Self-similar energies on p.c.f. self-similar fractals

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Abstract

On a large class of p.c.f. (finitely ramified) self-similar fractals with possibly little symmetry we consider the question of existence and uniqueness of a Laplace operator. By considering positive refinement weights (local scaling factors) which are not necessarily equal we show that for each such fractal, under a certain condition, there are corresponding refinement weights which support a unique self-similar Dirichlet form. As compared to previous results, our technique allows us to replace symmetry by connectivity arguments.

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1 Introduction

A long standing question in the area of analysis on fractals was and still is the existence of a "Laplace operator" on a given self-similar set. In [18] a general theory of analysis on p.c.f. self-similar sets is developed assuming the existence of a so called nondegenerate self-similar harmonic structure, or equivalently, a self-similar Laplace operator. Thus far the highly symmetric (affine) nested fractals are the only large class of finitely ramified fractals for which the existence of a Laplacian has been established [8, 20]. As soon as the symmetry assumptions are weakened there are no general results available

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and any particular example requires the solution of a non-linear eigenvalue problem.

In this paper we mimic a classical generalization to achieve new results: Instead of dealing only with flat spaces, one considers differentiable manifolds and looks for Laplace-Beltrami operators. One possible way to rephrase this in terms of self-similar fractals is: On a given fractal we do not insist on ("flat") refinement weights which are all equal, but we consider the set of all possible refinement weights (respecting self-similarity) and ask ourselves, which of them support a unique self-similar Dirichlet form and hence an associated "Laplace-Beltrami operator". A Dirichlet form is just the abstract version of a Dirichlet integral, and a refinement weight can be interpreted as an inverse length. Our result in this paper is to provide a flexible reduction technique which implies the existence of a self-similar "Laplace-Beltrami operator", makes only very mild assumptions on the set, and allows us to replace symmetry arguments partly or entirely by connectivity arguments. The price we have to pay is only vague information about the exact numerical values of the corresponding refinement weights.

More specifically, the notion of self-similarity for a Dirichlet form is defined up to a fixed set of positive energy renormalization (refinement) weights. If a fractal F satisfies the self-similarity relation

$$F = \Psi(F) := \bigcup_{i=1}^{k} \psi_i(F) \tag{1.1}$$

where ψ_i are continuous injections (see Section 3 for a formal definition), then the self-similarity relation for a Dirichlet form $\mathcal{E}$ is

$$\mathcal{E}(u,u) = \sum_{i=1}^{k} \rho_i \, \mathcal{E}(u \circ \psi_i, u \circ \psi_i). \tag{1.2}$$

where $\rho := (\rho_1, \ldots, \rho_k)$ are positive energy refinement weights. In previous work, the usual setup was that a p.c.f. fractal and a set of positive weights are given, and then the problems are existence and uniqueness of a self-similar Dirichlet form with given weights. Substantial progress in solving these problems has been made in [24, 25, 27, 28, 31, 35]. Moreover, situations of existence and non-uniqueness were also thoroughly studied. The existing method allows us to completely analyze relatively simple examples or highly symmetric classes of fractals. But the question of whether every p.c.f. self-similar set has a self-similar Dirichlet form remains open.

In studying this question on nested fractals the refinement weights are always assumed to be equal. However there exist many examples where there is no self-similar Dirichlet form if the weights are equal (for instance, the abc gaskets of [14], Example 1.3 below or graph-directed constructions as in [12, 13, 29, 30]). When a self-similar set is considered only topologically as a shift space with an equivalence relation, then every information on relations between the refinement weights is lost.

In this paper we deal with post critically finite (p.c.f.) fractals, defined in [16]. We give conditions which ensure that there exists a nonempty relatively open set of positive refinement weights $\mathcal{R} \subset \mathbb{R}_+^k$ such that existence and uniqueness of a self-similar Dirichlet form holds for each particular choice of weights ρ in $\mathcal{R}$. As compared to previous results, our criteria require only mild assumptions on symmetry and we also drop the requirement that any point of the "boundary" of the fractal is the fixed point of one of the defining contractions of the fractal. Thus the class of fractals for which we establish existence of a "Laplace-Beltrami operator" is much larger than the class of affine nested fractals. We can even deal with finitely ramified, graph-directed constructions as we will see in Section 7. We do not know whether or not such an $\mathcal{R}$ exists for all p.c.f. sets. We also do not know of any example not supporting at least one ρ for which a (possibly nonunique) self-similar Dirichlet form exists.

The main idea of this article is to allow refinement weights $\rho \in (0, +\infty]^k$ provided not too many (in the sense of (5.1) below) of its components are infinite. In terms of electrical networks we have short circuited $\psi_i(F)$, see [4]. Mathematically this collapses $\psi_i(F)$ into a single point and, consequently, simplifies the model. A rigorous calculus based on monotone convergence is developed in [31]. It reduces the analysis of infinite energies to the classical situation of finite weights. In this article, results on partly infinite weights will carry over to entirely finite weights by the use of Dini's theorem for sequences of monotonically increasing weights. We will extensively use [25, 31].

Note that our methods do not apply to infinitely ramified fractals such as the Sierpiński carpet. "Laplace operators" and Dirichlet forms have been constructed on such sets in [2, 3] under strong symmetry assumptions. Also, our method does not allow us to consider situations of existence and non-uniqueness. This case was recently solved by R. Peirone in [33] if the boundary of a p.c.f. fractal consist of three points (the case of two point boundary is trivial).

The organization of the article is as follows. In Section 2 we state our main results. We postpone the formal definition of a p.c.f. fractal and the main technical results in order not to burden the exposition with too many details. In Section 3 we reduce the existence and uniqueness of a self-similar Dirichlet form on the fractal to a finite dimensional nonlinear eigenvalue problem for the renormalization map Λ_ρ, depending on the refinement weights

$\rho = (\rho_1, \ldots, \rho_k)$. Section 4 uses Hilbert's projective metric on cones to reformulate the existence of a unique irreducible Λ_ρ-eigenvector as a unique irreducible zero of a nonnegative functional on a bounded convex domain $\mathbb{H}$. In Section 5 we prove Theorem 4, our main technical result, as an application of Dini's theorem. Section 6 contains various applications of Theorem 4 to fractals with a single connected component. We describe classes of fractals in Section 6 for which Theorem 4 implies existence of admissible weights without any symmetry assumptions, even on the boundary. Section 7 treats a fractal with finitely many connected components, a so called graph-directed set.

In the remainder of this section we illustrate our above arguments by examples. Expert readers might just skip this part.

Example 1.1 In the case of the nonsymmetric Sierpiński gasket, every local regular self-similar Dirichlet form has refinement weights ρ satisfying

$$\begin{array}{rcl} \rho_2^{-1} + \rho_3^{-1} & > & \rho_1^{-1}, \\ \rho_1^{-1} + \rho_2^{-1} & > & \rho_3^{-1}, \\ \rho_1^{-1} + \rho_3^{-1} & > & \rho_2^{-1}. \end{array} \tag{1.3}$$

Conversely, if ρ satisfies (1.3) then there exists a unique positive real number γ, and a local regular self-similar Dirichlet form, unique up to a positive multiplier, which satisfies (1.2) with weights $\gamma\rho$ (see [35, Sect. 5.2] and Example 6.1).

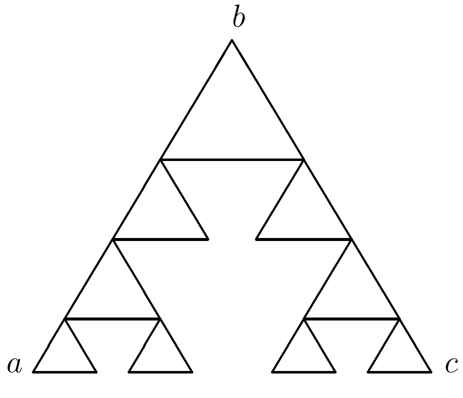

Figure 1: The second step in the construction of a symmetric cut Sierpiński gasket

Example 1.2 A cut Sierpiński gasket is a self-similar fractal which is constructed using three contractions with contraction ratios $\frac{2}{5}$, $\frac{3}{5}$ and $\frac{2}{5}$, as shown in Figure 1. We label its first generation copies clockwise starting with the bottom left triangle. Topologically, it can be obtained from the usual Sierpiński gasket by separating two smaller copies of the Sierpiński gasket at the junction point, and repeating it by self-similarity.

This fractal is a dendrite (that is, a topological tree). Its harmonic structures are investigated in [17]. A local regular self-similar Dirichlet form exists if and only if ρ satisfies

$$\rho_1{}^{-1} + \rho_2{}^{-1} = \rho_2{}^{-1} + \rho_3{}^{-1} = 1, \tag{1.4}$$

but it is not unique up to a positive multiplier. Thus the conditions for the energy refinement weights are quite different from the case of the Sierpiński gasket in Example 1.1. In particular, the set of weights for the cut Sierpiński gasket is a one dimensional curve, while it is a two dimensional surface for the Sierpiński gasket.

The main idea of obtaining (1.4) is as follows. The fractal F contains straight line segments from a to b and from c to b. According to [17], having a self-similar Dirichlet form on F is equivalent to defining a self-similar Dirichlet form on these two segments and then transferring it by self-similarity to other parts of the fractal, which in this case are also straight line segments of various lengths. Then the well known formula for the conductance of resistors in line gives (1.4). Note that (1.4) defines a self-similar Dirichlet form on each of the two straight line segments from a to b and from c to b up to two positive multipliers. Therefore for any fixed choice of ρ satisfying (1.4) the self-similar Dirichlet form is not unique up to a positive multiplier.

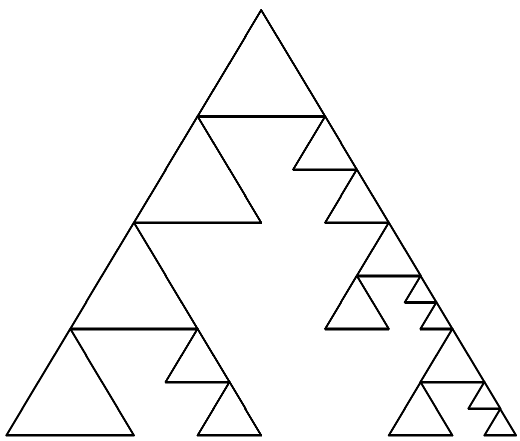

Figure 2: The second step in the construction of a non-symmetric cut Sierpiński gasket

Example 1.3 A non-symmetric cut Sierpiński gasket is a self-similar fractal which is constructed using four contractions with contraction ratios $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$, as shown in Figure 2. It can be obtained from the usual Sierpiński gasket by removing one of the nine triangles in the second step of the standard construction of the Sierpiński gasket, and repeating this procedure by self-similarity. Again we label the copies of the first generation approximation clockwise starting at the bottom left triangle. As in the Example 1.2, this fractal is a dendrite. The condition for the energy refinement weights are

$$\rho_1{}^{-1} + \rho_2{}^{-1} = \rho_2{}^{-1} + \rho_3{}^{-1} + \rho_4{}^{-1} = 1.$$

In particular, it is not possible to have equal weights.

2 Main results.

According to the definition (see the next section and [16, 18]), a p.c.f. self-similar set F satisfies (1.1), and there exists a finite subset $V_0 \subset F$ such that

$$\psi_v(F) \cap \psi_w(F) \subseteq \psi_v(V_0) \cap \psi_w(V_0), \tag{2.1}$$

for any two different words v and w of the same length. Here for any finite word $w \in \{1, \ldots, k\}^m$ we define $\psi_w := \psi_{w_1} \circ \ldots \circ \psi_{w_m}$ and call $F_w = \psi_w(F)$ an m-cell. It is assumed that V_0 is the minimal subset of F with this property. Since $\Psi(V_0)$, defined as in (1.1), separates the copies $\psi_1(F), \ldots, \psi_k(F)$ of the fractal, V_0 is often called the boundary of F.

It is well known that for any choice of positive measure weights $\mu_1, \ldots, \mu_k$ with $\sum_{i=1}^k \mu_i = 1$, there is a self-similar (Bernoulli) probability measure μ on F, which is uniquely determined by the self-similarity condition

$$\int_F u d\mu = \sum_{i=1}^k \mu_i \int_F u \circ \psi_i d\mu$$

(see, for instance, [18, Prop. 1.4.3]). If we consider the analogous question for a Dirichlet form, then the following definition is natural.

Definition 2.1 *We call positive energy refinement weights $\rho := (\rho_1, \ldots, \rho_k)$ admissible if there exist a unique positive real number γ such that the following holds. There is an irreducible, local, regular, self-similar Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(F, \mu)$, unique up to a positive multiplier, which gives positive capacity to all points and satisfies the $\gamma\rho$-self-similarity identity*

$$\mathcal{E}(u, u) = \gamma \sum_{i=1}^k \rho_i \, \mathcal{E}(u \circ \psi_i, u \circ \psi_i). \tag{2.2}$$

Note that if the weights ρ are admissible then $\gamma\rho \in (1, +\infty)^k$, according to [16, 18], which means that the so called harmonic structure is regular. Also note that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ does not depend on the choice of the measure μ. In fact, μ can be any finite Borel measure which is positive on each nonempty open subset of F (see [16, 18]). In Section 3 we will reduce the question of admissibility to the existence of a simple nonlinear eigenvector of a certain nonlinear renormalization map Λ_ρ.

We assume that a group $\mathfrak{G}$, which can be trivial, consisting of continuous bijections acts on the fractal F in a way which is compatible with the self-similar structure. This means that

$$\forall (\mathfrak{g}, i) \in \mathfrak{G} \times \{1, \ldots, k\}\, \exists (\mathfrak{h}, j) \in \mathfrak{G} \times \{1, \ldots, k\} : \mathfrak{g} \circ \psi_i = \psi_j \circ \mathfrak{h}. \tag{2.3}$$

We also assume that in this case $\mu_i = \mu_j$ and $\rho_i = \rho_j$. The latter implies $\mathcal{E}(u, u) = \mathcal{E}(u \circ \mathfrak{g}, u \circ \mathfrak{g})$, for all $\mathfrak{g} \in \mathfrak{G}$. Thus μ and $\mathcal{E}$ are also $\mathfrak{G}$-invariant.

Our main technical result, Theorem 4 in Section 5, is stated in terms of the nonlinear renormalization map Λ_ρ. For a p.c.f. self-similar set it implies the following result.

Remark 2.2 *When some weights in ρ are infinite we have to modify the notion of a regular Dirichlet form. We introduce the space of continuous functions $C_\rho(F)$ consisting of functions which are constant on F_w whenever the word w contains an i with $\rho_i = \infty$. When (5.1) is satisfied then $C_\rho(F)$ still separates the points of V_0.*

Theorem 1 *If there is a ($\mathfrak{G}$-invariant) admissible weight $\rho \in (0, +\infty]^k$ satisfying (5.1) then there exists a non-void, open set $\mathcal{R} \subset \mathbb{R}_+^k$ of ($\mathfrak{G}$-invariant) admissible weights (near ρ).*

This theorem can be restated as follows. *If, by collapsing a subset of cells of F ($\mathfrak{G}$-invariantly), one can obtain a structure which has admissible weights, then F also has admissible ($\mathfrak{G}$-invariant) finite weights.* Naturally, the possibility to collapse big parts of the fractal significantly simplifies the problem under consideration. Note though that the collapsed structure may not be a p.c.f. fractal. However there is still an associated self-similar graph to which our methods apply. The Theorems 1 and 4 will be generalized to finitely ramified, graph-directed fractals in Section 7.

Definition 2.3 *We call F* connected *when every pair of different points $x, y \in V_0$ is connected by a chain of overlapping* 1*-cells, that is, there exist $i_1, \ldots, i_r \in \{1, \ldots, k\}$ such that $x \in \psi_{i_1}(V_0)$, $y \in \psi_{i_r}(V_0)$ and $\psi_{i_j}(V_0) \cap \psi_{i_{j+1}}(V_0) \neq \emptyset$ for $1 \leq j < r$ (cf. [18, Cor. 1.6.5]).*

An m-cell is called a boundary m-cell *if it intersects V_0. Otherwise it is called an* interior m-cell.

*We say that F has m-*connected interior *if the set of interior m-cells is connected (by chains of m-cells as above), any boundary m-cell contains exactly one point of V_0, and the intersection of two different boundary m-cells is contained in an interior m-cell.*

We say that F has m-admissible weights if the refinement of F by Ψ^m has admissible weights. Now Theorem 4 in Section 5 implies

Theorem 2 *If $\mathfrak{G}$ acts transitively on the boundary of F, and F has m-connected interior, then m-admissible weights exist.*

Fortunately, any connected p.c.f. self-similar fractal F, with a group acting transitively on V_0, can be refined by some Ψ^m such that a connected interior appears by Lemma 6.4. Thus

Corollary 2.4 *When $\mathfrak{G}$ acts transitively on V_0, then a connected p.c.f. self-similar fractal F has a refinement with admissible weights.*

The following definition requires symmetries only among the boundary cells but allows a trivial $\mathfrak{G}$.

Definition 2.5 *The boundary of F is called* symmetric *if for some $m > 0$ and for any two boundary m-cells $\psi_u(F), \psi_v(F)$ the bijection $\psi_v\psi_u^{-1} : \psi_u(F) \to \psi_v(F)$ maps points of V_0 to points of V_0, and points of interior m-cells to points of interior m-cells.*

Theorem 2 can be generalized to this setting.

Theorem 3 *If the p.c.f. self-similar set F has m-connected interior and a symmetric boundary, then F has m-admissible weights.*

3 Definitions and reduction of the problem

The following definition of a p.c.f. self-similar set is due to Kigami [16, 18].

Definition 3.1 *Let F be a compact metrizable topological space and $S = \{1, ..., k\}$. Then $(F, S, \{\psi_i\}_{i\in S})$ is called a self-similar structure if $\psi_i : F \to F$ is a continuous injections for any $i \in S$, and there is a continuous surjection $\pi : S^{\mathbb{N}} \to F$ such that $\psi_i \circ \pi = \pi \circ \sigma_i$, where $\sigma_i(w_1w_2...) = iw_1w_2$.*

As before, for any finite word $w \in \{1, \ldots, k\}^m$ we define $\psi_w := \psi_{w_1} \circ \ldots \circ \psi_{w_m}$ and call $F_w = \psi_w(F)$ an m-cell.

The set $\mathcal{C} = \pi^{-1}\Big(\bigcup_{i,j\in S, i\neq j}\big(F_i \cap F_j\big)\Big)$ *is called the critical set, and* $\mathcal{P} = \bigcup_{n\geqslant 1} \sigma^n \mathcal{C}$ *is called the post critical set. Here* σ *is the shift map on* $S^{\mathbb{N}}$*.*

The fractal F*, or more precisely the self-similar structure* $(F, S, \{\psi_i\}_{i\in S})$*, is called post critically finite, or p.c.f. for short, if* $\mathcal{P}$ *is finite. Finally, the boundary of* F *is defined as* $V_0 = \partial F = \pi(\mathcal{P})$*.*

Let us call $V_n := \Psi^n(V_0)$ the vertices of generation $n \in \mathbb{N}$. The vertex sets V_0 and V_1 together with $\psi_1, \dots, \psi_k$ define the "combinatorial skeleton" or "ancestor" of F. The fractal F can be reconstructed from these data [16, App. A]. Our arguments in this article depend exclusively on these data and on the symmetry group $\mathfrak{G}$.

Suppose we have a conductance on V_0, that is, a function $c_0 : V_0^2 \to \mathbb{R}_+$ which is symmetric, vanishes on the diagonal and is $\mathfrak{G}$-invariant. The latter requires $c_0(\mathfrak{g}(x), \mathfrak{g}(y)) = c_0(x, y)$ for all $x, y \in V_0$ and $\mathfrak{g} \in \mathfrak{G}$. For c_0 and $f : V_0 \to \mathbb{R}$ we define the following quadratic form, or energy for short,

$$\mathcal{E}_0(f) := \frac{1}{2} \sum_{x,y\in V_0} (f(y) - f(x))^2 c_0(x, y). \tag{3.1}$$

By polarization it determines a bilinear form which is $\mathfrak{G}$-invariant and a discrete Dirichlet form in the sense of [9]. Let $\mathbb{D}$ be the cone of $\mathfrak{G}$-invariant Dirichlet forms on V_0 of the form (3.1). Let $1_x : V_0 \to \mathbb{R}$ denote the characteristic function of $x \in V_0$. The equation $c_0(x, y) = -\mathcal{E}_0(1_x, 1_y)$, for $x \neq y \in V_0$, defines a linear isomorphism $\mathcal{E}_0 \mapsto c_{\mathcal{E}_0}$ between Dirichlet forms and conductances [24, Sect. 2]. Hence we can represent $\mathbb{D}$ by a cone of conductances.

We choose $\mathfrak{G}$-invariant refinement weights $\rho := (\rho_1, \dots, \rho_k) \in (0, \infty)^k$ to define the refinement map Ψ_ρ for $\mathcal{E}_0$ and $f : V_1 \to \mathbb{R}$,

$$\mathcal{E}_1(f) := \Psi_\rho(\mathcal{E}_0)(f) := \sum_{i=1}^{k} \rho_i \cdot \mathcal{E}_0(f \circ \psi_i). \tag{3.2}$$

We force $\mathcal{E}_1$ to be $\mathfrak{G}$-invariant in assuming (2.3). Furthermore, we assume F to be connected. Disconnected p.c.f. fractals will be treated in Section 7.

For $\mathcal{E}_0 \in \mathbb{D}$ consider $\mathcal{E}_1$ and $f : V_0 \to \mathbb{R}$ to define the trace map $\mathrm{Tr}_{V_0} =: \mathrm{Tr}$,

$$\mathrm{Tr}(\mathcal{E}_1)(f) := \inf\{\mathcal{E}_1(g) | g : V_1 \to \mathbb{R}, g|_{V_0} = f\}. \tag{3.3}$$

Since $\mathcal{E}_1(\cdot)$ is a positive semidefinite quadratic form, a minimizing element g of the variational problem (3.3) always exists [34, Thm. 12.3]. The Dirichlet principle, [18, Thm. 2.1.6], tells us that a minimizing element solves the $\mathcal{E}_1$-Dirichlet problem on the "open set" $V_1 \setminus V_0$ for the "boundary data" f on the "boundary" V_0. We know that $\mathcal{E}_1$ is $\mathfrak{G}$-invariant, and thus $\mathrm{Tr}(\mathcal{E}_1)$ is too.

Finally, the renormalization map is $\Lambda_\rho := \mathrm{Tr} \circ \Psi_\rho$. It maps $\mathbb{D}$ into itself [27, Prop. 2.1(a)]. This map was first considered in this context in [14].

When $\mathcal{E}_0$ is an eigenvector of Λ_ρ then it is called ρ-self-similar. For $i = 1, 2$ an energy $\mathcal{E}_i$ is termed irreducible when it vanishes only on constants, that is, its kernel is minimal. When we define the graph $\Gamma(\mathcal{E}_i) := (V_i, E_i)$ by the edge set $E_i := \{\{x, y\} \subset V_i | c_{\mathcal{E}_i}(x, y) > 0\}$, then the minimum principle in [18, Thm. 3.2.5] shows that $\mathcal{E}_i$ is irreducible if and only if $\Gamma(\mathcal{E}_i)$ is connected. The cone $\mathbb{D}$ spans the real vector space $\mathbb{B} := \mathbb{D} - \mathbb{D}$ which we endow with the norm $\|\mathcal{E}_0\|_{\mathbb{B}} := \sup\{|\mathcal{E}_0(f)|; \|f\| = 1\}$. Set $\mathbb{P} := \{\mathcal{E} \in \mathbb{B} | \mathcal{E}(\cdot) \geq 0\}$ and call it the cone of positive semidefinite forms. Its interior $\mathbb{P}^\circ$ consists of forms with minimal kernel. Since Λ_ρ is positively homogeneous, we can restrict this map to the open, bounded, convex set $\mathbb{H} := H \cap \mathbb{D} \cap \mathbb{P}^\circ$.

Proposition 3.2 *A collection of refinement weights ρ is admissible if and only if*

$$\Lambda_\rho(\mathcal{E}) = \gamma \mathcal{E}$$

has a unique solution in $\mathbb{H}$ and $\gamma\rho \in (1, +\infty)^k$.

Proof: See Chapter 3 in [18] and especially [18, Prop. 3.1.3]. □

This result originated in [14], and in the context of p.c.f. fractals was first proved in [16]. It plays an important role in [24, 25, 27, 28, 31, 35] and other works.

4 Self-similar energies

A unique irreducible Λ_ρ-eigenvector will turn out to be the unique zero of a functional q_ρ, derived from a "hyperbolic" distance h on a projective space $\mathbb{H}$.

The cone $\mathbb{P}$ defines a partial ordering $\mathcal{E} \leq \mathcal{F}$ on $\mathbb{B}$ by $\mathcal{F} - \mathcal{E} \in \mathbb{P}$. It is the pointwise ordering of positive semidefinite quadratic forms. When the minimizing element of (3.3) is ambiguous, that is $\Gamma(\mathcal{E}_1)$ is reducible (not irreducible), we choose the unique one with minimal Euclidean norm and denote it by $H^{\mathcal{E}_1}_{V_1 \setminus V_0} f$.

Proposition 4.1 *Suppose F is connected and (2.3) holds. Then the following statements hold for $\rho \in (0, \infty)^k$.*

1. $\Lambda_\rho : \mathbb{P} \to \mathbb{P}, \mathbb{P}^\circ \to \mathbb{P}^\circ$.
2. $\Lambda_\rho(\alpha\mathcal{E}) = \alpha\Lambda_\rho(\mathcal{E})$ *and* $\Lambda_\rho(\mathcal{E} + \mathcal{F}) \geq \Lambda_\rho(\mathcal{E}) + \Lambda_\rho(\mathcal{F})$ *for all* $\mathcal{E}, \mathcal{F} \in \mathbb{P}$ *and* $\alpha \geq 0$.

3. *Λ_ρ is continuous on $\mathbb{D} \cup \mathbb{P}^\circ$.*

Proof: 1. The definitions imply $\Lambda_\rho : \mathbb{P} \to \mathbb{P}$. Let $\mathcal{M} \in \mathbb{D}$ be defined by the conductance which is identically 1. Suppose $0 = \Lambda_\rho(\mathcal{M})(f)$. Then $\mathcal{M}_1(H^{\mathcal{M}_1}_{V_1\setminus V_0} f) = 0$ and $H^{\mathcal{M}_1}_{V_1\setminus V_0} f$ must be constant by the connectedness of F and the minimum principle. Thus f is constant. For $\mathcal{E} \in \mathbb{P}^\circ$ there exists $\alpha > 0$ such that $\mathcal{M} \leq \alpha\mathcal{E}$.

2. This follows from the linearity of Ψ_ρ, the positive homogeneity of Tr and the fact that Tr is an infimum.

3. See [25, Thm. 2.2(h)] or the proof of Lemma 5.3 below. □

On $\mathbb{P}^\circ$ we define Hilbert's projective metric h. For $\mathcal{E}, \mathcal{F} \in \mathbb{P}^\circ$ there exists the biggest lower bound of $\mathcal{E}/\mathcal{F}$,

$$m(\mathcal{E}/\mathcal{F}) := \sup\{\alpha > 0 | \alpha\mathcal{F} \leq \mathcal{E}\} \in (0, +\infty).$$

Obviously the smallest upper bound $M(\mathcal{E}/\mathcal{F})$ equals $m(\mathcal{F}/\mathcal{E})^{-1}$. Define

$$h(\mathcal{E}/\mathcal{F}) := \ln \frac{M(\mathcal{E}/\mathcal{F})}{m(\mathcal{E}/\mathcal{F})}.$$

It is a pseudo distance on $\mathbb{P}^\circ$. For $\mathcal{E} \in \mathbb{B}$ we define $\mathrm{trace}(\mathcal{E}) := \sum_{x\in V_0} \mathcal{E}(1_x)$. It defines the affine hyperplane $H := \{\mathcal{E} \in \mathbb{B} | \mathrm{trace}(\mathcal{E}) = 1\}$.

Proposition 4.2 [25, Sect. 3] *On $\mathbb{P}^\circ$ Hilbert's projective metric satisfies:*

1. *$h(\alpha\mathcal{E}, \beta\mathcal{F}) = h(\mathcal{E}, \mathcal{F})$ for all $\alpha, \beta > 0$. Furthermore, $h(\mathcal{E}, \mathcal{F}) = 0$ if and only if there exists $\alpha > 0$ such that $\mathcal{E} = \alpha\mathcal{F}$.*
2. *For every $\mathcal{E}, \mathcal{F} \in \mathbb{P}^\circ$ the distance $h(\mathcal{E}, \mathcal{F})$ tends to ∞ when $\mathcal{F}$ tends to $\partial\mathbb{P}$.*
3. *$(H \cap \mathbb{P}^\circ, h)$ is a complete metric space.*
4. *The h- and the $\|\cdot\|$-topology coincide locally on $H \cap \mathbb{P}^\circ$.*

We want to find Λ_ρ-eigenvectors via the map $q_\rho : \mathbb{H} \to \mathbb{R}_+$,

$$q_\rho(\mathcal{E}) := h(\Lambda_\rho(\mathcal{E}), \mathcal{E}).$$

Denote the closed h-ball of radius $r > 0$ centered at $\mathcal{E} \in \mathbb{P}^\circ$ by $B_r(\mathcal{E})$.

Proposition 4.3 *Suppose F is connected and (2.3) holds. Then the following statements hold true for $\rho \in (0, \infty)^k$.*

1. *Λ_ρ is h-nonexpansive on $\mathbb{P}^\circ$, that is, lower q_ρ-level sets are Λ_ρ-invariant.*

2. *Λ_ρ has a unique eigenvector $\mathcal{F} \in \mathbb{H}$ if and only if $q_\rho|_\mathbb{H}$ vanishes only at $\mathcal{F}$.*

3. *Λ_ρ has multiple eigenvectors in $\mathbb{H}$ if and only if q_ρ vanishes on a connected set which accumulates at $\partial\mathbb{P}$.*

4. *When a Λ_ρ-forward orbit started in $\mathbb{H}$ is contained in $B_r(\mathcal{E})$ for some $r > 0$ and $\mathcal{E} \in \mathbb{H}$, then there exists a Λ_ρ-eigenvector in $B_{3r}(\mathcal{E}) \cap \mathbb{H}$.*

Proof: Statement 1 is Proposition 4.1 of [25]. Statement 2 is a consequence of Proposition 4.2(1). Observation 3 is basically due to Sabot [35]. The present version is Proposition 9 of [28]. Finally, Result 4 is a consequence of the proof of [32, Thm. 4.1]. □

5 Varying the refinement weights continuously

We will vary $\rho \in (0,\infty)^k$ continuously to see what happens to the Λ_ρ-eigenvalues. We even allow a limit in $(0,\infty]^k$, that is, we consider sequences increasing to infinity in some components. Since Tr is monotone there is no problem in defining the limit. Even better, as the components at infinity will have been short circuited the corresponding model will have a simpler structure.

Let $C(\mathbb{H})$ be the set of continuous real valued functions on $\mathbb{H}$ and denote the uniform norm on $C(\mathbb{H})$ by $\|\cdot\|_\infty$.

Proposition 5.1 *Let $(\rho_n)_n \subset (0,\infty)^k$ be such that $(\Lambda_{\rho_n})_n$ converges to Λ in $(C(\mathbb{H}), \|\cdot\|_\infty)$ for $n \to \infty$. When $q := h(\Lambda(\cdot),\cdot) : \mathbb{H} \to \mathbb{R}_+$ vanishes only at a single point, then there exists an $m \in \mathbb{N}$ such that Λ_{ρ_n} has a unique eigenvector in $\mathbb{H}$, for $n \geq m$.*

Proof: According to Proposition 4.2(3) the metric h is $\|\cdot\|$-continuous on $\mathbb{H}$. Since $\mathbb{B}$ is locally $\|\cdot\|$-compact, h is even locally uniformly continuous. The convergence of $(\Lambda_{\rho_n})_n$ to Λ is uniform by assumption. So $(q_{\rho_n})_n$ converges to q locally uniformly.

Again by assumption q vanishes in a point, say $\mathcal{F}$. By the first paragraph, there exists an $m \in \mathbb{N}$ such that for all $n \geq m$ we have $q_{\rho_n}(\mathcal{F}) \leq 1$. Proposition 4.3(1) shows that 1 is an upper bound for every step length $h(\Lambda_{\rho_n}^{l+1}(\mathcal{F}), \Lambda_{\rho_n}^{l}(\mathcal{F}))$, $l \in \mathbb{N}$. The continuity and positivity of q imply that it is bounded below by some $\varepsilon > 0$ on the annulus $A : \mathrm{cl}(B_3(\mathcal{F}) \setminus B_1(\mathcal{F}))$. The first paragraph shows that there exists an $m' \geq m$ such that q_{ρ_n} lies in the $\varepsilon/3$-neighborhood of q for all $n \geq m'$. Hence for $n \geq m'$, q_{ρ_n} is less than $\varepsilon/3$ in $\mathcal{F}$ and bounded below on A by $2\varepsilon/3$.

Fix $n \geq m'$ and consider a Λ_{ρ_n}-forward orbit started in $\mathcal{F}$. It is started on a $\varepsilon/3$-lower level set L of q_{ρ_n} and it stays in L forever by Proposition 4.3(1). But $L \cap A = \emptyset$ and Λ_{ρ_n} cannot jump across A, because its maximal step size is 1. Thus the orbit has to stay inside $B_1(\mathcal{F})$. Now Proposition 4.3(4) implies the existence of a Λ_{ρ_n}-eigenvector in $B_3(\mathcal{F})$. By the definition of A the fixed point is actually an element of $B_1(\mathcal{F})$. Suppose Λ_{ρ_n} has multiple fixed points. Then Proposition 4.3(3) and 4.2(2) imply that A intersects the 0-level line of q_{ρ_n} which is a contradiction. $\square$

Now we are able to prove our first result on admissible refinement weights.

Corollary 5.2 *The set of admissible refinement weights in* $(0, \infty)^k$ *is open.*

Proof: Let $(\rho_n)_n \subset (0, \infty)^k$ converge to $\rho \in (0, \infty)^k$. For every $0 < \varepsilon < 1$ there exists an $m \in \mathbb{N}$ such that $(1-\varepsilon)\rho \leq \rho_n \leq (1+\varepsilon)\rho$ for all $n \geq m$. Thus $(1-\varepsilon)\Lambda_\rho \leq \Lambda_{\rho_n} \leq (1+\varepsilon)\Lambda_\rho$ pointwise. This is the uniform convergence of $(\Lambda_{\rho_n})_n$ to Λ_ρ on $\mathbb{H}$ we need in Proposition 5.1. $\square$

Consider $(\rho_n)_n \subset (0, \infty)^k$ converging to $\rho \in (0, \infty]^k$. We write $\rho_n \leq \rho_{n+1}$ when $\rho_{n+1} - \rho_n \in [0, \infty)^k$. When this is true for all $n \in \mathbb{N}$ we call the sequence (component-wise) increasing. Applying the classical conventions of measure theory concerning the calculus of $+\infty$, we can easily define Ψ_ρ, Tr and Λ_ρ. For $\mathcal{E} \in \mathbb{H}$ some edges of $\Gamma(\Psi_\rho(\mathcal{E}))$ have an infinite conductance. The $\Psi_\rho(\mathcal{E})$-energy is finite if and only if the function in question is constant on every edge of infinite conductance. Thus the study of finite energies reduces to the study of functions which are constant on all infinite edges. This is the point of view of "short circuited electrical networks". Its physical aspects are nicely explained in [4]. A rigorous calculus in the present framework can be found in [31, Sect. 4]. The next lemma tells us how to keep all $\Lambda_\rho(\cdot)$-energies finite.

Lemma 5.3 *Let* $\rho \in (0, \infty]^k$ *and* $\mathcal{E} \in \mathbb{H}$. *Suppose:*

(5.1) *Every path in* $\Gamma(\Psi_\rho(\mathcal{E}))$ *which connects two distinct points of* V_0 *contains at least one edge with finite conductance.*

Then $\Lambda_\rho(\mathcal{E})(f) < \infty$, *for all* $f : V_0 \to \mathbb{R}$. *Furthermore,* $\Lambda_\rho \in C(\mathbb{D} \cup \mathbb{P}^\circ)$.

Proof: Let $f : V_0 \to \mathbb{R}$. By (5.1) it is possible to find a function $g : V_1 \to \mathbb{R}$ which is constant on all edges of $\Gamma(\Psi_\rho(\mathcal{E}))$ with infinite conductance but still satisfies $g|_{V_0} = f$. Hence Tr gives us a quadratic form which is finite by definition.

The continuity of Λ_ρ is a variant of the proof of [25, Thm. 2.2(h)]. Use the minimum principle to verify for $\mathcal{E}, \mathcal{F} \in \mathbb{H}$,

$$\begin{aligned}\Lambda_\rho(\mathcal{E})(f) - \Lambda_\rho(\mathcal{F})(f) &= \mathcal{E}_1(H^{\mathcal{E}_1}_{V_1\setminus V_0} f) - \mathcal{F}_1(H^{\mathcal{F}_1}_{V_1\setminus V_0} f) \\ &\leq (\mathcal{E}_1 - \mathcal{F}_1)(H^{\mathcal{F}_1}_{V_1\setminus V_0} f).\end{aligned}$$

Since $(\mathrm{Tr} \circ \Psi_\rho)(\mathcal{F})(f) < \infty$ by the first paragraph, $H^{\mathcal{F}_1}_{V_1\setminus V_0} f$ must be constant on all edges of $\Gamma(\Psi_\rho(\mathcal{F})) = \Gamma(\Psi_\rho(\mathcal{F}))$ with infinite conductance. Thus we can contract every connected region of such edges to a single point (new vertex). Eventually this procedure produces multiple edges with finite conductances. In this case we add up the finite values to form a new single edge. This does not change the energy [28, Lem. 17]. Now the joint infinities of $\mathcal{E}_1$ and $\mathcal{F}_1$ do not matter anymore and we can use the classical minimum principle to conclude that $H^{\mathcal{F}_1}_{V_1\setminus V_0} f$ is bounded by $\|f\|_\infty$. The convergence of the finite conductances gives the desired result. □

We prove the main result of this article.

Theorem 4 *Suppose ρ satisfies (5.1) and Λ_ρ has a unique eigenvector in $\mathbb{H}$. Then there exist finite admissible refinement weights.*

Proof: Let $(\rho_n)_n \subset (0,\infty)^k$ be an increasing sequence converging to $\rho \in (0,\infty]^k$. Consider the map $n \to \Lambda_{\rho_n}(\mathcal{E})(f)$ on the compact set

$$K := \overline{\mathbb{H}} \times \{f : V_1 \to \mathbb{R}; \|f\| = 1\}.$$

Since $(\rho_n)_n$ is increasing, the sequence $(\Lambda_{\rho_n}(\cdot)(\cdot))_n \subset C(K)$ also is. It increases to $\Lambda_\rho(\cdot)(\cdot) \in C(K)$ according to Lemma 5.3. Thus Dini's theorem implies the uniform convergence we need in Proposition 5.1. □

Assumption (5.1) in Theorem 4 guarantees $\Lambda_\rho(\mathcal{E})(f) < \infty$ for all $\mathcal{E} \in \mathbb{H}$ and $f : V_0 \to \mathbb{R}$. In the proof of Lemma 5.3 we saw that consequently $H^{\mathcal{E}_1}_{V_1\setminus V_0} f$ has to be constant on all edges of $\Gamma(\Psi_\rho(\mathcal{E}))$ with infinite conductance. Thus every connected region of such edges collapses into a single point (new vertex). This can substantially simplify the study of Λ_ρ.

Finally, we prepare a nonlinear version of Frobenius' famous theorem on the leading eigenvector of a nonnegative square matrix. Proposition 4.1(1) shows that we can consider the normalized version $\tilde{\Lambda}_\rho$ of Λ_ρ which stays on $\mathbb{H}$. The $\mathbb{P}$-part of $\mathcal{E} \in \mathbb{P} \setminus \{0\}$ is the set of all $\mathcal{A} \in \mathbb{P}$ for which there exists an $\alpha > 0$ such that $\frac{1}{\alpha}\mathcal{E} \leq \mathcal{A} \leq \alpha\mathcal{E}$. The $\mathbb{D}$-part of $\mathcal{E} \in \mathbb{D} \setminus \{0\}$ is the set of all $\mathcal{A} \in \mathbb{D}$ for which there exists an $\alpha > 0$ such that $\frac{1}{\alpha} c_{\mathcal{E}} \le c_{\mathcal{A}} \le \alpha c_{\mathcal{E}}$. Both types of parts are Λ_ρ-invariant if and only if they contain at least one element whose Λ_ρ-image lies again in the same part according to Lemma 16 and Corollary 4 of [31].

Proposition 5.4 (Frobenius) *Let $\rho \in (0,\infty]^k$ satisfy (5.1). When $\mathbb{D}^\circ$ is the only Λ_ρ-invariant $\mathbb{D}$-part, then Λ_ρ has a unique (up to positive multiples) eigenvector in $\mathbb{H} \cap \mathbb{D}^\circ$.*

Proof: Even when ρ has some infinite components but satisfies (5.1) it has nice properties, as listed in Proposition 4.1, according to Lemma 5.3.

Existence: The invariance of $\mathbb{D}^\circ$ implies the invariance of $\mathbb{P}^\circ$ which implies the existence of an eigenvector $\mathcal{F}$ with strictly positive eigenvalue in $\mathbb{D} \cap \mathbb{P}^\circ$ [19, Thm. 9.1]. Its $\mathbb{D}$-part must be invariant. So our assumptions imply $\mathcal{F} \in \mathbb{D}^\circ$.

Uniqueness: Suppose there exist different Λ_ρ-eigenvectors in $\mathbb{H}$. Then there exists a path of Λ_ρ-eigenvectors approaching $\partial\mathbb{D}$ by Proposition 4.3(3). They all have the same eigenvalue [25, Prop. 4.2]. Using the continuity of Λ_ρ we find an eigenvector at $\partial\mathbb{D}$. Again its $\mathbb{D}$-part must be invariant. This contradicts our assumptions. □

6 Applications of the main theorem

The main difficulty in finding Λ_ρ-eigenvectors in $\mathbb{H}$ lies in the fact that this set is relatively open. In a first step we will use Theorem 4 to simplify the eigenvalue problem. To the simplified problem we will, in a second step, apply established methods to find eigenvectors.

The idea of the next example is to collapse all interior cells in order to come up with a simplified configuration which also appears for a well studied fractal. Thus the simplified eigenvalue problem might have been solved already, that is, has a unique solution. In the latter case Theorem 4 provides finite admissible refinement weights. Of course this just indicates that there are admissible ($\mathfrak{G}$-invariant) weights which determine a non-degenerate self-similar Dirichlet form. This does not indicate that there exists an "optimal" set of weights which may reflect the geometry of the fractal in maximal detail. But whenever we embed a fractal in a space to "see" its symmetries we have the same problem (as the Vicsek set in $\mathbb{R}^2$ shows). When we look at fractals topologically as one sided shift spaces with an equivalence relation, then there is no geometry and thus no information on relations among the refinement weights.

Example 6.1 Consider a class of fractals with $|V_0| = 3$ which we could call generalized gaskets. Take the unit triangle in $\mathbb{R}^2$, divide the sides into n equal intervals and join the interval end points on different sides by lines parallel to the original edges of the triangle to generate $n(n+1)/2$ upward pointing triangles. Now remove any pattern provided the set has connected interior and each boundary cell is connected to two interior cells. We give an example in Figure 3. It is clear that if we put infinite weight on the interior cells the short circuiting reduces the fractal to a simplified model which we would also get when we start from the classical Sierpinski gasket. In the latter case the Y-Δ-transformation, [18, Lem. 2.1.15], is known to be a powerful tool.

Let $V_0 := \{v_1, v_2, v_3\}$ and denote the conductance $c(v_i, v_{i+1 \bmod 3}) \geq 0$ by β_i. Put the refinement weight $\rho_i > 0$ on the cell containing v_i, $i = 1, 2, 3$. Suppose on the collapsed graph we have short circuited the edge with the conductance $\beta_{i+1 \bmod 3}$ in the boundary cell containing v_i. Then the new graph is Y-shaped with the outer vertices V_0 and a single interior vertex, say v_4. Its conductances are

$$c(v_i, v_4) = \rho_i(\beta_i + \beta_{i+2 \bmod 3}) =: \alpha_i \qquad (i = 1, 2, 3).$$

Consider the corresponding Dirichlet form $\mathcal{E}_1$ on $V_1 := V_0 \cup \{v_4\}$ and use the Y-Δ transformation to calculate its trace $\mathcal{E}$ on V_0. For $i = 1, 2, 3$ the resulting conductances are

$$c_{\mathcal{E}}(v_i, v_{i+1 \bmod 3}) = \frac{\rho_i \rho_{i+1 \bmod 3}(\beta_i + \beta_{i+2 \bmod 3})(\beta_{i+1 \bmod 3} + \beta_i)}{(\rho_1 + \rho_3)\beta_1 + (\rho_1 + \rho_2)\beta_2 + (\rho_2 + \rho_3)\beta_3},$$

provided $\beta_1 + \beta_2 + \beta_3 > 0$. But this is the only case we are interested in because we want to find a $\lambda > 0$ such that $c_{\mathcal{E}}(v_i, v_{i+1 \bmod 3}) = \lambda \beta_i$, for $i = 1, 2, 3$. From the existence of λ we deduce

$$\rho_2\beta_1(\beta_2 + \beta_3) = \rho_3\beta_2(\beta_1 + \beta_3) = \rho_1\beta_3(\beta_1 + \beta_2). \tag{6.1}$$

For $\beta_1 = 0$ this implies $0 = \rho_3\beta_2\beta_3$. But another vanishing conductance disconnects V_0. We are looking for irreducible eigenvectors, so $\beta_1 > 0$. By the same reasoning $\beta_2, \beta_3 > 0$. Dividing (6.1) by $\beta_1\beta_2\beta_3$ and defining $r_i := \beta_i^{-1}$, for $i = 1, 2, 3$, we arrive at

$$\rho_2(r_2 + r_3) = \rho_3(r_1 + r_3) = \rho_1(r_1 + r_2).$$

Eliminating r_1 and then r_2 we conclude

$$\begin{aligned} (-\rho_1^{-1} + \rho_2^{-1} + \rho_3^{-1})r_2 &= (\rho_1^{-1} + \rho_2^{-1} - \rho_3^{-1})r_3, \\ (-\rho_1^{-1} + \rho_2^{-1} + \rho_3^{-1})r_1 &= (\rho_1^{-1} - \rho_2^{-1} + \rho_3^{-1})r_3. \end{aligned}$$

Thus we have unique (up to a positive multiple) solutions $r_1, r_2, r_3 > 0$ if and only if (1.3) holds.

We have identified unique irreducible Λ_ρ-eigenvector of the collapsed model provided (1.3) is met. This is well known for the classical Sierpinski gasket [35, Sect. 5.2]. Thus by Theorem 14 we have existence of admissible weights for all such generalized gaskets.

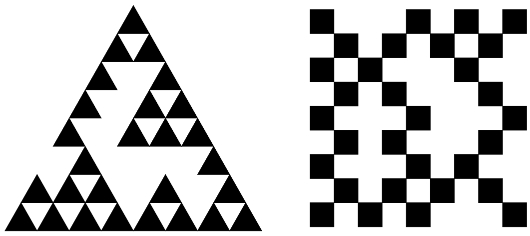

Figure 3: A non-symmetric Sierpinski gasket and Vicsek set

In Example 6.1 we found many admissible weights. When we only need one, we can enlarge the class of fractals under consideration significantly. The next theorem describes a class of fractals for which Theorem 4 implies existence of admissible weights without any symmetry assumptions, even on the boundary.

Theorem 5 *Suppose F is connected, has m-connected interior and V_0 has three elements. Then $\Psi^m(F)$ has admissible weights.*

Proof: We collapse all the interior m-cells and then apply the shortcut test to the collapsed structure (see [31] for more information). The result is a cloverleaf like network (with three leaves). We choose weight 1 for each boundary m-cell. The only reducible invariant $\mathbb{D}$-parts we have to check, if any, are defined by a single nonzero conductance. Then the eigenvalue of Λ_ρ is $\frac{1}{2}$. Short circuiting of the positive conductance and looking at the resulting renormalization problem for the other two conductances (being now positive and finite) yields the eigenvalue 1. So there exist admissible weights by the shortcut test and Theorem 4. □

The following example shows that the strategy of Example 6.1 does not always work. More precisely, one might end up in a well known non-unique situation. Then the assumptions of Theorem 4 cannot be met. When we make additional symmetry assumptions in order to obtain uniqueness, then the underlying fractal also has to have these symmetries. Otherwise the assumptions of Theorem 4 are once more violated.

Example 6.2 For the case $|V_0| = 4$ we can, similarly to Example 6.1, construct a family of generalized Vicsek sets. Take a $2n + 1 \times 2n + 1$ checkerboard pattern in a square in $\mathbb{R}^2$ and assume that the color of the squares which meet the boundary corners is black. Now remove any pattern of white squares which preserves the property of connected interior. An example is shown in Figure 3. In order to deduce from our results that there are admissible weights in this case we observe that by putting all the interior cells to infinite weight we have a symmetric fractal X - four squares joined at the central point. It resembles, but is different from, the "cut-square fractal" in [1]. It has admissible refinement weights when it is endowed with the symmetry group of a nested fractal, that is, the one of the Vicsek set. Indeed, the same collapsing strategy on the Vicsek set also results in X. Unfortunately, the right hand set in Figure 3 is not invariant under such a relatively big symmetry group. Even worse, a less symmetric Vicsek set has non-unique Λ_ρ-eigenvectors [23]. This effect reappears on a less symmetric X. Hence we cannot meet the requirements of Theorem 4.

Example 6.3 In Example 6.2 we can modify the usual construction of the classical Vicsek set so that the self-similar Dirichlet form is unique without any assumption on symmetry. We change the contractions but the classical Vicsek set stays the same. Note that the notion of admissible weights refers, strictly speaking, not to the fractal but to the self-similar structure. In this particular case we build the same Vicsek set using a different self-similar structure.

Suppose the boundary of the Vicsek set is $\{v_1, v_2, v_3, v_4\}$ with $v_i = (\pm 1, \pm 1)$, and the contractions are $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$. We assume that $\psi_i(v_i) = v_i$ for $i = 1, 2, 3, 4$. Let R_i be the rotation about the origin that moves v_1 into v_i and let $\widetilde{\psi_i} = \psi_i \circ R_i$. Then the iteration function system $\{\widetilde{\psi_1}, \widetilde{\psi_2}, \widetilde{\psi_3}, \widetilde{\psi_4}, \psi_5\}$ defines again the classical Vicsek set, but the self-similar Dirichlet form is unique because the boundary is symmetric (see Definition 2.5 and Theorem 3).

On a generalized Vicsek set, like the one on the right hand side of Figure 3, which may not have any symmetries but has a symmetric boundary we can find admissible weights for the same reason. But the use of the above $\{\widetilde{\psi_i} | 1 \leq i \leq 4\}$ for the Vicsek-like fractal in Figure 3 will change the fractal this time.

In the remainder of this section we will prove the Theorems 2 and 3.

Proof of Theorem 2: Choose the coupling weight 1 on all boundary cells and ∞ on all interior cells. Let $\mathcal{E} \in \partial\mathbb{D} \setminus \{0\}$. When there exists a $v \in V_0$ which is connected to an interior cell in $\Gamma(\mathcal{E}_1)$, then the same holds for all points of V_0 by the transitivity of $\mathfrak{G}$ and the $\mathfrak{G}$-invariance of $\mathcal{E}_1$. When there is no such connection for v, then again the same happens to all $v \in V_0$. In the first case the $\mathbb{D}$-part of $\mathcal{E}$ is mapped to $\mathbb{D}^\circ$. In the second case $\Lambda_\rho(\mathcal{E}) = 0$. Hence its $\mathbb{D}$-part is not Λ_ρ-invariant. On the other hand $\mathbb{D}^\circ$ is Λ_ρ-invariant by assumption. With Proposition 5.4, ρ is admissible. According to Theorem 4 there also exists a finite admissible weight. □

Proof of Theorem 3: Follow the proof of Theorem 2. □

Fortunately, it is quite easy to meet the technical assumptions of Theorem 2. A given fractal F with a given refinement map Ψ (on subsets of $\mathbb{R}^d$) can as well be constructed by Ψ^m, $m \geq 2$. The vertices of generation 0 are still given by the old V_0. But the vertices of the next generation are now the elements of the old V_m.

Lemma 6.4 *When the p.c.f. fractal F is connected and $\mathfrak{G}$ acts transitively on V_0, then there exists an $m \geq 1$ such that its interior is m-connected.*

Proof: According to the proof of Proposition 1.3.6 in [18], the diameter of an m-cell tends to zero for increasing m. Since V_0 contains only finitely many points, there exists a positive lower bound on the distance between two different points of V_0. Thus there exists an $m \in \mathbb{N}$ such that each n-cell contains at most one point of V_0 for $n \geq m$. For the same reason two different boundary m-cells are disjoint for big m.

When the p.c.f. fractal F is connected and $\mathfrak{G}$ acts transitively on V_0, then $F \setminus V_0$ is arcwise connected [18, Prop. 1.6.9, Thm. 1.6.2]. Since F is compact this shows that there exists an $m \in \mathbb{N}$ such that F has n-connected interior for all $n \geq m$. □

7 Graph-directed fractals

The results of Sections 4 and 5 also hold for finitely ramified, graph-directed fractals as introduced in [22, 13]. These are a straightforward generalization of p.c.f. self-similar fractals allowing them to have finitely many connected components. Examples of such fractals can be found in [13] and in the 'House' fractal below, which is considered in [10]. The existence of a suitable self-similar energy on such fractals was assumed in [13] and proved for some simple examples. Two strongly connected examples, the diamond fractal and the so called 'Hany' fractal were treated in [29, 30] where existence of a Laplace operator was proved using similar ideas to those used here.

The proofs of the results in Sections 4 and 5 remain literally the same, provided irreducible forms are replaced by forms with a minimal kernel. On disconnected fractals this kernel consists of all functions which are constant on what we will call the "building blocks of the fractal" below. Especially, Theorem 4 remains true for finitely ramified graph-directed fractals.

As an example, we will use this observation to construct a Dirichlet form on two different fractals, with a common boundary, the 3-gasket in the left column of Figure 4 and the Vicsek fractal in the middle column of the same figure. They overlap along the top side of the square to form a house as in the right column of Figure 4. Such construction problems are also considered in [21, 15, 12]. As the two fractals are both nested fractals the construction of a Dirichlet form on the house fractal is covered by [12]. We discuss this fractal as a graph directed construction and show that our results here give an alternative existence theorem for a Dirichlet form. The advantage of our approach is that we could give an existence result for non-symmetric fractals with a common boundary, provided their boundary structure matches in allowing us to write down a graph directed description of the combined structure.

As a graph directed construction the house has three "building blocks", the triangle "T", the square "S" and the composite "H" in the shape of a house. We need 20 contractions $\psi_1, \dots, \psi_{20}$ by $\frac{1}{3}$ to define the house as a graph directed construction. Set $V := \{T, S, H\}$ and $\mathbb{R}^2_v := \mathbb{R}^2 \times \{v\}$, for $v \in V$. Let us say that $\psi_1, \dots, \psi_6 : \mathbb{R}^2_T \to \mathbb{R}^2_T$ define the scaled triangles inside the triangle. The scaled squares inside the square are produced by $\psi_7, \dots, \psi_{11} : \mathbb{R}^2_S \to \mathbb{R}^2_S$. The scaled triangles inside the house are given by $\psi_{12}, \dots, \psi_{15} : \mathbb{R}^2_T \to \mathbb{R}^2_H$, the scaled squares inside the house by $\psi_{16}, \dots, \psi_{18} : \mathbb{R}^2_S \to \mathbb{R}^2_H$ and the scaled houses inside the house by $\psi_{19}, \psi_{20} : \mathbb{R}^2_H \to \mathbb{R}^2_H$. Let $e \in E := \{1, \dots, 20\}$ and consider $\psi_e : \mathbb{R}^2_w \to \mathbb{R}^2_v$. Define the domain of ψ_e by $\mathrm{dom}(e) := w$ and the target by $\mathrm{tar}(e) := v$. We think of e as pointing from $\mathrm{tar}(e)$ to $\mathrm{dom}(e)$, the convention used by Mauldin and Williams. This defines the construction graph (V, E) on the left hand side of Figure 5. For

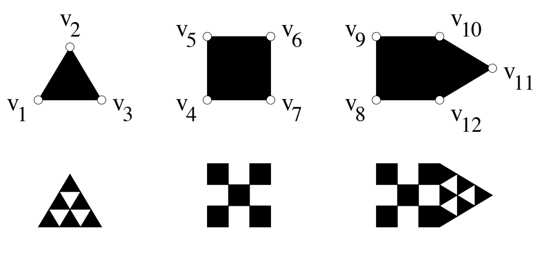

Figure 4: The first two construction steps of the house fractal.

$M \subset \mathbb{R}^2 \times V$ define the refinement map by

$$\Psi(M) := \bigcup_{e \in E} \psi_e(M \cap \mathbb{R}^m_{\mathrm{dom}(e)}).$$

According to [5, Thm. 4.3.5] there exist non-void compacts $K_v \subset \mathbb{R}^2_v$, for $v \in V$, such that

$$K_v = \bigcup_{e \in E, \mathrm{tar}(e)=v} \psi_e(K_{\mathrm{dom}(e)}).$$

Thus $F := K_T \cup K_S \cup K_H$ is a fixed point of Ψ and therefore self-similar.

A subgraph of (V, E) is termed strongly connected, when every two vertices are connected by a directed path. Hence (V, E) has three strongly connected components T, S and H, see the right graph in Figure 5. Since our construction graph satisfies the "open set condition", [5, Thm. 6.4.8] shows that we could calculate a Hausdorff dimension for every strongly connected component via [22, Thm. 3]. Here it is simpler to use the self-similarity dimension in [7, Thm. 9.3]. This is possible because all our strongly connected components consist of a single vertex with a (multiple) loop. The T-component defines the 3-gasket K_T with 6 subtriangles and Hausdorff dimension $\frac{\ln 6}{\ln 3}$. The S-component defines the Vicsek fractal K_S with Hausdorff dimension $\frac{\ln 5}{\ln 3}$. The H-component defines the classical middle third Cantor set with Hausdorff dimension $\frac{\ln 2}{\ln 3}$, the intersection of the 3-gasket and the Vicsek set.

To define self-similar measures on F one can use [6, Thm. 3.3.16].

Let us denote the extremal points of the initial shapes as indicated in the first line of Figure 4 by $V_0 := \{v_1, \ldots, v_{12}\}$. The n-th generation is again $V_n := \Psi^n(V_0)$, for $n \in \mathbb{N}$. Consider a conductance $c_0 : V_0^2 \to \mathbb{R}_+$ which assumes only the values d_1, d_2 on the triangle and the square as listed below:

$$\begin{array}{lll} d_1 & \text{on every} & \{v_i, v_j\} \subset \{v_1, v_2, v_3\}, \\ d_2 & \text{on every} & \{v_i, v_j\} \subset \{v_4, \ldots, v_7\}. \end{array}$$

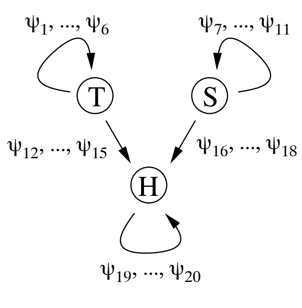

Figure 5: The construction graph of the house fractal and its 3 strongly connected components.

As a nested fractal the 3-gasket defines a self-similar fractal Laplacian when all its conductances are equal. This is the reason for the choice of d_1. The same holds for the Vicsek fractal, which justifies the choice of d_2. On the house we want c_0 to be invariant under the reflection in the line through v_{11} and $\frac{v_8+v_9}{2}$. This requires the following choices:

$$\begin{array}{lll} d_3 & \text{on} & \{v_8, v_{12}\}, \{v_9, v_{10}\}, \\ d_4 & \text{on} & \{v_{10}, v_{11}\}, \{v_{11}, v_{12}\}, \\ d_5 & \text{on} & \{v_8, v_{11}\}, \{v_9, v_{11}\}, \\ d_6 & \text{on} & \{v_8, v_{10}\}, \{v_9, v_{12}\}, \\ d_7 & \text{on} & \{v_{10}, v_{12}\}, \\ d_8 & \text{on} & \{v_8, v_9\}. \end{array}$$

On each building block K_v we define a symmetric, discrete Dirichlet form $\mathcal{E}_0^v(\cdot,\cdot)$ by

$$\mathcal{E}_0^v(f) := \frac{1}{2} \sum_{x,y \in V_0 \cap K_v} (f(y) - f(x))^2 c_0(x,y), \tag{7.1}$$

for $v \in V$ and $f : V_0 \to \mathbb{R}$. Then $\mathcal{E}_0 := \mathcal{E}_0^T + \mathcal{E}_0^S + \mathcal{E}_0^H$ defines a Dirichlet form on V_0. To define the refinement map on energies we choose refinement weights $\eta_e > 0$, for $e \in E$. For $f : V_1 \to \mathbb{R}$ we define a Dirichlet form on V_1 by

$$\mathcal{E}_1(f) := \Psi_\eta(\mathcal{E}_0)(f) := \sum_{e \in E} \eta_e \cdot \mathcal{E}_0^{d(e)}(f \circ \psi_e).$$

The trace of $\mathcal{E}_1$ on V_0 for $f : V_0 \to \mathbb{R}$ is again given by (3.3), and the renormalization map is also given by $\Lambda_\eta := \mathrm{Tr} \circ \Psi_\eta$.

Once more we want to find refinement weights $\eta := (\eta_T, \eta_S, \eta_H)$ for which there exists a unique (up to positive multiples) solution of

$$\Lambda_\eta(\mathcal{E}) = \gamma \mathcal{E} \qquad (\gamma \in \mathbb{R}, \mathcal{E} \in \mathbb{D} \cap \mathbb{P}^\circ). \tag{7.2}$$

Again the triangle, the square and the house define irreducible subproblems. When we evaluate Λ_η only on conductances between points of the triangle, $\{v_1, v_2, v_3\}$, then $\eta_T = \frac{15}{7}$ is the only refinement weight guaranteeing a Λ_η-fixed point [11]. Doing the same on the square, $\{v_4, \ldots, v_7\}$, we find the only possible refinement weight $\eta_S = 3$ according to [23]. Therefore let us fix η_T and η_S to be these values. Only η_H is left as a free variable in (7.2).

We define $\mathbb{D}$ and $\mathbb{P}$ as in Section 3. Then the kernel of $\mathcal{E} \in \mathbb{P}^\circ$ consists of functions which are constant on the triangle $\{v_1, v_2, v_3\}$, the square $\{v_4, \ldots, v_7\}$ and on the house $\{v_8, \ldots, v_{12}\}$. Suppose $\mathcal{E}$ has the conductances $d_1 = d_2 > 0$. Then $\Lambda_\eta(\mathcal{E})$ also has the conductances $d_1 = d_2$ because of our specific choice of η_T and η_S. Hence the corresponding subcone $\mathbb{P}_* \subset \mathbb{P}$ is Λ_η-invariant. We restrict our attention to $\mathbb{P}_*$. Choosing $\eta_H := \infty$ we deduce that $\Lambda_\eta(\mathbb{D} \cap \mathbb{P}_*)$ is a single ray in $\mathbb{P}_*^\circ$. We have found an admissible weight in $(0, \infty]^3$. Theorem 4 then guarantees an admissible weight in $(0, \infty)^3$.

Consequently it is possible to construct a graph directed self-similar energy form on the coupled 3-gasket and the Vicsek fractal. The positivity of the conductances of the corresponding eigenvector shows that the continuous model derived from these data is a diffusion which can travel with positive probability back and forth across the interface of the two component fractals.

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Abstract

On a large class of p.c.f. (finitely ramified) self-similar fractals with possibly little symmetry we consider the question of existence and uniqueness of a Laplace operator. By considering positive refinement weights (local scaling factors) which are not necessarily equal we show that for each such fractal, under a certain condition, there are corresponding refinement weights which support a unique self-similar Dirichlet form. As compared to previous results, our technique allows us to replace symmetry by connectivity arguments.