal{H})$ under the SC-assumption and in $C^1(\operatorname{Dom}\Delta_\mu)$, under the weak main assumption.

The following propositions are interpretations of Theorems 2.3– 2.8 in terms of analysis on fractals.

Proposition 4.1. *For μ a.e. nonjunction point x,*

$$\|M_{[x]_n}\| = \varnothing(\alpha_+^n). \tag{4.1}$$

Proposition 4.2. *Suppose F satisfies the SC-assumption and $h \in \mathcal{H}$, $h \neq 0$. Then $\alpha_+ > \alpha_2$ and*

$$\|h_{[x]_n}\| = \|M_{[x]_n}h\| = \varnothing(\alpha_+^n), \tag{4.2}$$

for μ a.e. nonjunction point x.

Proposition 4.3. *For μ a.e. nonjunction point x there exists a subspace $\mathcal{H}_x^- \subset \mathcal{H}$ of codimension one such that*

$$\|M_{[x]_n}h\| = \varnothing(\alpha_+^n), \tag{4.3}$$

for $h \notin \mathcal{H}_x^-$, and

$$\limsup_{n\to\infty} \frac{1}{n}\log\|M_{[x]_n}h^-\| \leqslant \alpha_2, \tag{4.4}$$

for $h^- \in \mathcal{H}_x^-$. For any nonzero $h \in \mathcal{H}$, $h \notin \mathcal{H}_x^-$, μ a.e. .

The subspace $\mathcal{H}_x^-$ corresponds to the orthogonal complement of $\bar{Z}$ in Theorem 2.8. We will denote by $\mathcal{H}_x^+$ the orthogonal complement of $\mathcal{H}_x^-$ and byP_x^- and P_x^+ the orthogonal projections onto $\mathcal{H}_x^-$ and $\mathcal{H}_x^+$ respectively. Also denote by h_x^+ an element of $\mathcal{H}_x^+$ of norm one. The property in proposition 4.3 is what we will use to prove differentiability so we make the following definition.

Definition 4.4. We say that $x \in F$ is weakly generic if there is a subspace $\mathcal{H}_x^- \subset \mathcal{H}$ of co-dimension one such that

$$\|M_{[x]_n}h^-\| = o\|M_{[x]_n}\|_{n\to\infty} \tag{4.5}$$

for any $h^- \in \mathcal{H}_x^-$.

Proposition 4.5. *$x \in F$ is weakly generic if and only if there is a subspace $\mathcal{H}_x^- \subset \mathcal{H}$ of co-dimension one such that*

$$\|M_{[x]_n} h^-\| = o\|M_{[x]_n} h\|_{n\to\infty} \tag{4.6}$$

for any $h^- \in \mathcal{H}_x^-$ and $h \notin \mathcal{H}_x^-$.

Proof. Necessarily $\|M_{[x]_n} h_x^+\| = O\|M_{[x]_n}\|_{n\to\infty}$, since if not $\|M_{[x]_n} h\| = o(\|M_{[x]_n}\|)$ for any $h \in \mathcal{H}$. The proposition follows immediately since if $h \notin \mathcal{H}_x^-$ then $P_x^+ h \neq 0$. □

Clearly μ a.e. x is weakly generic if F satisfies the SC-assumption.

Proposition 4.6. *If $x \in F$ is weakly generic and $f = u(h_1, \ldots, h_l) \in C^1(\mathcal{H})$ then $\frac{df}{dh}$ exists for any $h \notin \mathcal{H}_x^-$ with*

$$\frac{df}{dh} = \sum_{j=1}^{l} \frac{\partial u}{\partial h_j} \frac{dh_j}{dh}. \tag{4.7}$$

If $h' \in \mathcal{H}$ then

$$\frac{dh'}{dh} = \frac{< h', h_x^+ >}{< h, h_x^+ >}, \tag{4.8}$$

and in particular $h' \in \mathcal{H}_x^-$ if and only if $\frac{dh'}{dh_x^+} = 0$.

Proof. Because of Proposition 1.3 it is enough to show that $\frac{dh'}{dh}$ exists for any $h' \in \mathcal{H}$. Write $h' = a_x h + h^-$ with $h^- \in \mathcal{H}_x^-$. Then since

$$(h'(y) - h'(x))|_{F_{[x]_n}} = a_x(h(y) - h(x)) + (M_{[x]_n} h^-(\psi_{[x]_n}^{-1} y) - M_{[x]_n} h^-(\psi_{[x]_n}^{-1} x)), \tag{4.9}$$

it is clear from Proposition 4.5 that $\frac{dh'}{dh}(x) = a_x = \frac{<h', h_x^+>}{<h, h_x^+>}$ and (4.8) follows. □

Theorem 4.7. *Suppose F satisfies the SC-assumption. Then for any nonzero $h \in \mathcal{H}$ and any $f = u(h_1, \ldots, h_l) \in C^1(\mathcal{H})$ we have that $\frac{df}{dh}(x)$ exists for μ a.e. x and is given by (4.7).*

Proof. This follows immediately from Proposition 4.3 and Proposition 4.6 since μ a.e. x is weakly generic. □

Theorem 4.8. *Suppose (F, μ) satisfies the weak main assumption and h is a non-constant harmonic function. Then for μ-almost every x the derivative $\frac{df}{dh}(x)$ exists for any function $f = u(g_1, \ldots, g_l) \in C^1(\mathrm{Dom}\, \Delta_\mu)$ and is given by*

$$\frac{df}{dh} = \sum_{j=1}^{l} \frac{\partial u}{\partial g_j} \frac{dg_j}{dh}. \tag{4.10}$$

Moreover, there exists C such that if $f \in \mathrm{Dom}\, \Delta_\mu$, then for μ a.e. x

$$\left|\frac{df}{dh}\right| \leqslant \left|\frac{d(Hf)}{dh}\right| + C \frac{\|\Delta f\|_\infty}{|< h, h_x^+ >|} \sum_{n=0}^{\infty} (n+1) r_{[x]_n} \mu_{[x]_n} \|M_{[x]_n}^{-1*} h_x^+\|. \tag{4.11}$$

We first state and prove two Lemmas.

Lemma 4.9. *Suppose $u \in L^\infty(F)$ has support in a cell F_w. Then*

$$Osc_{F_{[w]_k}} Gu \leqslant C(k+1) r_{[w]_k} \mu_w \|u\|_\infty \tag{4.12}$$

for $k = 0, 1, ..., n = |w|$.

Proof. It will be enough to show that

$$|Gu(x) - Gu(x_0)| \leqslant C(k+1) r_{[w]_k} \mu_w \|u\|_\infty \tag{4.13}$$

for $x \in F_{[w]_k}$ and $x_0 \in V_{[w]_k}$. This can be done by using properties of the Green's function

$$g(x,y) = \sum_{v \in \varnothing \cup W^*} r_v \Psi(\psi_v^{-1}(x), \psi_v^{-1}(y)). \tag{4.14}$$

For the exact definition of Ψ, see [12]. We only need that it is continuous and harmonic on 1-cells.

Since we consider points in $F_{[w]_k}$ and u has support in F_w we are only concerned about x and y in $F_{[w]_k}$. For those, $\Psi(\psi_v^{-1}(x), \psi_v^{-1}(y)) = 0$ in case $|v| \geqslant k$ and $[v]_k \neq [w]_k$, and in case $|v| < k$ and $[w]_{|v|} \neq v$. The properties of Ψ also makes $\Psi(\psi_v^{-1}(x_0), \psi_v^{-1}(y)) = 0$ for all $|v| \geqslant k$. In all

$$\begin{aligned} |g(x_0,y) - g(x,y)| &\leqslant \sum_{m=0}^{k-1} r_{[w]_m} |\Psi(\psi_{[w]_m}^{-1}(x_0), \psi_{[w]_m}^{-1}(y)) - \Psi(\psi_{[w]_m}^{-1}(x), \psi_{[w]_m}^{-1}(y))| \\ &+ \left| \sum_{v \in \phi \cup W^*} r_v r_{[w]_k} \Psi(\psi_{vw}^{-1}(x), \psi_{vw}^{-1}(y)) \right|. \end{aligned} \tag{4.15}$$

The difference in the first term is, by the definition of Ψ, bounded by a constant times the difference of the value of 1-harmonic functions at the points $\psi_{[w]_m}^{-1}(x_0)$ and $\psi_{[w]_m}^{-1}(x)$. Both points lie in the cell $F_{[w]_{m,k}}$, and the difference is thus bounded by a constant times $r_{[w]_{m,k}}$ since the largest eigenvalue of A_i' is less or equal to r_i , see [12, Appendix A], and the first term is bounded by $Ckr_{[w]_k}$. The second term is $r_{[w]_k} g(\psi_{[w]_k}^{-1} x, \psi_{[w]_k}^{-1} y) \leqslant r_{[w]_k} \|g\|_\infty$ and we conclude that

$$\begin{aligned} |Gu(x) - Gu(x_0)| &\leqslant \int_F |g(x,y) - g(x_0,y)||u(y)| d\mu(y) \\ &\leqslant C(k+1) r_{[w]_k} \int_{F_w} |u(y)| d\mu(y) \leqslant C(k+1) r_{[w]_k} \mu_w \|u\|_\infty. \end{aligned} \tag{4.16}$$

□

Lemma 4.10. *Suppose F satisfies the SC-assumption. Given any nonconstant $h, h' \in \mathcal{H}$, we have for μ a.e. $x \in F$ that*

$$\sup_{y \in F_{[x]_n}} \left| h'(y) - h'(x) - \frac{dh'}{dh}(x)(h(y) - h(x)) \right| \leqslant c_{n,x} \frac{\|h\| \|h'\|}{| < h, h_x^+ > |}, \tag{4.17}$$

where

$$\limsup \frac{1}{n} \log c_{n,x} \leqslant \log \alpha_2. \tag{4.18}$$

Proof. Let x be such that $h \notin \mathcal{H}_x^-$. This holds for μ a.e. x. Since, in the proof of Proposition 4.6, $h^- = P_x^- h' - \frac{<h',h_x^+>}{<h,h_x^+>} P_x^- h$, it follows from (4.9) that for $y \in F_{[x]_n}$

$$\left| h'(y) - h'(x) - \frac{dh'}{dh}(h(y) - h(x)) \right| \leqslant \|M_{[x]_n} h^-\| \tag{4.19}$$

$$\leqslant \frac{\|h\| \|h'\|}{| < h, h_x^+ > |} \left(\frac{\|M_{[x]_n} P_x^- h'\|}{\|h'\|} + \frac{\|M_{[x]_n} P_x^- h\|}{\|h\|} \right).$$

Now, by Proposition 4.3

$$\limsup_n \frac{1}{n} \log \|M_{[x]_n} h_-\| \leqslant \log \alpha_2 \tag{4.20}$$

for any $h_- \in \mathcal{H}_x^-$. Thus

$$c_{n,x} = 2 \sup_{h_- \in \mathcal{H}_x^-} \frac{\|M_{[x]_n} h_-\|}{\|h_-\|} \tag{4.21}$$

satisfies (4.18) and (4.17) follows from (4.19). □

Proof of Theorem 4.8. In view of Proposition 1.3 it is enough to suppose $f \in \operatorname{Dom} \Delta_\mu$. It is clear from Theorem 4.7 that we can suppose $f = Gu$. We also assume $x \in F$ is weakly generic, $r_{[x]_n} \mu_{[x]_n} = \emptyset(\gamma^n)$ and $h \notin \mathcal{H}_x^-$ with $\|M_{[x]_n} h\| = \emptyset(\alpha_+^n)$.

Denote $B_{[x]_n} = F_{[x]_{n-1}} \setminus F_{[x]_n}$ and let $u^{[x]_n}$ be the restriction of u to $B_{[x]_n}$ so that

$$f = \sum_{n=1}^{\infty} G u^{[x]_n}. \tag{4.22}$$

Since $u^{[x]_n} = 0$ on $F_{[x]_n}$, $Gu^{[x]_n}$ is harmonic on $F_{[x]_n}$ and thus $\frac{d(Gu^{[x]_n})}{dh}$ exists and our aim is to show that

$$\frac{df}{dh} = \sum_{n=1}^{\infty} \frac{d(Gu^{[x]_n})}{dh}. \tag{4.23}$$

To prove convergence of the right hand side of (4.23) we show that

$$\left| \frac{d(Gu^{[x]_n})}{dh} \right| = \emptyset((\gamma/\alpha_+)^n), \tag{4.24}$$

which is enough by Lemma 2.2. Let $v^{[x]_n}$ be the function in $\mathcal{H}$ that corresponds to $(Gu^{[x]_n})_{[x]_n}$ and note that

$$\frac{d(Gu^{[x]_n})}{dh}(x) = \frac{d(v^{[x]_n})}{d(M_{[x]_n} h)}(\psi_{[x]_n}^{-1}(x)) = \frac{\left\langle v^{[x]_n}, h^+_{\psi_{[x]_n}^{-1}(x)} \right\rangle}{\left\langle M_{[x]_n} h, h^+_{\psi_{[x]_n}^{-1}(x)} \right\rangle}, \tag{4.25}$$

where the last equality follows from (4.8). According to Lemma 2.2 we obtain (4.24) by showing that the absolute value of the denominator of the right hand side of (4.25) is $\emptyset(\alpha_+^n)$ and that the absolute value of the numerator is $\emptyset(\gamma^n)$.

From Theorem 2.8 it follows that there is $\tilde{h} \in \mathcal{H}$ such that

$$h_x^+ = \lim_{n \to \infty} \frac{M_{[x]_n}^* \tilde{h}}{\|M_{[x]_n}^* \tilde{h}\|} \tag{4.26}$$

and

$$h^+_{\psi_w(x)} = \lim_{n\to\infty} \frac{M^*_{w[x]_n}\tilde{h}}{\|M^*_{w[x]_n}\tilde{h}\|}, \tag{4.27}$$

consequently

$$h^+_{\psi^{-1}_{[x]_n}(x)} = \frac{M^{-1*}_{[x]_n} h^+_x}{\|M^{-1*}_{[x]_n} h^+_x\|}. \tag{4.28}$$

Note that

$$\|M^{-1*}_{[x]_n} h^+_x\| = \sup_{\|h\|=1} < h, M^{-1*}_{[x]_n} h^+_x >= \sup_{\|k\|=1} < \frac{M_{[x]_n} k}{\|M_{[x]_n} k\|}, M^{-1*}_{[x]_n} h^+_x > \tag{4.29}$$

$$= \sup_{\|k\|=1} \frac{< k, h^+_x >}{\|M_{[x]_n} k\|} = \frac{< k, h^+_x >}{\|M_{[x]_n} k\|}$$

for some $k \notin \mathcal{H}^-_x$. Since $\|M_{[x]_n}\| = \emptyset(\alpha^n_+)$ it then follows by Lemma 2.2 that

$$\|M^{-1*}_{[x]_n} h^+_x\| = \emptyset((1/\alpha_+)^n). \tag{4.30}$$

and

$$\left|\left\langle M_{[x]_n} h, h^+_{\psi^{-1}_{[x]_n}(x)}\right\rangle\right| = \frac{|< h, h^+_x >|}{\|M^{-1*}_{[x]_n} h^+_x\|} = \emptyset(\alpha^n_+). \tag{4.31}$$

The numerator has the bound

$$|< v^{[x]_n}, h^+_{\psi^{-1}_{[x]_n}(x)} >| \leqslant C\, Osc(v^{[x]_n}) \leqslant C(n+1) r_{[x]_n} \mu_{[x]_n} \|u\|_\infty = \emptyset(\gamma^n), \tag{4.32}$$

where the last inequality follows from Lemma 4.9 and the last equality follows from Lemma 2.2. Thus, the right hand side of (4.23) converges and (4.11) follows from (4.31) and (4.32) as soon as we have shown (4.23).

For $y \in F_{[x]_k}$ we must show

$$\left| Gu(y) - Gu(x) - \sum_{n=1}^{\infty} \frac{d(Gu^{[x]_n})}{dh}(h(y) - h(x)) \right| = o(\|M_{[x]_k} h\|). \tag{4.33}$$

We write

$$\left| Gu(y) - Gu(x) - \sum_{n=1}^{\infty} \frac{d(Gu^{[x]_n})}{dh}(h(y) - h(x)) \right| \tag{4.34}$$

$$\leqslant \left| \sum_{n=1}^{k} (Gu^{[x]_n}(y) - Gu^{[x]_n}(x)) - \sum_{n=1}^{k} \frac{d(Gu^{[x]_n})}{dh}(h(y) - h(x)) \right|$$

$$+ \left| \sum_{n=k+1}^{\infty} (Gu^{[x]_n}(y) - Gu^{[x]_n}(x)) \right|$$

$$+ \left| \sum_{n=k+1}^{\infty} \frac{d(Gu^{[x]_n})}{dh}(h(y) - h(x)) \right|.$$

Lemma 4.9 and Lemma 2.2 implies that the second term is estimated from above by

$$C(k+1)r_{[x]_k}\mu_{[x]_k} = \emptyset(\gamma^k) = o(\|M_{[x]_k}h\|). \tag{4.35}$$

The third term is $\emptyset(\gamma^k) = o(\|M_{[x]_k}h\|$ since $|h(y)-h(x)| = \emptyset(\alpha_+^k)$ and

$$\sum_{n=k+1}^{\infty} \frac{d(Gu^{[x]_n})}{dh} = \emptyset((\gamma/\alpha_+)^k)$$

by Lemma 2.2 and (4.24). Remains the first term which we write

$$\left|\sum_{n=1}^{k} Gu^{[x]_n}(y) - Gu^{[x]_n}(x) - \frac{d(Gu^{[x]_n})}{dh}(h(y)-h(x))\right| \tag{4.36}$$

Suppose that we fix a (large) constant M, which is to be chosen later, and that the integers from 1 to k are divided into M subintervals $[jk/M,(j+1)k/M]$. From the arguments below it is evident that without loss of generality we can assume that k is an integer multiple of M, say $k = Mm$. So we write the sum in (4.36) as M sums of $m = k/M$ addends each, and have to show that for each $j = 1, \ldots, M$ we have

$$\left|\sum_{n=m(j-1)+1}^{jm} Gu^{[x]_n}(y) - Gu^{[x]_n}(x) - \frac{d(Gu^{[x]_n})}{dh}(h(y)-h(x))\right| = o(\|M_{[x]_k}h\|). \tag{4.37}$$

If we denote

$$h_j = \sum_{n=m(j-1)+1}^{jm} Gu^{[x]_n} \tag{4.38}$$

then we have to show

$$\left|\sum_{n=m(j-1)+1}^{jm} h_j(y) - h_j(x) - \frac{dh_j}{dh}(h(y)-h(x))\right| = o(\|M_{[x]_k}h\|). \tag{4.39}$$

Note that h_j is harmonic on $F_{[x]_{jm}}$. By Lemma 4.9 we have $\|h_j\| = \emptyset(\gamma^{m(j-1)})$ and Lemma 4.10 then implies that the left hand side of (4.39) is bounded by $\emptyset\left(\gamma^{m(j-1)}\alpha_2^{m(M-j)}\right)$. Let $\widetilde{\alpha} = \max\{\gamma,\alpha_2\}$ and $\varepsilon = \frac{1}{2}(\alpha_+ - \widetilde{\alpha}) > 0$. If we have that

$$M > \frac{\log\gamma}{\log\widetilde{\alpha} - \log(\widetilde{\alpha}+\varepsilon)} \tag{4.40}$$

then

$$\gamma^{j-1}{\alpha_2}^{M-j} \leqslant \widetilde{\alpha}^M\gamma^{-1} < (\widetilde{\alpha}+\varepsilon)^M = (\alpha_+ - \varepsilon)^M \tag{4.41}$$

which implies

$$\emptyset\left(\gamma^{m(j-1)}\alpha_2^{m(M-j)}\right) = o\big((\alpha_+ - \varepsilon)^k\big)_{k\to\infty} \tag{4.42}$$

and this completes the proof. □

The next corollary is an analog of Fermat's theorem about stationary points in our context.

Corollary 4.11. *Suppose (F, μ) satisfies the weak main assumption. Then for any nonconstant harmonic function h there exists a set F' of full μ-measure such that if $f = u(g_1, \dots, g_l) \in C^1(\mathrm{Dom}\,\Delta_\mu)$ has a local maximum at $x \in F'$, then $\frac{df}{dh}(x) = 0$.*

Proof. Let F'' be the set of full μ-measure such that, according to Theorem 4.8, the derivative $\frac{df}{dh}(x)$ exists for any $f \in C^1(\mathrm{Dom}\,\Delta_\mu)$. There exists $w \in W_*$ such that the cell F_w does not contain any boundary points. We define F' as the set of all x such that $x \in F''$ and there are infinitely many n such that $[x]_{n,n+k} = w$, $|w| = k$. Obviously F' is a set of full μ-measure.

Non-negative harmonic functions satisfy a Harnack inequality [12, Proposition 3.2.7] on F_w,

$$\max_{y \in F_w} h(y) \leqslant c \min_{y \in F_w} h(y), \tag{4.43}$$

for some $c > 1$. Suppose h is a harmonic function with a zero in F_w. Applying (4.43) on $\max_F h - h$ and $h - \min_F h$ gives

$$\max_F h \geqslant \frac{1}{c-1} \mathrm{Osc}_{F_w}(h) \tag{4.44}$$

and

$$\min_F h \leqslant \frac{1}{1-c} \mathrm{Osc}_{F_w}(h). \tag{4.45}$$

Suppose $f \in C^1(\mathrm{Dom}\,\Delta_\mu)$ has a local maximum at $x \in F'$. Since $x \in F'$ we can choose a subsequence n_l for which $[x]_{n_l, n_l+k} = w$. Then, for l large enough, we have for $y \in F_{[x]_{n_l}}$ that

$$f(y) - f(x) = \frac{df}{dh}(x)(h(y) - h(x)) + o(\|M_{[x]_{n_l}} h\|) \leqslant 0. \tag{4.46}$$

Using (4.44) on $h_{[x]_{n_l}}(y) - h(x)$ we get

$$\begin{aligned} \max_{y \in F_{[x]_{n_l}}} (h(y) - h(x)) &= \max_{y \in F} (h_{[x]_{n_l}}(y) - h(x)) \geqslant \frac{1}{c-1} \mathrm{Osc}_{F_w}(h_{[x]_{n_l}}) \\ &= \frac{1}{c-1} \mathrm{Osc}_{F_{[x]_{n_l+k}}}(h) \geqslant C \|M_{[x]_{n_l+k}} h\| \geqslant \frac{C}{\|M_w^{-1}\|} \|M_{[x]_{n_l}} h\|. \end{aligned} \tag{4.47}$$

So that by (4.46) we must have $\frac{df}{dh}(x) \leqslant 0$. In the same way (4.45) implies

$$\min_{y \in F_{[x]_{n_l}}} (h(y) - h(x)) \leqslant -\frac{C}{\|M_w^{-1}\|} \|M_{[x]_{n_l}} h\|, \tag{4.48}$$

which together with (4.46) implies $\frac{df}{dh}(x) \geqslant 0$. □

For the next theorem recall that a point $x \in F$ is called *periodic* if it is a fixed point of some ψ_w, $w \in W_*$.

Theorem 4.12. *Let $x = \psi_w(x) \in F$ be a periodic point. Suppose M_w has a dominating eigenvalue λ and the corresponding eigenvector is denoted by h_λ. If $|\lambda| > r_w \mu_w$ then the local derivative $\dfrac{df}{dh_\lambda}(x)$ exists for any $f \in C^1(\mathrm{Dom}\,\Delta_\mu)$. In particular, if x is a boundary fixed point then the normal derivative $\partial_N f(x)$ exists for any $f \in C^1(\mathrm{Dom}\,\Delta_\mu)$.*

Proof. In order to prove this one can adapt the proof of Theorem 4.8 defining $B_{w^n} = F_{w^{n-1}} \setminus F_{w^n}$, where $w^n = \underbrace{w \dots w}_{n \text{ times}}$ and use

$$f = \sum_{n=1}^{\infty} G u^{w^n}. \tag{4.49}$$

The condition $|\lambda| > r_w \mu_w$ is necessary to have convergence of $\sum_{n=1}^{\infty} \frac{d(Gu^{w^n})}{dh_\lambda}$.

For a boundary fixed point $x = \psi_i(x)$ this condition is always fulfilled since $\lambda = \lambda_2 = r_i$ in this case. □

The next corollary is another analog of Fermat's theorem.

Corollary 4.13. *If x is a non-boundary periodic point, the assumptions of Theorem 4.12 hold, and $f = u(g_1, \dots, g_l) \in C^1(\operatorname{Dom} \Delta_\mu)$ has a local maximum at x, then $\frac{df}{dh_\lambda}(x) = 0$.*

Proof. The proof is the same as that of Corollary 4.11 and uses Theorem 4.8 and Theorem 4.12. □

The result of Theorem 4.12 partially improves Theorem 3.2 in [3] where it was shown in the case of the Sierpiński gasket that $\partial_2 f$ and $\partial_3 f$ exist for any $f \in \operatorname{Dom} \Delta$. Namely, under the assumption that M_w has two real eigenvalues $\lambda_2 > \lambda_3$, two local derivatives at periodic points of the Sierpiński gasket were defined in [3]. If $h_2, h_3 \in \mathcal{H}$ are any harmonic functions corresponding to these eigenvalues and

$$Hf_{[x]_n} = a_{1n} + a_{2n} h_{2,[x]_n} + a_{3n} h_{3,[x]_n} \tag{4.50}$$

then

$$\partial_2 f(x) = \lim_{n \to \infty} a_{2n} \text{ and } \partial_3 f(x) = \lim_{n \to \infty} a_{3n} \tag{4.51}$$

if the limits exists. Note that the notation λ_2 for the leading eigenvalue is used in [3] because $\lambda_1 = 1$ denotes the leading eigenvalue of the matrix A_w.

For arbitrary p.c.f. fractals, local derivatives $\partial_2, \dots, \partial_{N_0}$ can be defined analogously to (4.51) at any periodic point $x = \psi_w(x)$ such that M_w has distinct real eigenvalues $|\lambda_2| > \dots > |\lambda_{N_0}|$ with corresponding harmonic functions $h_2, \dots, h_{N_0}$. Periodic points of this type are weakly generic and $\mathcal{H}_x^-$ is spanned by $h_3, \dots, h_{N_0}$, but the rate of decrease for $h \notin \mathcal{H}_x^-$ is $\|M_{[x]_n} h\| = \mathcal{O}(\sigma^n)$ for $\sigma = \lambda_2^{1/|w|}$ instead of $\mathcal{O}(\alpha_+^n)$.

It should be noted that if $x = \psi_i(x)$ is a boundary point then ∂_2 equals, for an appropriate choice of h_2, the normal derivative ∂_N. For the Sierpiński gasket, ∂_3 equals the tangential derivative ∂_T, for an appropriate choice of h_3. For periodic points on the Sierpiński gasket where M_w has two complex conjugate eigenvalues local derivatives ∂^+ and ∂^- were defined in [1] using the eigenvectors. It was also shown that there are infinitely many periodic points with this property. Such periodic points are not weakly generic. Actually for any nonconstant $h \in \mathcal{H}$, $\|M_{[x]_n} h\| = O((\sqrt{3}/5)^n)$ and h is only differentiable with respect to harmonic functions that are proportional to h. The local behavior at such points is thus truly different from the generic behavior.

5. Directions on p.c.f. fractals

In this section we prove the geography is destiny principle for large classes of functions and use it to obtain a result on the pointwise behavior of eccentricities. We begin by giving a precise formulation of the principle. It was formulated for the first time in [15] for harmonic functions on the Sierpiński gasket. For harmonic functions it holds under the SC-assumption.

For any $h \in l(V_0)$, $h \neq 0$ we define the *direction* $\mathrm{Dir} h$ as the element in the projective space $\mathbb{P}(\mathcal{H})$ corresponding to $P_{\mathcal{H}} h$. This definition extends to any function f defined on F, and nonconstant on the boundary, through $\mathrm{Dir} f = \mathrm{Dir} f|_{V_0}$. $\mathbb{P}(\mathcal{H})$.

Proposition 5.1. *Suppose F satisfies the SC-assumption. Then for any nonconstant harmonic functions $h_1, h_2 \in \mathcal{H}$*

$$\lim_{n\to\infty} \rho(\mathrm{Dir} h_1|_{F_{[x]_n}}, \mathrm{Dir} h_2|_{F_{[x]_n}}) = 0 \tag{5.1}$$

for μ a.e. x.

Proof. This follows from Theorem 2.9. □

In fact, the convergence in (5.1) is even exponential by (2.10).

If f is differentiable with respect to h with nonzero derivative at a point x, then the difference in direction of $f_{[x]_n}$ and $h_{[x]_n}$ will tend to zero. Note that by definition of the derivative, $\mathrm{Dir} f_{[x]_n}$ exists for n large enough if $\frac{df}{dh}(x) \neq 0$.

Proposition 5.2. *Suppose $\frac{df}{dh}(x)$ exists and is different from zero. Then*

$$\lim_{n\to\infty} \rho(\mathrm{Dir} f_{[x]_n}, \mathrm{Dir} h_{[x]_n}) = 0 \tag{5.2}$$

Proof. This is clear since $f(y) - f(x) = c(h(y) - h(x)) + o(\|M_{[x]_n} h\|)$ implies

$$\rho(\mathrm{Dir} f_{[x]_n}, \mathrm{Dir} h_{[x]_n}) = \rho(\mathrm{Dir}(c h_{[x]_n} + o(\|M_{[x]_n} h\|)), \mathrm{Dir} h_{[x]_n}) \to 0. \tag{5.3}$$

□

The above Proposition together with Theorem 4.8 immediately gives the following broad extension of the geography is destiny principle.

Theorem 5.3. *Suppose (F, μ) satisfies the weak main assumption and that $f \in C^1(\mathrm{Dom}\, \Delta_\mu)$ and $h \in \mathcal{H}$ is a nonconstant harmonic function. Then*

$$\lim_{n\to\infty} \rho(\mathrm{Dir} f_{[x]_n}, \mathrm{Dir} h_{[x]_n}) = 0 \tag{5.4}$$

for μ a.e. x outside the set where $\frac{df}{dh}(x) = 0$.

Remark 5.4. From (4.8) and (4.11) it follows that there is C' so that

$$\{x : \frac{df}{dh}(x) = 0\} \subset \{x : | < Hf, h_x^+ > | < C'\varepsilon\} \tag{5.5}$$

for any $f = Hf + G\Delta f$ with $\|\Delta f\|_\infty < \varepsilon$ and $\|h\| = 1$. Note that

$$\mu\{x : < Hf, h_x^+ >= 0\} = 0$$

and so informally one can write $\mu\{x : \frac{df}{dh}(x) = 0\} \to 0$ as $\|\Delta f\|_\infty \to 0$. This can be restated as follows. Given any $Hf \neq 0$ and $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$, such that

$$\mu\{x : \frac{df}{dh}(x) = 0\} < \delta(\varepsilon)$$

for any $f = Hf + G\Delta f$ with $\|\Delta f\|_\infty < \varepsilon$ and $\|h\| = 1$.

In [15] the *eccentricity* $e(h)$ of a nonconstant harmonic function h on the Sierpiński gasket were defined as

$$e(h) = \frac{h(q_1) - h(q_0)}{h(q_2) - h(q_0)}, \tag{5.6}$$

where q_i, $i = 0, 1, 2$ are the boundary points labeled so that $h(q_0) \leqslant h(q_1) \leqslant h(q_2)$. Note that the eccentricity is the same for harmonic functions corresponding to the same element in $\mathcal{H}$. The concept of eccentricity extends to any F with three boundary points and any function defined on F and nonconstant on the boundary.

It was shown in [15] that there is a measure on $[0, 1]$ such that for any nonconstant harmonic function, the distribution of eccentricities of the restrictions h_w to cells of a fixed level $|w| = n$ converges in the Wasserstein metric to this measure. This result was extended to functions with Hölder continuous Laplacian in [16].

If, instead of the global distribution of local eccentricities, we look at the behavior of the eccentricities on neighborhoods of a point, the geography is destiny principle applies. Since $e(-f) = 1 - e(f)$ we define an equivalence relation on $[0, 1]$ by $e \sim e'$ if and only if $e = e'$ or $e = 1 - e'$. We denote by $\bar{e}$ the equivalence class of e and let $d(\bar{e}, \bar{e}') = \min_{x\sim e, x'\sim e'} |x - x'|$ be the natural distance on $[0, 1]/\sim$.

Corollary 5.5. *If F satisfies the SC-assumption then for any nonconstant harmonic functions h, h'*

$$\lim_{n\to\infty} d(\bar{e}(h_{[x]_n}), \bar{e}(h'_{[x]_n})) = 0, \tag{5.7}$$

for μ a.e. x. If (F, μ) satisfies the weak main assumption then for any $f, f' \in C^1(\operatorname{Dom}\Delta_\mu)$ and nonconstant $h \in \mathcal{H}$ we have

$$\lim_{n\to\infty} d(\bar{e}(f_{[x]_n}), \bar{e}(f'_{[x]_n})) = 0 \tag{5.8}$$

for μ a.e. x outside the set where $\frac{df}{dh}$ or $\frac{df'}{dh}$ are zero.

Proof. Since $\bar{e}$ depends continuously on the direction these results follow immediately from Theorem 5.3. □

6. Derivatives and gradients

In this section we clarify the relation between the derivative and the *gradient* of a function on F defined in [21]. We will restrict attention to cases where (F, μ) satisfies the strong main assumption.

For a nonjunction point $x \in F$, let $\operatorname{Grad}_{[x]_n} f = M^{-1}_{[x]_n} P_{\mathcal{H}} H f_{[x]_n}$. The gradient of f at x is defined as

$$\operatorname{Grad}_x f = \lim_{n\to\infty} \operatorname{Grad}_{[x]_n} f, \tag{6.1}$$

if the limit exists. In [21] the gradient was defined for sequences $\omega \in \Omega$, so at junction points there are several "directional" gradients defined, but for non-junction points $\operatorname{Grad}_x f$ is defined unambiguously.

Immediately from the definition we have

Proposition 6.1. *If $h \in \mathcal{H}$ then $\operatorname{Grad}_x h$ exists for all x and $\operatorname{Grad}_x h = h$.*

In [21, Theorem 1] the following estimate was proved for any harmonic structure on a p.c.f. fractal.

$$\|\mathrm{Grad}_{[x]_{n+1}} f - \mathrm{Grad}_{[x]_n} f\| \leqslant C\|\Delta f\|_\infty r_{[x]_n}\mu_{[x]_n}\|M^{-1}_{[x]_n}\|. \tag{6.2}$$

It implies the following theorem.

Theorem 6.2. *There exists a constant C such that for any $f \in Dom\Delta$ with $\|\Delta f\|_\infty < \infty$ and any $x \in F \setminus V_*$ with*

$$\sum_{n\geqslant 1} r_{[x]_n}\mu_{[x]_n}\|M^{-1}_{[x]_n}\| < \infty, \tag{6.3}$$

$\mathrm{Grad}_x f$ *exists and*

$$\|P_{\mathcal{H}}Hf - \mathrm{Grad}_x f\| \leqslant C\|\Delta f\|_\infty \sum_{n\geqslant 1} r_{[x]_n}\mu_{[x]_n}\|M^{-1}_{[x]_n}\|. \tag{6.4}$$

Also, for any $n > 0$

$$\|P_{\mathcal{H}}Hf - \mathrm{Grad}_{[x]_n} f\| \leqslant C\|\Delta f\|_\infty \sum_{k=1}^{n} r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_k}\|. \tag{6.5}$$

From Theorem 6.2 we can immediately deduce the following lemma.

Lemma 6.3. *If (F,μ) satisfies the strong main assumption, then for any function $f \in \mathrm{Dom}\,\Delta_\mu$, $\mathrm{Grad}_x f$ exists for μ-almost all $x \in F$.*

Proof. The upper Lyapunov exponent of the matrices M_j^{-1} with respect to the measure μ is $1/\alpha_-$ and so the series (6.3) converges exponentially μ-almost everywhere. □

The next lemma uses the central limit theorem and large deviations results for products of random matrices. We will use it to show that $\mathrm{Grad}_x f$ is the unique function in $\mathcal{H}$ that best approximates f in neighborhoods of x.

Lemma 6.4. *Suppose (F,μ) satisfies the strong main assumption. Then for any $\varepsilon > 0$*

$$\sum_{k\geqslant n} r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_{n,k}}\| = o\big((\gamma+\varepsilon)^n\big)_{n\to\infty} \tag{6.6}$$

for μ a.e. x.

Proof. By the Borel-Cantelli lemma this follows if for any $\delta > 0$

$$\sum_{n=1}^{\infty} \mu\Big\{x : (\gamma+\varepsilon)^{-n} \sum_{k\geqslant n} r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_{n,k}}\| > \delta\Big\} < \infty. \tag{6.7}$$

Since $r_{[x]_n}\mu_{[x]_n} = \emptyset(\gamma^n)$ for μ a.e. x it is then enough, by Lemma 2.2 i), to show that

$$\sum_{n=1}^{\infty} \mu\left\{x : \left(\frac{\gamma-\varepsilon/2}{\gamma+\varepsilon}\right)^n \sum_{k\geqslant n} r_{[x]_{n,k}}\mu_{[x]_{n,k}}\|M^{-1}_{[x]_{n,k}}\| > \delta\right\} \tag{6.8}$$

$$= \sum_{n=1}^{\infty} \mu\left\{x : \left(\frac{\gamma-\varepsilon/2}{\gamma+\varepsilon}\right)^n \sum_{k=1}^{\infty} r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_k}\| > \delta\right\}$$

$$= \sum_{n=1}^{\infty} \mu \left\{ x : \sum_{k=1}^{\infty} r_{[x]_k} \mu_{[x]_k} \|M_{[x]_k}^{-1}\| > \delta \left(\frac{\gamma + \varepsilon}{\gamma - \varepsilon/2} \right)^n \left(\frac{1-\beta}{\beta} \right) \sum_{k=1}^{\infty} \beta^k \right\} < \infty,$$

where the first equality follows from self-similarity and $1 > \beta > \frac{\gamma}{\alpha_-}$ is a fixed number. Thus, it is enough to show that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu \left\{ x : r_{[x]_k} \mu_{[x]_k} \|M_{[x]_k}^{-1}\| > \delta \left(\frac{\gamma + \varepsilon}{\gamma - \varepsilon/2} \right)^n \left(\frac{1-\beta}{\beta} \right) \beta^k \right\} \tag{6.9}$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \mu \left\{ x : \log \left(r_{[x]_k} \mu_{[x]_k} \|M_{[x]_k}^{-1}\| \right) - k \log \left(\frac{\gamma}{\alpha_-} \right) > c_0 + nc_1 + kc_2 \right\} < \infty,$$

where $c_1, c_2 > 0$. Assuming $1 - \beta > \beta - \frac{\gamma}{\alpha_-}$ we have $c_0 + kc_2 > 0$ and the last inner sum can then be estimated from above by

$$\frac{1}{c_1} \int_{B_k} b_k(x)\, d\mu(x) \leqslant \frac{1}{c_1} \sqrt{\mu(B_k)} \|b_k(x)\|_{L^2_\mu} \tag{6.10}$$

where

$$b_k(x) = \log \left(r_{[x]_k} \mu_{[x]_k} \|M_{[x]_k}^{-1}\| \right) - k \log \left(\frac{\gamma}{\alpha_-} \right) \tag{6.11}$$

and

$$B_k = \{ x : b_k(x) > c_0 + kc_2 \}. \tag{6.12}$$

By Theorem 2.10 the L^2_μ-norm of $b_k(x)$ grows polynomially while $\mu(B_k)$ decreases exponentially, which completes the proof. □

Theorem 6.5. *Suppose (F, μ) satisfies the strong main assumption and $f \in \operatorname{Dom} \Delta_\mu$. Then for any $\varepsilon > 0$ and μ a.e. x*

$$f(y) = f(x) + \operatorname{Grad}_x f(y) - \operatorname{Grad}_x f(x) + o\big((\gamma + \varepsilon)^n\big)_{y \to x}, \tag{6.13}$$

where $y \in F_{[x]_n}$.

Proof. The proof follows the same ideas as the proof of Theorem 4.8, but is actually simpler. We assume that $f = Gu$ and let u_n be u multiplied by the indicator function of $F_{[x]_n}$. For $y \in F_{[x]_n}$ we have that

$$G(u-u_n)(y) - G(u-u_n)(x) - (\operatorname{Grad}_x G(u-u_n)(y) - \operatorname{Grad}_x G(u-u_n)(x)) = 0 \tag{6.14}$$

since $G(u - u_n)$ is harmonic on $F_{[x]_n}$. Thus, we have to show that, for $y \in F_{[x]_n}$,

$$Gu_n(y) - Gu_n(x) - (\operatorname{Grad}_x Gu_n(y) - \operatorname{Grad}_x Gu_n(x)) = o((\gamma + \varepsilon)^n). \tag{6.15}$$

Lemma 4.9 implies

$$\|Gu_n(y) - Gu_n(x)\|_{L^\infty(F_{[x]_n})} = o((\gamma + \varepsilon)^n), \tag{6.16}$$

and it follows that

$$\|\operatorname{Grad}_{[x]_n} Gu_n(y) - \operatorname{Grad}_{[x]_n} Gu_n(x)\|_{L^\infty(F_{[x]_n})} = o((\gamma + \varepsilon)^n) \tag{6.17}$$

by the maximum principle applied to the harmonic function $(\operatorname{Grad}_{[x]_n} Gu_n)_{[x]_n}$, because its boundary values coincide with those of $(Gu_n)_{[x]_n}$. Hence it suffices to bound

$$\|\operatorname{Grad}_{[x]_n} Gu_n(y) - \operatorname{Grad}_{[x]_n} Gu_n(x) - (\operatorname{Grad}_x Gu_n(y) - \operatorname{Grad}_x Gu_n(x))\|_{L^\infty(F_{[x]_n})}$$
$$\leqslant 2 \|\operatorname{Grad}_{[x]_n} Gu_n - \operatorname{Grad}_x Gu_n\|_{L^\infty(F_{[x]_n})}$$

$$
\begin{aligned}
&\leqslant 2\sum_{k=n}^{\infty} \|\mathrm{Grad}_{[x]_k} Gu_n - \mathrm{Grad}_{[x]_{k+1}} Gu_n\|_{L^\infty(F_{[x]_n})} \\
&= 2\sum_{k=n}^{\infty} \|\mathrm{Grad}_{[x]_{n,k}} (Gu_n)_{[x]_n} - \mathrm{Grad}_{[x]_{n,k+1}} (Gu_n)_{[x]_n}\|_{L^\infty(F)} \\
&\leqslant C\sum_{k=n}^{\infty} \|\Delta(Gu_n)_{[x]_n}\|_\infty r_{[x]_{n,k}}\mu_{[x]_{n,k}}\|M^{-1}_{[x]_{n,k}}\| \\
&\leqslant C\|u\|_\infty \sum_{k=n}^{\infty} r_{[x]_n}\mu_{[x]_n} r_{[x]_{n,k}}\mu_{[x]_{n,k}}\|M^{-1}_{[x]_{n,k}}\| = o((\gamma+\varepsilon)^n),
\end{aligned}
$$

where we used that $(\mathrm{Grad}_{[x]_k} Gu_n)_{[x]_n} = \mathrm{Grad}_{[x]_{n,k}}(Gu_n)_{[x]_n}$, the estimate (6.2) and Lemma 6.4. □

As an immediate consequence we obtain the following Corollary, which makes it straightforward to prove μ a.e. differentiability at points where $\mathrm{Grad}_x f$ exists.

Corollary 6.6. *Suppose (F,μ) satisfies the strong main assumption and $f \in \mathrm{Dom}\,\Delta_\mu$. Then for μ a.e. x*

$$f(y) = f(x) + \mathrm{Grad}_x f(y) - \mathrm{Grad}_x f(x) + o\big(\|M_{[x]_n} h\|\big)_{y\to x}, \tag{6.18}$$

for any nonconstant $h \in \mathcal{H}$.

The same result for $\mathrm{Grad}_x f$, or rather the tangent $T_1(f)$, on the Sierpiński gasket was proved in [19, Section 7] under the stronger assumption (3.6).

We can now state the relations between the derivative and the gradient.

Proposition 6.7. *Suppose (F,μ) satisfies the strong main assumption, $f \in \mathrm{Dom}\,\Delta_\mu$ and h is a nonconstant harmonic function. Then the following assertions hold.*

1. *For μ a.e. x such that $\mathrm{Grad}_x f = 0$, we have that $\dfrac{df}{dh}(x) = 0$.*
2. *For μ a.e. x such that $\mathrm{Grad}_x f \neq 0$, we have that $\dfrac{df}{d\mathrm{Grad}_x f}(x) = 1$.*
3. *For μ a.e. x*

$$\frac{df}{dh}(x) = \frac{< \mathrm{Grad}_x f, h_x^+ >}{< h, h_x^+ >}. \tag{6.19}$$

In particular for μ a.e. x we have

$$\frac{df}{dh_x^+}(x) =< \mathrm{Grad}_x f, h_x^+ >, \tag{6.20}$$

$$\left|\frac{df}{dh}(x)\right| = \frac{\|P_x^+ \mathrm{Grad}_x f\|}{\|P_x^+ h\|} \tag{6.21}$$

and $\frac{df}{dh}(x) = 0$ if and only if $\mathrm{Grad}_x f \in \mathcal{H}_x^-$.

Proof. The first two statements are obvious from Corollary 6.6. For the third, we know $h \notin \mathcal{H}_x^-$ for μ a.e. x, and in that case

$$
\begin{aligned}
f(y) - f(x) &= \mathrm{Grad}_x f(y) - \mathrm{Grad}_x f(x) + o\big(\|M_{[x]_n} h\|\big)_{y\to x} \\
&= \frac{< \mathrm{Grad}_x f, h_x^+ >}{< h, h_x^+ >}(h(y) - h(x)) + o\big(\|M_{[x]_n} h\|\big)_{y\to x}.
\end{aligned}
\tag{6.22}
$$

□

As formulated, Theorem 5.3 on geography is destiny, raises the question about where the derivative is different from zero. Our next results relates this to the same question on the gradient.

Lemma 6.8. *Suppose* (F,μ) *satisfies the strong main assumption. Then for any* $\varepsilon > 0$ *there is* $\delta(\varepsilon) > 0$ *with* $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ *such that if*

$$\frac{\|\Delta f\|_\infty}{\|P_{\mathcal{H}} Hf\|} < \varepsilon, \tag{6.23}$$

then

$$\mu\{x : \mathrm{Grad}_x f \in \mathcal{H}_x^-\} < \delta(\varepsilon). \tag{6.24}$$

In particular, $\mu\{x : \mathrm{Grad}_x f \neq 0\} > 1 - \delta(\varepsilon)$.

Proof. For simplicity assume $\|P_{\mathcal{H}} Hf\| = 1$ and $\|\Delta f\|_\infty < \varepsilon < \frac{1}{4}$. Define

$$F_\varepsilon = \{x \;:\; C\sum_{n\geqslant 1} r_{[x]_n}\mu_{[x]_n}\|M^{-1}_{[x]_n}\| < \varepsilon^{-\frac{1}{2}}\}, \tag{6.25}$$

where C is the constant in the estimate (6.2). Note that $\lim_{\varepsilon\to 0}\mu(F_\varepsilon) = 1$ by the strong main assumption. From (6.4) we have for any $x \in F_\varepsilon$ that

$$\|P_{\mathcal{H}} Hf - \mathrm{Grad}_x f\| \leqslant \sqrt{\varepsilon}, \tag{6.26}$$

so $\mathrm{Grad}_x f \neq 0$ and

$$\rho(\mathrm{Dir} P_{\mathcal{H}} Hf, \mathrm{Dir}\mathrm{Grad}_x f) < 2\sqrt{\varepsilon} \tag{6.27}$$

for all $x \in F_\varepsilon$. Let $V \subset \mathbb{P}(\mathcal{H})$ be the set of directions orthogonal to $P_{\mathcal{H}} Hf$, and let $V_\varepsilon = \{v_0 \in \mathbb{P}(\mathcal{H}) : \inf_{v\in V}\rho(v_0, v) < \varepsilon\}$. If $x \in F_\varepsilon$ and $\mathrm{Grad}_x f \in \mathcal{H}_x^-$ then by (6.27) we see that $\rho(\mathrm{Dir} h_x^+, v) < 2\sqrt{\varepsilon}$ for all $v \in V$. It follows that

$$\begin{aligned} \mu\{x : \mathrm{Grad}_x f \in \mathcal{H}_x^-\} &\leqslant \mu\{x \in F_\varepsilon : \mathrm{Grad}_x f \in \mathcal{H}_x^-\} + 1 - \mu(F_\varepsilon) \\ &\leqslant \mu\{x : \mathrm{Dir} h_x^+ \in V_{2\sqrt{\varepsilon}}\} + 1 - \mu(F_\varepsilon) \\ &= \nu(V_{2\sqrt{\varepsilon}}) + 1 - \mu(F_\varepsilon) \end{aligned} \tag{6.28}$$

where the measure ν is a μ-invariant measure on $\mathbb{P}(\mathcal{H})$, which means that

$$\nu(A) = \sum_{i=1}^{m}\int_{\mathbb{P}(\mathcal{H})} 1_A(\mathrm{Dir}(A_i' h))d\nu(\mathrm{Dir} h), \tag{6.29}$$

for any Borel set A in $\mathbb{P}(\mathcal{H})$. A theorem of product of random matrices says that if μ is supported on a strongly irreducible semigroup such measure ν has the property that hyperplanes have zero ν-measure [4, Proposition III.2.3]. Thus $\lim_{\varepsilon\to 0}\nu(V_{2\sqrt{\varepsilon}}) = \nu(V) = 0$. □

Theorem 6.9. *If* (F,μ) *satisfies the strong main assumption, then for any* $f \in \mathrm{Dom}\,\Delta_\mu$,

$$\mathrm{Grad}_x f \notin \mathcal{H}_x^- \tag{6.30}$$

for μ *a.e.* x *with* $\mathrm{Grad}_x f \neq 0$.

Proof. For simplicity assume $\|\Delta f\|_\infty < 1$. Define

$$F_\varepsilon = \{x \; : \; \|\mathrm{Grad}_x f\| > \varepsilon\} \tag{6.31}$$

and

$$F_{n,\varepsilon} = \{x \; : \; \|\mathrm{Grad}_{[x]_n} f\| > \varepsilon \text{ and } r_{[x]_n}\mu_{[x]_n}\|M^{-1}_{[x]_n}\| < \varepsilon^2\}. \tag{6.32}$$

Clearly

$$\lim_{n\to\infty} \mu(F_\varepsilon \setminus F_{n,\varepsilon}) = 0 \tag{6.33}$$

and

$$\lim_{\varepsilon\to 0} \mu(F_0 \setminus F_\varepsilon) = 0. \tag{6.34}$$

Then for any $x \in F_{n,\varepsilon}$ we have

$$\frac{\|\Delta f_{[x]_n}\|_\infty}{\|P_{\mathcal{H}} H f_{[x]_n}\|} = \frac{\|M^{-1}_{[x]_n}\|\|\Delta f_{[x]_n}\|_\infty}{\|M^{-1}_{[x]_n}\|\|M_{[x]_n}\mathrm{Grad}_{[x]_n} f\|} \leqslant \frac{r_{[x]_n}\mu_{[x]_n}\|M^{-1}_{[x]_n}\|}{\|\mathrm{Grad}_{[x]_n} f\|} < \varepsilon. \tag{6.35}$$

Here we can use Lemma 6.8 for each $f_{[x]_n}$ together with

$$\mathrm{Grad}_x f_{[x]_n} = M_{[x]_n}\mathrm{Grad}_{\,\psi_{[x]_n}(x)} f$$

and $M^{-1}_{[x]_n}\mathcal{H}^-_x = \mathcal{H}^-_{\psi_{[x]_n}(x)}$, to obtain that

$$\begin{aligned} \delta(\varepsilon) &> \mu\{x : \mathrm{Grad}_x f_{[x]_n} \in \mathcal{H}^-_x\} \\ &= \mu\{x : M_{[x]_n}\mathrm{Grad}_{\,\psi_{[x]_n}(x)} f \in \mathcal{H}^-_x\} \\ &= \mu\{x : \mathrm{Grad}_{\,\psi_{[x]_n}(x)} f \in M^{-1}_{[x]_n}\mathcal{H}^-_x\} \\ &= \mu\{x : \mathrm{Grad}_{\,\psi_{[x]_n}(x)} f \in \mathcal{H}^-_{\psi_{[x]_n}(x)}\} \\ &= \mu_w^{-1}\mu\{y \in F_w : \mathrm{Grad}_{\,y} f \in \mathcal{H}^-_y\}. \end{aligned} \tag{6.36}$$

Therefore,

$$\begin{aligned} &\mu\{x \in F_{n,\varepsilon} : \mathrm{Grad}_x f \in \mathcal{H}^-_x\} \\ &= \sum \mu\{x \in F_w : \mathrm{Grad}_x f \in \mathcal{H}^-_x\} < \sum \mu_w \delta(\varepsilon) = \mu(F_{n,\varepsilon})\delta(\varepsilon), \end{aligned} \tag{6.37}$$

where the sum is over all $w \in W_n$ such that $F_w \subset F_{n,\varepsilon}$. Thus,

$$\mu\{x \in F_\varepsilon : \mathrm{Grad}_x f \in \mathcal{H}^-_x\} < \limsup \mu(F_\varepsilon \setminus F_{n,\varepsilon}) + \mu(F_{n,\varepsilon})\delta(\varepsilon) < \delta(\varepsilon) \tag{6.38}$$

and

$$\mu\{x \in F_0 : \mathrm{Grad}_x f \in \mathcal{H}^-_x\} = 0. \tag{6.39}$$

□

We can now formulate geography is destiny with conditions on the gradient.

Corollary 6.10. *Suppose* (F,μ) *satisfies the strong main assumption,* $f \in \mathrm{Dom}\,\Delta_\mu$ *and* h *is a nonconstant harmonic function. Then*

$$\lim_{n\to\infty} \rho(\mathrm{Dir} f_{[x]_n}, \mathrm{Dir} h_{[x]_n}) = 0 \tag{6.40}$$

for μ *a.e.* x *where* $\mathrm{Grad}_x f \neq 0$

Proof. Theorem 6.9, Proposition 6.7 and Theorem 5.3. □

The next corollary is one more analog of Fermat's theorem.

Corollary 6.11. *Suppose* (F, μ) *satisfies the strong main assumption. Then there exists a set* F' *of full* μ*-measure such that if* $f = u(g_1, \ldots, g_l) \in C^1(\mathrm{Dom}\,\Delta_\mu)$ *has a local maximum at* $x \in F'$*, then* $\mathrm{Grad}_x f = 0$*.*

Proof. The proof is the same as that of Corollary 4.11 and uses Theorem 6.5. □

Similarly to Corollary 4.13, we can obtain an analogous corollary for nonboundary periodic points under the assumption $r_w \mu_w \|M_w^{-1}\| < 1$. The existence of the gradient in such a case is guaranteed by Theorem 6.2.

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