The spectrum of the Laplacian on the pentagasket

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Abstract. The spectrum of the fully symmetric Laplacian on the fractal pentagasket is studied by theoretical and experimental methods. We show how to construct derived eigenspaces of high multiplicity for both Dirichlet and Neumann spectrum starting from primitive Neumann eigenspaces. We prove that both spectra may be parceled into groups of five dimensional spaces which decompose in a prescribed way under the D_5 symmetry group. We show that D_5 invariant eigenfunctions possess additional local symmetries that in particular force them to be constant along the Cantor set bordering the inner deleted pentagon. Numerical approximations for eigenfunctions and eigenvalues obtained using the finite element method are reported. We formulate several conjectures based on this data. More data can be found at http://www.mathlab.cornell.edu/∼sas60/

1. Introduction

The pentagasket (see Figure 1.1) is one of the simplest examples of a nested fractal [Li] for which the method of spectral decimation [Sh2] does not hold. The construction of a Laplacian that is symmetric under the dihedral symmetry group D_5 may be described exactly using the method of Kigami [Ki] as a limit of graph Laplacians for a sequence of graphs approximating the fractal. Let V_0 denote the vertices of a regular pentagon (say the fifth roots of unity in $\mathbb{C}$) and let F_i denote the contractive similarity $F_i x = \rho x + (1-\rho)q_i$ for $q_i \in V_0$ and $\rho = 1/\tau$ ($\tau =$ golden ratio $\frac{\sqrt{5}+1}{2}$), so that the images of the pentagon intersect at the points $F_i q_{i+2} = F_{i+1} q_{i-1}$ (cyclic notation). The pentagasket is the unique nonempty compact set K satisfying

$$K = \bigcup_{i=0}^{4} F_i K. \tag{1.1}$$

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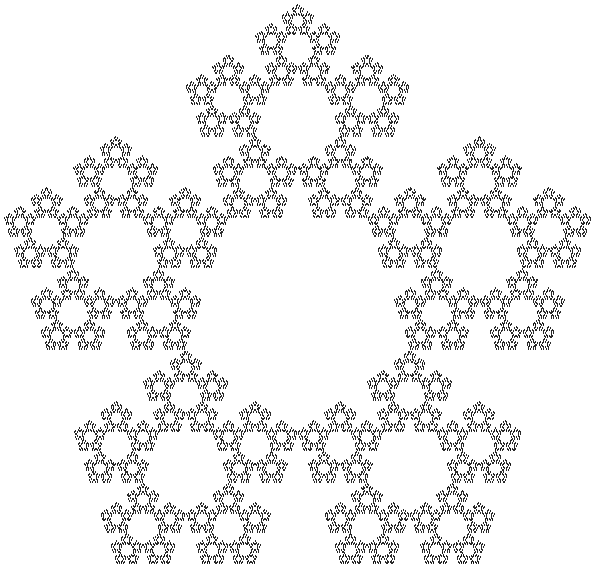


Figure 1.1: The pentagasket

The sequence of graphs Γ_m with vertices V_m and edge relation $x \sim_m y$ is defined inductively by taking Γ_0 to be the complete graph on V_0, and letting

$$V_m = \bigcup_{j=0}^{4} F_i V_{m-1} \tag{1.2}$$

and saying $x \sim_m y$ if and only if there exists i so that $x = F_i x'$, $y = F_i y'$ with $x' \sim_{m-1} y'$. Note that the representation $x = F_i x'$ for $x \in V_m$ and $x' \in V_{m-1}$ may not be unique. If x has two such representations we say x is a junction vertex; otherwise, a nonjunction vertex. We also declare V_0 to be the boundary of all the graphs Γ_m, and of K. We use u and v to denote functions on K, which may be restricted to V_m.

We define a sequence of energy forms $\mathcal{E}_m$ on Γ_m inductively by

$$\begin{aligned} \mathcal{E}_0(u,v) = p \sum_{j=0}^{4} (u(q_{j+1}) - u(q_j))(v(q_{j+1}) - v(q_j)) \\ + q \sum_{j=0}^{4} (u(q_{j+2}) - u(q_j))(v(q_{j+2}) - v(q_j)) \end{aligned} \tag{1.3}$$

for $p = \frac{\sqrt{161}-7}{16} \approx .35536\ldots$ and $q = \frac{1}{2} - p \approx .14446\ldots$. Then $\mathcal{E}_m$ is defined by

$$\mathcal{E}_m(u,v) = r^{-1} \sum_{i=0}^{4} \mathcal{E}_{m-1}(u \circ F_i, v \circ F_i) \tag{1.4}$$

for $r = \frac{\sqrt{161}-9}{8} \approx .46107\ldots$. The numbers p, q, r are chosen so that the restriction of $\mathcal{E}_m$ to V_{m-1} is equal to $\mathcal{E}_{m-1}$, meaning that for any u defined on V_{m-1}, the harmonic extension $\widetilde{u}$ to V_m, which is by definition the extension of u that minimizes the energy $\mathcal{E}_m(\widetilde{u},\widetilde{u})$, has the property that $\mathcal{E}_m(\widetilde{u},\widetilde{u}) = \mathcal{E}_{m-1}(u,u)$. (Up to a constant multiple, the given values of p, q, r are the only ones that yield this property [Sa].) This means that the energy

$$\mathcal{E}(u,u) = \lim_{m\to\infty} \mathcal{E}_m(u,u) \tag{1.5}$$

is well–defined in $R_+ \cup \{\infty\}$ for any continuous function on K, and we may define the domain of $\mathcal{E}$ to be the functions of finite energy, and harmonic functions on K to be those that minimize $\mathcal{E}_m(u,u)$ at each level m, subject to the boundary values $u(q_j)$.

Let μ denote the self–similar probability measure on K satisfying

$$\mu(A) = \frac{1}{5}\sum_{i=0}^{4} \mu(F_i^{-1}A). \tag{1.6}$$

We may then define the Laplacian Δ by the weak formulation as follows: Suppose f is continuous on K and $u \in \text{dom } \mathcal{E}$. Then $\Delta u = f$ means

$$\mathcal{E}(u,v) = -\int_K f v d\mu \tag{1.7}$$

for all $v \in \text{dom}_0\mathcal{E}$ (the subscript 0 denotes functions vanishing on the boundary). There is also an equivalent pointwise definition. Let $x = F_w q_j$ be a nonjunction vertex in V_m, where $w = (w_1,\ldots,w_m)$ is a word of length m, and $F_w = F_{w_1}\circ\cdots\circ F_{w_m}$. Let

$$\begin{aligned}\Delta_m u(x) = \frac{5^{m+1}}{r^m}(&p(u(F_w q_{j+1}) + u(F_w q_{j-1})) \\ &+ q(u(F_w q_{j+2}) + u(F_w q_{j-2})) - u(F_w q_j)).\end{aligned} \tag{1.8}$$

For a junction vertex $\Delta_m u(x)$ is defined as the average of the two forms of (1.8) corresponding to the two representations of x. Then

$$\Delta u(x) = \lim_{m\to\infty} \Delta_m u(x) \tag{1.9}$$

for any $x \in V_* = \cup V_m$, and the limit is uniform. Conversely, the existence of a uniform limit (1.9) implies $u \in \text{dom } \Delta$. See [Ki] for the full description of this equivalence. Normal derivatives may also be defined for $u \in \text{dom } \Delta$ at each boundary point

$$\begin{aligned}\partial_n u(q_j) = \lim_{m\to\infty} r^{-m}(&u(q_j) - p(u(F_j^m q_{j+1}) + u(F_j^m q_{j-1})) \\ &- q(u(F_j^m q_{j+2}) + u(F_j^m q_{j-2})))\end{aligned} \tag{1.10}$$

and the definition may be localized to any vertex in V_m and cell $F_w K$ containing it. Note that at junction points there are two distinct normal derivatives, and they sum to zero for $u \in \text{dom } \Delta$. At nonjunction points, the normal derivative vanishes

for any function in dom Δ, as can be seen by comparing the scaling factors in (1.8) and (1.10).

An important technical tool is the Gauss–Green formula

$$\int_K (u\Delta v - v\Delta u)d\mu = \sum_{\partial K} (u\partial_n v - v\partial_n u), \tag{1.11}$$

which also may be localized to any cell $F_w K$. Also, we have the following gluing principle: if $\Delta u = f$ on each cell $F_w K$ of order m, where u and f are continuous on K, then $\Delta u = f$ on K if and only if the two normal derivatives sum to zero at each junction point in V_m. We will refer to these as *matching conditions*. Also, the Laplacian scales by a factor of $5r^{-1}$, namely

$$\Delta(u \circ F_i) = 5r^{-1}(\Delta u) \circ F_i. \tag{1.12}$$

Under Dirichlet ($u\big|_{\partial K} = 0$) or Neumann ($\partial_n u\big|_{\partial K} = 0$) boundary conditions, $-\Delta$ becomes a nonnegative self–adjoint operator. Because the inverse operator is compact, the spectrum consists of a countable number of eigenfunctions with nonnegative eigenvalues tending to infinity [Ki]. In this paper we attempt to find out as much as possible about these two spectra and their associated eigenfunctions. For some nested fractals, such as the Sierpinski gasket SG, there is a theory called "spectral decimation" ([Sh1,2], [FS]) which says that eigenfunctions of Δ on K restrict to eigenfunctions of Δ_m on Γ_m, with a change in eigenvalues. This is false for the pentagasket (our data confirms this). Since the spectral decimation method was the primary tool for studying these spectra on SG in [DSV], [GRS] and [T], we need to develop new methods in this context. In fact, many of the ideas we use are fairly generic, and may be expected to extend to a wider class of nested fractals, all of which possess dihedral symmetry groups.

To attack the problem experimentally, we use the finite element method to obtain numerical approximations of eigenvalues and eigenfunctions. This method was worked out in detail for the Sierpinski gasket in [GRS], based on earlier work in [SU], but it extends easily to the pentagasket. The results are reported in Section 4, along with a number of conjectures. In fact, all the results proved in Sections 2 and 3 were first conjectured on the basis of the numerical evidence.

Because of the scaling property (1.12), any eigenspace with eigenvalue λ has the potential to produce a "derived" eigenspace with eigenvalue $5\lambda/r$ through a process of gluing scaled down copies $u \circ F_i^{-1}$ of eigenfunctions. The only obstacle is to match values and normal derivatives at junction points, and to obtain the required boundary conditions. In Section 2 we show how to do this starting from any nonconstant Neumann eigenspace. Eigenvalues not obtained in this way are called *primitive*.

One of the main tools we use is the classification of eigenspace in terms of representations of D_5. Since the Laplacian is invariant under D_5, so are the eigenspaces, so each eigenspace may be decomposed into a direct sum of irreducible representations types. We denote by π_+ the trivial representation, π_- the

alternating representation (also 1–dimensional), and $\pi_{2,1}$ and $\pi_{2,2}$ the two inequivalent 2–dimensional representations. (For the convenience of the reader we review the basic properties of these representations at the beginning of Section 2.) In Section 2 we show that each primitive Neumann eigenspace contains no π_+ component, and at most one component each of the other types. The experimental evidence leads us to conjecture that all primitive eigenspaces are irreducible. We also give explicit lower bounds for the dimensions of the derived eigenspaces, and we conjecture that these are in fact always the exact dimensions.

In Section 3 we show that all eigenvalues (enumerated with multiplicities) may be sorted into groups of five that we call *quintets*. The terminology here is deliberately chosen to suggest the structure of the common musical string quintet, scored for two violins, two violas and one cello. Our quintets are composed of a $\pi_{2,1}$ and a $\pi_{2,2}$ representation, and either a π_+ or a π_- representation. On the other hand, string quintets are not always scored for the same instruments, as for instance the famous Schubert quintet for two violins, one viola, and two cellos. We conjecture that in the pentagasket there are no Schubert quintets: the eigenvalue of the one dimensional representation is never sandwiched between the others.

In Section 5 we show that eigenfunctions associated with the trivial representation π_+ all (in the case of multiplicity one) possess some additional local symmetries. In Figure 1.1 you will be able to identify a central pentagon lying in the complement of the pentagasket, that we will refer to as the central deleted pentagon (note that this is not a component of the complement, since its component also contains additional "spikes"). At any level m, there are $5 \cdot 2^{m-1}$ cells of order m bordering this pentagon. The local symmetries are described by a simultaneous reflection of each of these cells. As a consequence of these symmetries, the eigenfunction must be constant along the Cantor set that borders the central deleted pentagon. A similar constancy was observed in [DSV] for SG, but was proved by algebraic methods that are not available here. In fact, the local symmetries result is also true on SG.

2. Derived eigenvalues

We review briefly the representation theory of D_5 in the concrete setting of spaces of functions on K. Let $\mathcal{H}$ denote a one-dimensional space of functions on K. We say $\mathcal{H}$ corresponds to the trivial representation π_+ if $f(gx) = f(x)$ for every $g \in D_5$, and $\mathcal{H}$ corresponds to the alternating representation π_- if $f(gx) = f(x)$ for every rotation g, and $f(Rx) = -f(x)$ for every reflection R. Note that in the π_- case, f vanishes on every 5 element orbit of D_5.

Next consider a two-dimensional space $\mathcal{H}$. We say $\mathcal{H}$ corresponds to the representation $\pi_{2,1}$ (respectively $\pi_{2,2}$) if $\mathcal{H}$ splits $\mathcal{H}_+ \oplus \mathcal{H}_-$, where $f \in \mathcal{H}_\pm$ satisfies $f(wx) = e^{\pm 2\pi i/5} f(x)$ (respectively $f(wx) = e^{\pm 4\pi i/5} f(x)$) where w denotes rotation through angle $2\pi/5$ and $f(x) \in \mathcal{H}_+$ if and only if $f(Rx) \in \mathcal{H}_-$ if R is any reflection.

An eigenvalue λ (Dirichlet or Neumann) is called *derived* if $\lambda r/5$ is a Dirichlet or Neumann eigenvalue; otherwise it is called primitive. Here we will show how to build derived eigenspaces out of primitive Neumann eigenspaces, and certain primitive Dirichlet eigenspaces.

Algorithm 2.1: Suppose that λ is a primitive Neumann eigenvalue of multiplicity one associated with the alternating representation π_-. Then there is an eigenfunction u satisfying $u(x) = u(Rx)$ for any rotation in D_5, and $u(x) = -u(Rx)$ for any reflection in D_5. The reflection skew–symmetry implies that u vanishes at the boundary points, so u is also a Dirichlet eigenfunction with eigenvalue λ. For any fixed $m \geq 1$ and every word w of length m, the function $u(F_w^{-1}x)$ on F_wK and zero elsewhere is a joint Neumann and Dirichlet eigenfunction with eigenvalue $(5r^{-1})^m\lambda$. These functions are clearly linearly independent (in fact orthogonal since they have disjoint support), so the derived eigenspace with eigenvalue $(5r^{-1})^m\lambda$ has multiplicity at least 5^m.

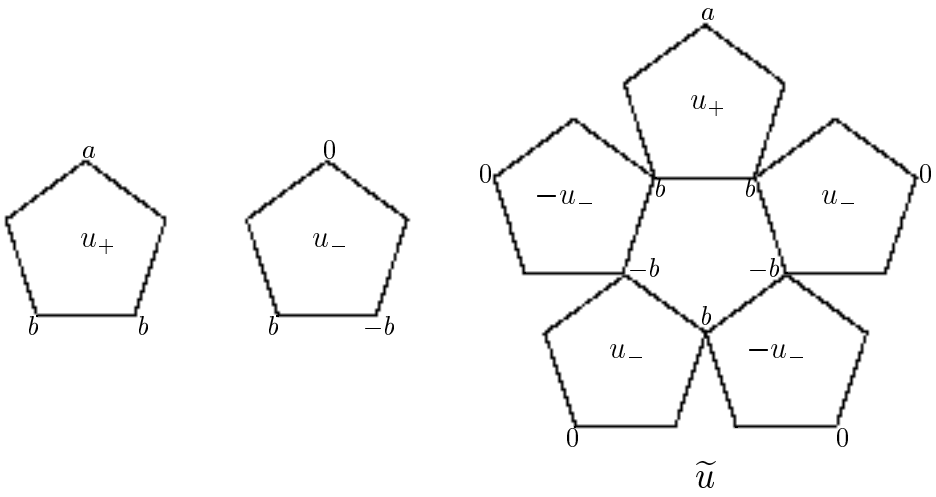


Figure 2.1: u_+ and u_- normalized so that $u_+(q_3) = u_-(q_3) = b$, and then contracted and spread around to form $\widetilde{u}$.

Algorithm 2.2: Consider a primitive Neumann eigenvalue λ of multiplicity two corresponding to either of the representations $\pi_{2,1}$ or $\pi_{2,2}$. Choose a reflection, say R_0, that fixes the boundary point q_0, and denote by u_+ and u_- the eigenfunctions in the eigenspace that are symmetric and skew–symmetric with respect to R_0 (take any eigenfunction in the space and pass to its symmetric and skew–symmetric parts). Figure 2.1 shows how to glue together various rotations of $u_\pm(F_0^{-1}x)$ to obtain a Neumann eigenfunction $\widetilde{u}$ with eigenvalue $5r^{-1}\lambda$ that vanishes at the boundary points q_1, q_2, q_3, q_4. Note that we have to normalize $u_\pm$ to have the same value at q_3. (In the degenerate case when either $u_+(q_3) = 0$ or $u_-(q_3) = 0$ there is a simpler alternative.) We only have to check that the values of $\widetilde{u}$ at the 5

junction points match up; the matching of normal derivatives is automatic since they all vanish. Once we have $\widetilde{u}$ we can take its rotated copies to obtain 5 linearly independent Neumann eigenfunctions with eigenvalue $5r^{-1}\lambda$. At this level there are no apparent Dirichlet derived eigenvalues, assuming the nondegeneracy condition $a = u_+(q_0) \neq 0$.

For $m > 1$ we can use $\widetilde{u}$ to build both Dirichlet and Neumann derived eigenspaces with eigenvalue $(5r^{-1})^m\lambda$. Choose any vertex $y \in V_{m-1}$. If y is not a junction point then there is a unique representation $y = F_w q_j$ with $|w| = m-1$. We construct u_y supported in $F_w K$ by composing a suitable rotation of $\widetilde{u}$ (depending on q_j) with F_w^{-1}. If y is a junction point then there are 2 such representations, $y = F_w q_j = F_{w'} q_{j'}$, and u_y will be supported in $F_w K \cup F_{w'} K$, on each piece being the same as for a nonjunction point. Note that as long as y is not in V_0, u_y will be both a Dirichlet and Neumann eigenfunction, but for $y \in V_0$ it is just a Neumann eigenfunction. Clearly all the functions u_y are linearly independent. Thus we have the lower bounds $\#V_{m-1} = \frac{5}{4}(3 \cdot 5^{m-1} + 1)$ and $\#V_{m-1} - 5$ for the multiplicities of the derived Neumann and Dirichlet eigenspaces.

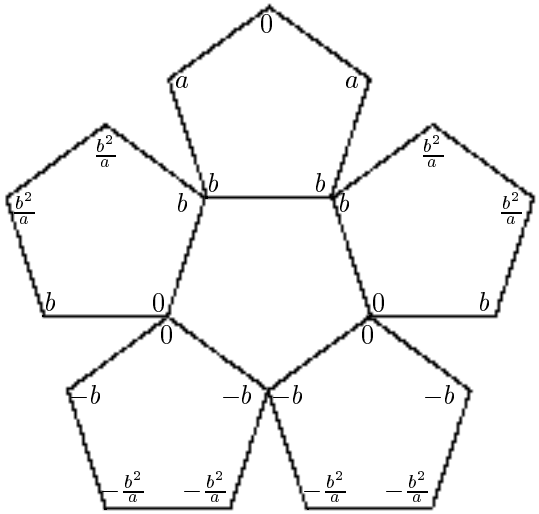


Figure 2.2: How to glue 5 modified copies of $\widetilde{u}$ in $F_0 K$.

Algorithm 2.3: Let λ be a Dirichlet eigenvalue of multiplicity one corresponding to the trivial representation π_+, and let u be an eigenfunction. Write $u(F_0 q_1) = a$ and $u(F_0 q_2) = b$. Let $\widetilde{u}$ denote $u \circ F_0^{-1}$ restricted to $F_0 F_0 K$. We build a single derived Dirichlet eigenfunction with eigenvalue $5\lambda/r$ by gluing 25 copies of $\widetilde{u}$, multiplied by appropriate constants (b/a and $-b/a$), and rotated appropriately, in each of the 25 cells of level 2. Figure 2.2 shows the pattern on the single cell $F_0 K$ of level 1, and this is repeated symmetrically on all 5 cells of level 1. We note that the π_+ symmetry of u forces the normal derivative to vanish at $F_0 q_2$

and F_0q_3, and F_0q_1 and F_0q_4 are nonjunction points so the normal derivative vanishes there automatically. Thus the gluing conditions are verified, and we have an eigenfunction. It is also clear that it is Dirichlet eigenfunction corresponding to the representation π_+. By iterating this algorithm, we may obtain derived Dirichlet eigenspaces with eigenvalue $(5r^{-1})^m\lambda$ for any m, with multiplicity at least one.

Next we turn to results that go the other way, putting upper bounds on multiplicities of eigenspaces. For example, the reason we did not describe derived eigenspaces starting from π_+ primitive Neumann eigenspaces is that there are none, except for the trivial case of constants.

Lemma 2.4. *A primitive Neumann eigenspace, other than the constants, contains no π_+ components.*

Proof: Let u be a nonconstant D_5 invariant Neumann eigenfunction with eigenvalue λ. Consider the function $u\circ F_0$. This has eigenvalue $r\lambda/5$. We claim it is a Neumann eigenfunction, and this will contradict the assumption that λ is primitive.

So it suffices to show that the normal derivative of u vanishes at the points F_0q_j. This is already true at $F_0q_0 = q_0$. At the two junction points F_0q_2 and F_0q_3 it follows from the group invariance, specifically under the reflections fixing those points, and the matching condition for normal derivatives at those points: the first makes the normal derivatives on both sides equal, while the second makes them sum to zero. The remaining two points, F_0q_1 and F_0q_4, are nonjunction points, so the normal derivative automatically vanishes. Q.E.D.

Theorem 2.5. *A primitive Neumann eigenspace cannot contain more than one component of the same type (π_-, $\pi_{2,1}$ or $\pi_{2,2}$).*

Proof: Assume that we had two components of one of the same type. Note that each such component contains a skew symmetric function with respect to the reflection fixing q_0, and it is not hard to see that these functions will be linearly independent when restricted to F_0K. By taking a linear combination of these functions we may obtain a nonzero function u on F_0K whose normal derivatives vanish at F_0q_2 and F_0q_3. The vanishing of the normal derivatives at the other points F_0q_0, F_0q_1, F_0q_4 follows as in Lemma 2.4, so $u\circ F_0$ gives a Neumann eigenfunction with eigenvalue $r\lambda/5$, contradicting the hypothesis that λ is primitive. Q.E.D.

Note that the same is true for Dirichlet eigenspaces, but now there is one more possible component type, π_+. The argument is the same for the other types (the vanishing of the normal derivative at q_0 comes from the skew–symmetry in this case). For the π_+ case we have the vanishing of the normal derivative at F_0q_2 and F_0q_3 as shown in the proof of Theorem 2.5, so we only need the normal derivative to vanish at q_0, which we can arrange in a 2–dimensional space.

3. Quintets

In this section we will show that the eigenvalues $\{\lambda_j\}$, repeated according to multiplicities, may be grouped in 5's so that the corresponding eigenspaces correspond

to representations $\pi_{2,1}$, $\pi_{2,2}$ and either π_+ or π_-. This is valid for both Dirichlet and Neumann eigenvalues. We will give the arguments in the Neuman case, but the arguments are easily adapted to the other case simply by imposing the vanishing condition of the boundary on all spaces considered. Because of the occurrence of eigenvalues of high multiplicity it must be understood that the statement only holds for the proper choice of ordering eigenfunctions within each eigenspace.

Another way to state the result is to create three separate lists of eigenvalues, $\{\lambda_j^0\}$ being the increasing list of eigenvalues associated to π_+ or π_- eigenfunctions, $\{\lambda_j^1\}$ and $\{\lambda_j^2\}$ being the same for $\pi_{2,1}$ and $\pi_{2,2}$ representations, with the convention that each $\pi_{2,k}$ space contributes a single entry in the $\{\lambda_j^k\}$ sequence, although it contributes two entries in the $\{\lambda_k\}$ sequence.

Theorem 3.1. *$\lambda_j^k \le \lambda_{j+1}^\ell$ for all j,k,ℓ.*

Corollary 3.2. *The sequence $\{\lambda_j\}$ may be spit into consecutive quintets of values $\{\lambda_j^0,\lambda_j^1,\lambda_j^1,\lambda_j^2,\lambda_j^2\}$ in the appropriate order.*

Proof: The eigenvalues $\{\lambda_j\}$ are characterized by the minimax principle

$$\lambda_j = \min_{\substack{\dim L=j \\ L\subseteq\mathcal{H}}} \max_{u\in L} R(u) \tag{3.1}$$

where $\mathcal{H}$ is the Hilbert space of functions of finite energy on K and $R(u)$ is the Rayleigh quotient

$$R(u) = \mathcal{E}(u,u)/\|u\|_2^2. \tag{3.20}$$

Similarly, the sequences $\{\lambda_j^k\}$ may be characterized in the same way with $\mathcal{H}$ replaced by a subspace of functions with appropriate symmetries. It is convenient to work with complex valued functions and consider the rotation subgroup of D_5, since then the irreducible representations are one dimensional, and pairs of them are combined to produce the $\pi_{2,1}$ and $\pi_{2,2}$ representations of D_5. As mentioned in Section 2, each irreducible $\pi_{2,1}$ space contains a complex valued function satisfying $u(wx) = wu(x)$, and each irreducible $\pi_{2,2}$ space contains one satisfying $u(wx) = w^2u(x)$, where $w = e^{2\pi i/5}$. Thus

$$\lambda_j^k = \min_{\substack{\dim L=j \\ L\subseteq\mathcal{H}_k}} \max_{u\in L} R(u) \quad \text{for} \quad k = 1,2 \tag{3.3}$$

with $\mathcal{H}_k = \{u\in\mathcal{H} : u(wx) = w^k u(x)\}$. We also have (3.3) holding for $k=0$ if we define $\mathcal{H}_0$ to be the direct sum of the spaces of functions with π_+ and π_- symmetry.

Next we want to localize (3.3) to the cell F_0K, since each function in any of the spaces $\mathcal{H}_k$ is uniquely determined by its restriction to this cell, and $R(u\circ F_0) = r^{-1}R(u)$ (for $k=0$ this requires the observation that π_+ functions are even and π_- functions are odd on F_0K with respect to the reflection fixing q_0, and so are orthogonal with respect to both the energy and L^2 norm). Let $\widetilde{\mathcal{H}}_k$ denote the space

of restrictions to F_0K of $\mathcal{H}_k$ functions. Then

$$\lambda_j^k = r \min_{\substack{\dim L=j \\ L\subseteq\widetilde{\mathcal{H}}_k}} \max_{u\in L} R(u\circ F_0) \tag{3.4}$$

is the local analog of (3.3). In order for this to be useful we need a more intrinsic description of the local spaces $\widetilde{\mathcal{H}}_k$. This is easy:

$$\begin{aligned}\widetilde{\mathcal{H}_k} = \{&u \text{ defined on } F_0K \text{ with } u\circ F_0 \text{ of finite energy and}\\ &u(F_0q_3) = w^k u(F_0q_2)\} \text{ for } k=0,1,2.\end{aligned} \tag{3.5}$$

For $k=1,2$ the condition $u(wx)=w^ku(x)$ implies $u(F_0q_3)=w^ku(F_0q_2)$ since $F_0q_3 = wF_0q_2$, and conversely, if it holds we can extend u to K by defining $u(w^nx)=w^{kn}u(x)$ and this gives a continuous function of finite energy. Note that we are exploiting here the fact that we may work with functions of finite energy, so we do not have to worry about matching normal derivatives when we define functions by gluing. For $\widetilde{\mathcal{H}}_0$ we separate functions into their even and odd parts. For the even part the condition $u(F_0q_3)=u(F_0q_2)$ is automatic, and this allows us to extend u to a π_+ function by setting $u(w^nx)=u(x)$. For the odd part, the condition $u(F_0q_3)=u(F_0q_2)$ is equivalent to $u(F_0q_3)=u(F_0q_2)=0$, and this is clearly necessary and sufficient for the extension $u(w^nx)=u(x)$ to define a continuous π_+ function.

Now the spaces $\widetilde{\mathcal{H}}_k$ all have a common subspace $\widetilde{\mathcal{H}}$ of codimension one, defined by $\widetilde{\mathcal{H}} = \{u \text{ defined on } F_0K \text{ with } u\circ F_0 \text{ of finite energy and } u(F_0q_3)=u(F_0q_2)=0\}$.

The estimate $\lambda_j^k \le \lambda_{j+1}^\ell$ for $k\neq\ell$ is now easy. By (3.4) there exists a subspace $L'\subseteq\widetilde{\mathcal{H}}_\ell$ with $\dim L' = j+1$ such that

$$\lambda_{j+1}^\ell = r\max_{u\in L'} R(u\circ F_0). \tag{3.6}$$

Then $L = L'\cap\widetilde{\mathcal{H}}$ has dimension at least j and so any j dimensional subspace of L is a candidate for the minimum in the expression (3.4) for λ_j^k. Thus

$$\lambda_j^k \le r\max_{u\in L} R(u\circ F_0) \le r\max_{u\in L'} R(u\circ F_0) = \lambda_{j+1}^\ell.$$

The Corollary is an immediate consequence, since if we arrange the values $\{\lambda_j^0,\lambda_j^1,\lambda_j^1,\lambda_j^2,\lambda_j^2\}$ in increasing order, and then list the ordered quintets in the order of j, we will get a complete increasing list of all eigenvalues. Q.E.D.

It is also of interest to sort the $\{\lambda_j^0\}$ sequence of eigenvalues into two sequences $\{\lambda_j^+\}$ and $\{\lambda_j^-\}$ corresponding to the π_+ and π_- representations.

Theorem 3.3. *$\lambda_j^+ \le \lambda_j^-$ for all j.*

Proof: Let $\widetilde{\mathcal{H}}_0 = \widetilde{\mathcal{H}}_+ \oplus \widetilde{\mathcal{H}}_-$ be the decomposition into even and odd functions. Then the analog of (3.4) is

$$\lambda_j^\pm = r \min_{\substack{\dim L=j \\ L\subseteq\widetilde{\mathcal{H}}_\pm}} \max_{u\in L} R(u\circ F_0). \tag{3.7}$$

We now define a one-to-one mapping from $\widetilde{\mathcal{H}}_-$ into $\widetilde{\mathcal{H}}_+$ that preserves $R(u \circ F_0)$, namely $u \mapsto \chi u$ where χ is $+1$ on the right half and -1 on the left half of $F_0 K$. This obviously takes odd functions to even functions, and does not violate the requirement $u(F_0 q_3) = u(F_0 q_2)$ since this value is already zero for $u \in \widetilde{\mathcal{H}}_-$. The fact that it preserves the Rayleigh quotient is almost obvious. The approximate energy at level m is unchanged, except in the topmost cell $F_0^m K$, which straddles the symmetry line, so passing to the even version χu may decrease the contribution to the energy. However, it is known that the energy contribution from individual cells goes to zero in the limit as $m \to \infty$, so the actual energy is the same.

If we choose L to achieve the minimum in (3.7) for λ_j^-, then the image of L under the mapping is a candidate for the minimum in (3.7) for λ_j^+, and this gives $\lambda_j^+ \le \lambda_j^-$. Q.E.D.

Theorem 3.4. *There exists a constant c such that $\lambda_j^- \le \lambda_{j+[\log_5 j]+c}^+$.*

Proof: Fix a value n, and define subspaces $\widetilde{\mathcal{H}}_\pm^n$ of $\widetilde{\mathcal{H}}_\pm$ as the even and odd functions on $F_0 K$ that vanish at the following $n+7$ points: $F_0 q_2$, $F_0 q_3$, $F_0^{n+1} q_k$ $(k = 0, \dots, 4)$ and $F_0^k F_2 q_4$ $(k = 1, \dots, n)$ (see Figure 3.1).

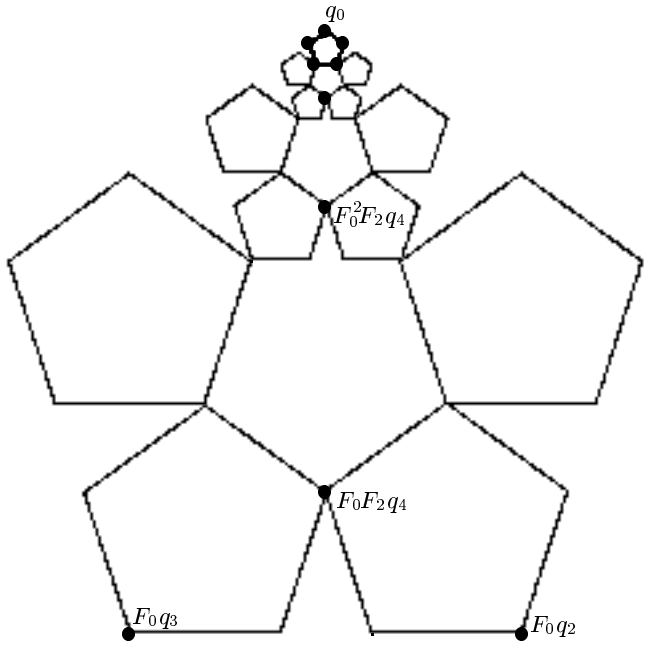


Figure 3.1: The black dots are the $n+7$ points (with $n = 3$) where the functions under consideration in the proof of Theorem 3.4 vanish.

Denote by $\{\lambda_{j,n}^\pm\}$ the eigenvalues defined by (3.4) with $\widetilde{\mathcal{H}}_k$ replaced by $\widetilde{\mathcal{H}}_\pm^n$. Clearly we have

$$\lambda_j^{\pm} \le \lambda_{n,j}^{+} \le \lambda_{j+n+7}^{+} \tag{3.8}$$

for any n and j. We will show that

$$\lambda_{n,j}^{-} = \lambda_{n,j}^{+} \tag{3.9}$$

holds for a suitable choice of n, which implies

$$\lambda_j^{-} \le \lambda_{j+n+7}^{+}. \tag{3.10}$$

We claim that the condition

$$\lambda_{j+n+7}^{+} \le \left(\frac{5}{r}\right)^n \cdot 13.0659\ldots \tag{3.11}$$

($13.0659\ldots$ is the first Dirichlet eigenvalue) implies (3.9). The reason is that any function u in $\widetilde{\mathcal{H}}_{\pm}^n$ that achieves the minimax in (3.4) must vanish identically on $F_0^{n+1}K$ (if not then the function $u \circ F_0^{n+1}$ would beat the lowest Dirichlet eigenvalue). This means that the map $u \mapsto \chi u$ considered in the proof of Theorem 3.3 will map the relevant portion of $\widetilde{\mathcal{H}}_{-}^n$ to $\widetilde{\mathcal{H}}_{+}^n$.

It remains to show that (3.11) holds for the choice

$$n = [\log_5 j] + N \tag{3.12}$$

for a suitable constant N, since then (3.10) is the desired result for $c = N + 7$. For this we require the asymptotic Weyl type estimate of Kigami and Lapidus [KL]

$$\lambda_j \le c_1 j^{2/d} \tag{3.13}$$

for suitable constant c_1 and spectral dimension

$$d = 2\log 5/\log(5/r). \tag{3.14}$$

(We give experimental data confirming this in Section 4.) In view of Theorems 3.1 and 3.3 the estimate (3.13) holds for λ_j^{+} as well, with a different constant. That means

$$\begin{aligned}\lambda_{j+n+7}^{+} &\le c_1(j+n+7)^{\frac{\log(5/r)}{\log 5}} \\ &\le c_1'\left(\frac{5}{r}\right)^{n-N}\end{aligned}$$

and by taking N large enough we obtain (3.11). Q.E.D.

Standard arguments allow us to compare Dirichlet and Neumann eigenvalues, not just for the full sequence $\{\lambda_j(D)\}$ and $\{\lambda_j(N)\}$, but for the other sequences defined above, since the Dirichlet boundary conditions are imposed by 5 conditions on $\mathcal{H}$, but only a single vanishing condition on $\widetilde{\mathcal{H}}_k$ because of the symmetry. Thus

$$\begin{cases} \lambda_j(N) \le \lambda_j(D) \le \lambda_{j+5}(N) \\ \lambda_j^k(N) \le \lambda_j^k(D) \le \lambda_{j+1}^k(N) \\ \lambda_j^{\pm}(N) \le \lambda_j^{\pm}(D) \le \lambda_{j+1}^{\pm}(N). \end{cases} \tag{3.15}$$

4. Experimental results

To obtain numerical approximations of eigenvalues and eigenfunctions, we use the finite element method. This is described in detail in [GRS] for SG, and the extension to the pentagasket is straightforward. We use the spline spaces $S(\mathcal{H}_0, V_m)$ of piecewise harmonic and $S(\mathcal{H}_1, V_m)$ of piecewise biharmonic functions with nodes at V_m, as described in [SU]. The functions in $S(\mathcal{H}_0, V_m)$ are required to be continuous at the points in V_m, while the functions in $S(\mathcal{H}_1, V_m)$ are also required to satisfy the matching condition for normal derivatives at the junction points in V_m. One new idea, which does not arise on SG, is that we also require functions in $S(\mathcal{H}_1, V_m)$ to have vanishing normal derivatives at nonjunction vertices in $V_m \setminus V_0$. This is not essential, but since the functions we are approximating have this property, it is inefficient to increase the dimension of $S(\mathcal{H}_1, V_m)$ by admitting functions that won't really help in the approximation process. It is extremely easy to weed out such functions from the "better basis" described in [SU], since each basis function vanishes together with its normal derivative at all but one point of V_m and at the exceptional point either has zero value or zero normal derivative. We denote by a subscript 0 the spline spaces of functions vanishing on V_0.

The finite element method produces approximations to λ_j obtained by restricting the spaces L in (3.1) to the spline spaces $S(\mathcal{H}_j, V_m)$ for Neumann and $S_0(\mathcal{H}_j, V_m)$ for Dirichlet eigenvalues. It is clear then that it always yields overestimates, and these improve as the spline spaces are enlarged. Of course we don't use the modified (3.1) directly, but rather solve an equivalent generalized eigenvalue problem for the matrix equation

$$Ec = \lambda Gc. \tag{4.1}$$

Here $c = \{c_j\}$ are the coefficients of the expansion

$$u = \sum c_j \varphi_j \tag{4.2}$$

in terms of the basis $\{\varphi_j\}$ of the spline space,

$$E_{jk} = \mathcal{E}(\varphi_j, \varphi_k) \tag{4.3}$$

is the energy matrix and

$$G_{jk} = \int \varphi_j \varphi_k d\mu \tag{4.4}$$

is the Gramm matrix for the basis. Only the lower portion of the approximate spectrum at any fixed level m is reliable, but by examining the approximations on several levels we can obtain some confidence in the results. Indeed, it should be possible to extrapolate the data to obtain guesses for the actual eigenvalues with greater precision, but we have not indulged in such speculation.

In Tables 4.1 and 4.2 we present a summary of approximations to the first 475 eigenvalues of Neumann and Dirichlet type, respectively, based on biharmonic splines on levels one to four. These eigenvalues are grouped according to multiplicities into 79 distinct Neumann and 121 distinct Dirichlet eigenvalues. Our approximations for the lowest eigenvalues are likely correct to 8 decimal places,

while the highest eigenvalues are overestimated by about 1.8, less than .005% (since 79_N is a derived eigenvalue, we can estimate it quite accurately by multiplying the eigenvalue for 26_N by $5/r$). We also observe that the approximation error is cut by a factor of $10^{-3} \approx (r/5)^3$ as we increase the level by one. This is consistent with the expectations for biharmonic splines in [SU]. We believe that both tables accurately represent the multiplicities and constitute two complete "cycles" of the spectra, as described below. More data is available at the website http://www.mathlab.cornell.edu/~sas60/ .

Table 4.1. Neumann Spectrum

#	Mult	Type	Total	Weyl Ratio	Eigenvalue(4)	Eigenvalue(3)	Eigenvalue(2)	Eigenvalue(1)
1	1	P+	1		.00000000	.00000000	.00000000	.00000000
2	2	P1	3	.647364	9.69015264	9.69015264	9.69015266	9.69016586
3	2	P2	5	.584329	24.03200182	24.03200182	24.03200229	24.03242183
4	5	D(2)	10	.431576	105.08281350	105.08281364	105.08297928	105.18868180
5	1	P-	11	.416183	127.70139845	127.70139876	127.70176749	127.92285712
6	2	P1	13	.458395	141.74517199	141.74517246	141.74571060	142.03226446
7	2	P2	15	.438861	186.87991151	186.87991292	186.88141810	187.53817189
8	5	D(3)	20	.467463	260.60996742	260.60997274	260.61546342	261.62235109
9	2	P2	22	.423255	347.69197858	347.69199556	347.70938322	352.99610177
10	2	P1	24	.383267	458.13680955	458.13686048	458.18617801	470.87806007
11	1	P-	25	.366366	520.30820135	520.30828575	520.38723592	538.52222451
12	20	D'(4)	45	.388419	1139.54837590	1139.55021119	1140.77665966	3234.28695495
13	1	P-	46	.380484	1213.79303977	1213.79542486	1215.37421255	
14	2	P1	48	.384414	1273.24113080	1273.24400500	1275.08514322	
15	2	P2	50	.385374	1347.60359385	1347.60717470	1349.80090386	
16	5	D-(5)	55	.416183	1384.83084452	1384.83484311	1387.23240288	
17	5	D(6)	60	.423134	1537.12558200	1537.13155795	1540.43252984	
18	1	P-	61	.422007	1581.45615305	1581.46287251	1585.13865198	
19	2	P1	63	.430363	1611.37238646	1611.37957292	1615.22848836	
20	2	P2	65	.427331	1705.47558256	1705.48448706	1710.00835668	
21	5	D(7)	70	.409604	2026.57974124	2026.59698227	2034.47396582	
22	2	P2	72	.408571	2120.84041973	2120.86123848	2129.92431816	
23	2	P1	74	.399805	2280.76368683	2280.79153459	2291.88759979	
24	1	P-	75	.395893	2360.68983407	2360.72167778	2372.77766058	
25	20	D'(8)	95	.444090	2826.12980119	2826.19144608	2837.78208106	
26	1	P-	96	.378657	3634.57154506	3634.73703190	3690.90844804	
27	2	P1	98	.383468	3677.85450369	3678.02761554	3734.65971814	
28	2	P2	100	.386527	3745.23083089	3745.41605260	3802.67705853	
29	5	D(9)	105	.404017	3770.47240328	3770.66226708	3828.11047674	
30	2	P2	107	.357732	4642.93928718	4643.35503523	4757.23017002	
31	2	P1	109	.362825	4673.15878632	4673.58494438	4787.09633878	
32	1	P-	110	.365210	4691.05736156	4691.48977096	4804.81018235	
33	5	D(10)	115	.367298	4968.16841133	4968.70342502	5106.34326977	
34	1	P-	116	.369745	4983.02616764	4983.56737542	5120.60047609	
35	2	P1	118	.375929	4986.78120792	4987.32392195	5124.28142135	
36	2	P2	120	.382016	4992.27699289	4992.82194005	5129.71801405	
37	5	D-(11)	125	.366366	5642.37330024	5643.22945861	5839.89051178	
38	95	D"(12)	220	.379787	12357.61156091	12370.91693680	35074.86672450	
39	5	D-(13)	225	.372212	13162.74809515	13179.86892475		
40	2	P2	227	.365473	13702.23503861	13721.74135816		
41	2	P1	229	.368551	13710.07071867	13729.61839289		
42	1	P-	230	.370044	13716.44306349	13736.02565625		
43	5	D(14)	235	.376404	13807.42569155	13827.43904849		
44	1	P-	236	.374863	13979.23808084	14000.07558797		
45	2	P1	238	.377061	14033.03363435	14054.12843613		
46	2	P2	240	.378166	14146.60113731	14168.25575990		
47	5	D(15)	245	.377666	14613.84145189	14637.79931475		
48	2	P2	247	.379056	14710.61243842	14735.14722972		
49	2	P1	249	.378568	14915.82064530	14941.55244167		
50	25	D'-(16)	274	.414669	15017.54893770	15043.54876822		
51	1	P-	275	.415852	15035.23979015	15061.66935712		
52	20	D'(17)	295	.416081	16669.09873571	16705.23144654		
53	1	P-	296	.415133	16809.55633813	16846.92920864		
54	2	P1	298	.415984	16926.62279724	16964.82840324		
55	2	P2	300	.416292	17076.42006514	17115.67589470		
56	5	D-(18)	305	.422006	17149.84006889	17189.70128290		

57	5	D(19)	310	.423531	17474.26577412	17516.47350070
58	1	P-	311	.423326	17570.38829877	17613.46764493
59	2	P1	313	.424989	17635.27046541	17678.73544013
60	2	P2	315	.424388	17839.77860894	17884.71189503
61	5	D(20)	320	.420755	18494.76714773	18544.84173924
62	2	P2	322	.420260	18698.78937686	18750.84619244
63	2	P1	324	.417849	19032.55324712	19087.64761315
64	1	P-	325	.416737	19195.19564054	19251.71478056
65	20	D'(21)	345	.403750	21977.00494913	22064.11356908
66	1	P-	346	.396983	22630.87138230	22726.62667641
67	2	P1	348	.397997	22738.85988943	22835.79925996
68	2	P2	350	.398110	22923.05849412	23021.82410889
69	5	D(22)	355	.402893	22999.23335016	23098.72994911
70	2	P2	357	.391234	24222.26245920	24336.54625200
71	2	P1	359	.392096	24344.09806353	24459.93651024
72	1	P-	360	.392446	24412.27813093	24528.98625509
73	5	D(23)	365	.394399	24733.56223712	24854.02215766
74	1	P-	366	.394470	24827.42174086	24949.13754703
75	2	P1	368	.396500	24839.03807745	24960.87151306
76	2	P2	370	.398498	24853.56400354	24975.55063505
77	5	D-(24)	375	.395890	25600.34758001	25731.08613888
78	95	D"(25)	470	.439409	30648.03623569	30765.88600141
79	5	D-(26)	475	.374701	39416.13797770	40025.27703504

Table 4.2. Dirichlet Spectrum

#	Mult	Type	Total	Weyl Ratio	Eigenvalue(4)	Eigenvalue(3)	Eigenvalue(2)	Eigenvalue(1)
1	1	P+	1	.169464	13.86011628	13.86011628	13.86011633	13.86018062
2	2	P1	3	.315994	28.03042329	28.03042329	28.03042421	28.03138921
3	2	P2	5	.338232	54.00718123	54.00718124	54.00719368	54.01842873
4	2	P2	7	.297598	107.44766157	107.44766173	107.44784906	107.57500520
5	2	P1	9	.357383	118.87620478	118.87620502	118.87648356	119.05334896
6	1	P-	10	.378349	127.70139845	127.70139876	127.70176749	127.92285712
7	1	D+(1)	11	.372820	150.30310332	150.30310392	150.30380102	150.67715815
8	2	P1	13	.405427	170.01567754	170.01567852	170.01680511	170.55511213
9	2	P2	15	.386385	225.67034186	225.67034490	225.67371975	226.87551674
10	2	P2	17	.338728	330.10341086	330.10342471	330.11782241	335.24070935
11	2	P1	19	.319320	424.75902406	424.75906178	424.79626039	435.32040145
12	1	P+	20	.333261	430.17881774	430.17885740	430.21789508	440.69400759
13	2	P2	22	.363474	435.64687051	435.64691221	435.68787321	446.08393290
14	2	P1	24	.380989	462.19967723	462.19972998	462.25075636	474.84414619
15	1	P-	25	.366366	520.30820135	520.30828575	520.38723592	538.52222451
16	15	D'(2N)	40	.345261	1139.54837590	1139.55023719	1140.83018662	
17	1	P-	41	.339127	1213.79303977	1213.79542486	1215.37421255	
18	2	P1	43	.347119	1258.33635927	1258.33910665	1260.11454485	
19	2	P2	45	.359892	1275.83870270	1275.84160301	1277.69766425	
20	1	P+	46	.363915	1296.53169218	1296.53478139	1298.49318047	
21	2	P1	48	.375348	1319.05076848	1319.05407344	1321.12896342	
22	5	D-(6)	53	.401050	1384.83084452	1384.83484311	1387.23240288	
23	2	P2	55	.413379	1398.77026623	1398.77442508	1401.28464409	
24	2	P2	57	.401204	1541.51633000	1541.52241074	1544.90680162	
25	2	P1	59	.411293	1563.70579734	1563.71222712	1567.25742239	
26	1	P-	60	.415089	1581.45615305	1581.46287251	1585.13865198	
27	1	D'+(7)	61	.413491	1629.93026199	1629.93782157	1633.98623055	
28	2	P1	63	.419529	1673.38383415	1673.39221049	1677.79018684	
29	2	P2	65	.414261	1785.76876812	1785.77955793	1791.12787545	
30	2	P2	67	.385816	2075.25686047	2075.27619608	2083.97712052	
31	2	P1	69	.379569	2220.70749106	2220.73262922	2231.11290044	
32	1	P+	70	.382607	2241.91427253	2241.94035240	2252.58695646	
33	2	P2	72	.390736	2265.77336345	2265.80053347	2276.75041654	
34	2	P1	74	.397679	2298.84987979	2298.87861672	2310.21929226	
35	1	P-	75	.395893	2360.68983407	2360.72167778	2372.77766058	
36	15	D'(3N)	90	.420717	2826.12980165	2826.19371727	2838.50967582	
37	1	P-	91	.358935	3634.57154506	3634.73703190	3690.90844804	
38	2	P1	93	.364661	3666.54294780	3666.71407468	3723.19707377	
39	2	P2	95	.371401	3682.66804522	3682.84206851	3739.48930674	
40	1	P+	96	.374345	3696.73805374	3696.91463082	3753.74780172	
41	2	P1	98	.381348	3708.17051970	3708.34919250	3765.31957623	
42	2	P2	100	.385512	3759.84509491	3760.03347510	3817.45659340	
43	2	P2	102	.342290	4617.35940876	4617.76653073	4731.80659423	
44	2	P1	104	.347496	4647.00838824	4647.42555158	4761.18711526	
45	1	D+(12)	105	.349924	4664.98375251	4665.40708833	4779.00972117	
46	1	P-	106	.351930	4691.05736156	4691.48977096	4804.81018235	
47	2	P1	108	.358507	4692.27255983	4692.70540518	4805.90762599	

48	2	P2	110	.365106	4693.03092735	4693.46404543	4806.58741313
49	2	P2	112	.357684	4968.82441084	4969.35974813	5106.97267436
50	2	P1	114	.363803	4974.24775291	4974.78532749	5112.16897042
51	1	P-	115	.366558	4983.02616764	4983.56737542	5120.60047609
52	1	P+	116	.369478	4988.36643930	4988.90985342	5125.84545637
53	2	P1	118	.375288	4999.39304056	4999.94103694	5136.61898228
54	2	P2	120	.381193	5008.24946525	5008.80114896	5145.39594398
55	5	D-(15)	125	.366366	5642.37330024	5643.22945861	5839.89051178
56	90	D'(16)	215	.371155	12357.61184022	12371.49198151	
57	5	D-(17)	220	.363941	13162.74809515	13179.86892475	
58	2	P2	222	.358271	13654.22955418	13673.52205907	
59	2	P1	224	.361166	13672.84671357	13692.22558598	
60	1	P+	225	.362156	13707.65423100	13727.19726719	
61	1	P-	226	.363608	13716.44306349	13736.02565625	
62	2	P1	228	.365975	13763.74828069	13783.55875110	
63	2	P2	230	.368496	13801.87786566	13821.87605833	
64	2	P2	232	.369222	13939.29794589	13959.93165586	
65	2	P1	234	.372045	13959.27486285	13980.01144062	
66	1	P-	235	.373275	13979.23808084	14000.07558797	
67	1	D+(20)	236	.373408	14059.99757003	14081.23501516	
68	2	P1	238	.375507	14119.11211439	14140.64658101	
69	2	P2	240	.376409	14244.46458775	14266.64610251	
70	2	P2	242	.372210	14662.25387398	14686.55700976	
71	2	P1	244	.372267	14838.68684465	14863.98380270	
72	1	P+	245	.373397	14861.97257587	14887.39861905	
73	2	P2	247	.376020	14886.88414208	14912.44796644	
74	2	P1	249	.378242	14934.85318143	14960.69362717	
75	25	D'-(22)	274	.414669	15017.54893770	15043.54876822	
76	1	P-	275	.415852	15035.23979016	15061.66935712	
77	15	D'(6N)	290	.409029	16669.09889799	16705.45579011	
78	1	P-	291	.408120	16809.55633813	16846.92920864	
79	2	P1	293	.409495	16896.58789235	16934.59153517	
80	2	P2	295	.411722	16931.12864877	16969.38334212	
81	1	P+	296	.412426	16973.21565827	17011.78148049	
82	2	P1	298	.414448	17019.63063466	17058.54132032	
83	5	D-(26)	303	.419238	17149.84006889	17189.70128290	
84	2	P2	305	.421436	17184.18609802	17224.32081320	
85	2	P2	307	.419277	17483.81165786	17526.21851764	
86	2	P1	309	.421226	17531.92837201	17574.70848411	
87	1	P-	310	.421965	17570.38829877	17613.46764493	
88	1	D"+(27)	311	.421624	17675.51641463	17719.41853065	
89	2	P1	313	.422814	17769.79906728	17814.44516368	
90	2	P2	315	.421586	18015.67616623	18062.29392666	
91	2	P2	317	.415175	18602.76543082	18654.19295988	
92	2	P1	319	.413195	18910.25203488	18964.30727431	
93	1	P+	320	.413867	18952.47139278	19006.88780841	
94	2	P2	322	.415760	18999.29348843	19054.11056691	
95	2	P1	324	.417328	19067.73876935	19123.14887218	
96	1	P-	325	.416737	19195.19564054	19251.71478056	
97	15	D'(7N)	340	.397899	21977.00716610	22065.93976097	
98	1	P-	341	.391247	22630.87138230	22726.62667641	
99	2	P1	343	.392594	22711.82894147	22808.50135976	
100	2	P2	345	.394393	22753.64682884	22850.79723689	
101	1	P+	346	.395130	22788.23696822	22885.78309274	
102	2	P1	348	.397070	22817.49309002	22915.37581621	
103	2	P2	350	.397606	22966.06553185	23065.68658567	
104	2	P2	352	.386786	24126.71256436	24239.83820897	
105	2	P1	354	.387708	24244.38151475	24358.98646001	
106	1	D+(32)	355	.388070	24312.24858701	24427.70343585	
107	1	P-	356	.388085	24412.27813093	24528.98625509	
108	2	P1	358	.390097	24427.94784089	24544.86463138	
109	2	P2	360	.392142	24440.33598099	24557.42162984	
110	2	P2	362	.391108	24738.18004160	24858.73581342	
111	2	P1	364	.392881	24774.39516933	24895.42506023	
112	1	P-	365	.393392	24827.42174086	24949.13754704	
113	1	P+	366	.394298	24843.48898733	24965.39687062	
114	2	P1	368	.395955	24889.66485079	25012.15404172	
115	2	P2	370	.397809	24917.31467991	25040.14365939	
116	5	D-(35)	375	.395890	25600.34758001	25731.08613888	
117	90	D"(36)	465	.434734	30648.06079069	30781.61860163	
118	5	D-(37)	470	.370757	39416.13797770	40025.27703504	
119	2	P2	472	.370094	39770.03363026	40382.63837857	
120	2	P1	474	.371563	39785.72156665	40398.51394371	
121	1	P+	475	.372168	39814.05709411	40427.22504674	

Each table assigns a number to the distinct eigenvalues, and reports the multiplicity (we adjusted a tolerance for identifying slightly differing values as the same in order to obtain accurate sorting) and the running total, the eigenvalue counting function $\rho(x)$ for x equal to that eigenvalue. We identify the type of eigenspace according to the following code: P stands for primitive, and D for derived eigenspace. We write $P+$, $P-$, $P1$, $P2$ according to the representations π_+, π_-, π_1, π_2 of D_5 acting on the eigenspace. For the D types, we indicate in parenthesis the number of the previous eigenvalue from which it is derived (in the Dirichlet table $2N$ refers to the Neumann eigenvalue numbered 2, etc.). We write $D-$ or $D+$ to denote an eigenspace derived from a $P-$ or $P+$ eigenspace, and we add primes to indicate how far along in the derived sequence it is. Thus, for example, eigenvalue 50 in the Neuman spectrum is labeled $D' - (16)$, since it is derived from eigenvalue 16 in the Neumann spectrum, which in turn is derived from eigenvalue 5, which is a $P-$ eigenspace.

The tables also report the approximation of the Weyl ratio, which is defined by

$$WR(x) = \rho(x)/x^{\beta} \tag{4.5}$$

where

$$\beta = \frac{\log 5}{\log 5 - \log r} \tag{4.6}$$

(in the tables, x is the eigenvalue). According to a result of Kigami and Lapidus [KL], $WR(x)$ is asymptotically a periodic function of $\log x$ of period $\log 5/r$ that is bounded and strictly positive, but not continuous. In Figure 4.1 we display the graphs of the data points $(\log x, WR(x))$ for Neumann and Dirichlet eigenvalues. We used more data in creating these graphs than is reported in the tables. The accuracy of the approximations is degraded toward the upper end. Nevertheless, the graphs give striking confirmation to the predictions, and give a reasonable image of what the periodic function looks like.

An examination of the types of eigenspaces in the tables reveals that although there are twelve possible ordering of the types within a quintet of unequal values, only four of them actually occur. We denote by $(+,1,2)$ the quintet type in which the $P+$ eigenvalue is the smallest and the $P2$ eigenvalue the largest, and similarly for other permutations and quintets containing a $P-$ eigenvalue.

Conjecture 4.1: Every quintet of unequal eigenvalues is either of type $(+,1,2)$, $(2,1,+)$, $(-,1,2)$ or $(2,1,-)$.

A slight clarification is required for ambiguous cases, such as eigenvalues 20–23 of the Dirichlet spectrum. The multiplicity 5 eigenspace of 22 is a sum of π_-, π_1 and π_2 representations. The π_2 part needs to be grouped with eigenvalues 20 and 21 to make a $(+,1,2)$ quintet, while the π_- and π_1 parts need to be grouped with eigenvalue 23 to make a $(-,1,2)$ quintet with the first two eigenvalues actually equal.

Note that it is possible to find other sequences of three consecutive eigenspaces. For example, eigenvalues 11–13 of the Dirichlet spectrum show a π_1 followed by a π_+ followed by a π_2, but these do not constitute a quintet. There are also many

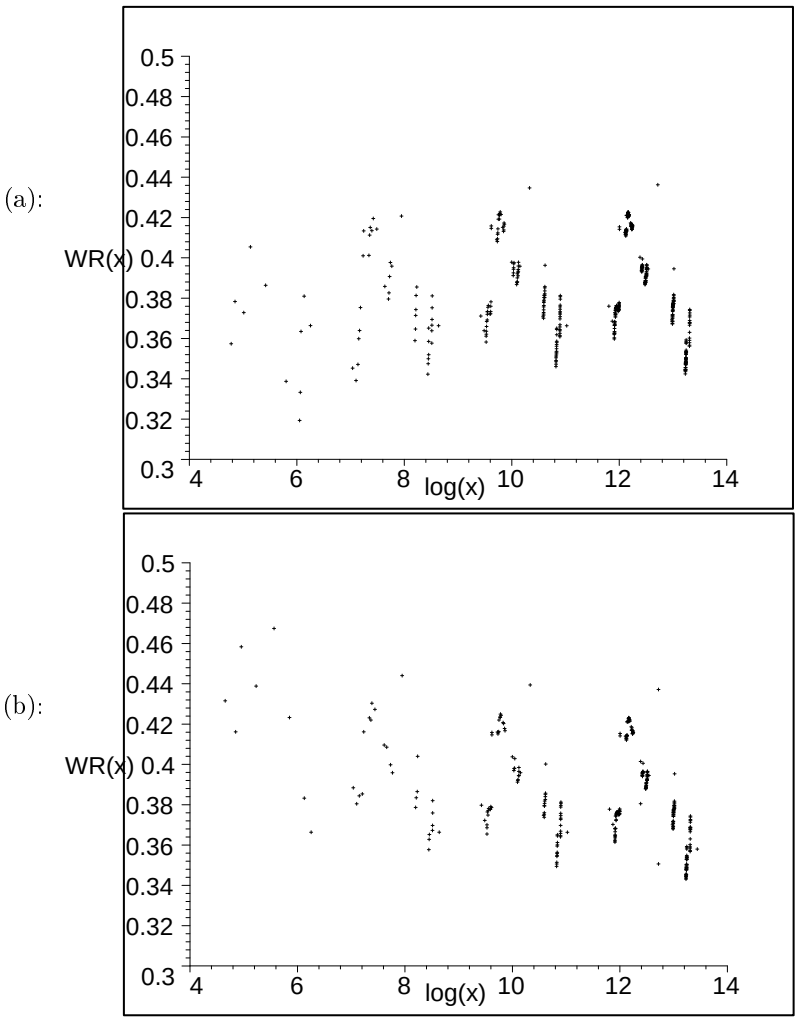


Figure 4.1. The Weyl ratio $WR(x)$ as a function of $\log x$ for (a) Dirichlet and (b) Neumann eigenvalues. The asymptotic periodicity is quite apparent in comparing the $[10, 12]$ and the $[12, 14]$ intervals.

omitted sequences. For example, there are never two consecutive π_1 eigenspaces, yet there are often consecutive π_2 eigenspaces.

We are not able to predict the location of the eigenvalues in the $D-$ derived eigenspaces, but aside from that, each of the spectra appears to be organized into cycles, with each cycle subdivided into three subcycles. The exact order of the eigenvalues in a subcycle is not always the same, and there may perhaps be some periodicity to the ordering in the subcycles, but with data covering 26 subcycles

we do not have clear evidence for such a pattern. The Tables presented in this paper cover only six subcycles; the full data may be found at the website.

We begin with a description of the Dirichlet spectrum. The first two subcycles of a cycle, which we denote by SCD, have the form

$$P+, P1, P2, P2, P1, P-, D+, P1, P2, P2, P1, P+, P2, P1, P-, D', P-, P1, P2.$$

Aside from D', this subcycle consists of six quintets. Variations in the order of the subcycle involve either changing the first $P+$ to $P-$ or transposing the first $P-$ and the adjacent $D+$. The third subcycle, denoted SCD$'$, has the form either

$$P+, P1, P2, P2, P1, P-, D+, P1, P2, P2, P1, P+, P-, P1, P2, D^{(k)}, P2, P1, P-,$$

or

$$P+, P1, P2, P2, P1, D+, P-, P1, P2, P2, P1, P-, P+, P1, P2, D^{(k)}, P2, P1, P+,$$

where $D^{(k)}$ denotes an eigenvalue in a derived sequence starting with a P1 or P2 Neumann eigenvalue, with $k \geq 2$.

For the Neumann spectrum, the basic form SCN for the first two subcycles is

$$P-, P1, P2, D, P-, P1, P2, D, P2, P1, P-, D'.$$

Here D denotes the first derived eigenvalue from a $P1$ or $P2$ eigenspace (recall these are precisely the ones that appear in the Neumann spectrum only), and D' denotes the second eigenvalue in such a sequence. The third subcycle SCN$'$ has the same form except that D' is replaced by $D^{(k)}$, with $k \geq 2$. Each subcycle contains three quintets, and variations in the order allow a reversal of order in each of the quintets separately. The first subcycle is exceptional because the first eigenvalue is $P+$ rather than $P-$, and it is the only $P+$ eigenspace in the Neumann spectrum.

Conjecture 4.2: Aside from sporadic insertions of $D-$ eigenspaces, the Dirichlet spectrum is organized into cycles of the form SCD, SCD, SCD' and the Neumann spectrum is organized into cycles of the form SCN, SCN, SCN' (with the exception of the first eigenvalue), with variations in the order of subcycles as described above.

Note that Conjecture 4.2 implies Conjecture 4.1. We can actually say a little bit more. For example, the order k of the derived space $D^{(k)}$ in every third subcycle can be predicted in terms of the number of the cycle, as it is determined by the earlier D' and $D^{(k)}$ eigenspaces in the spectrum. Note that there is an alignment of the eigenvalues associated to the $P-$, D' and $D^{(k)}$ eigenspaces of both spectra. Each subcycle contains either a D' or $D^{(k)}$ eigenspace which is aligned between the two spectra. Subcycles contain on average three $P-$ eigenspaces, but the alignment sometimes crosses subcycle divisions. There is also an alignment of the counting function $\rho(x)$ at the end of each subcycle. Because of the sporadic insertions, we are not able to predict what this common value is. Although the Dirichlet subcycle is longer, 19 as opposed to 12 for the Neumann subcycle, the Neumann multiplicities are higher, which allows the counting functions to remain in sync. The numerical value of the $D^{(k)}$ eigenvalues in cycle number n (subcycle number

$3n$) is exactly $5/r$ times the value of the D' or $D^{(k)}$ eigenvalue in the subcycle number n. Thus the eigenvalues in subcycle number n are roughly proportional to n^α for $\alpha = \log(5/r)/\log 3$.

Concerning the $D-$ eigenspaces, in the Neumann spectrum they appear to come only in the second subcycle in between the third ($P2$) and fourth (D) eigenspace, and two in the third subcycle bracketing a D' eigenspace. The $D-^{(k)}$ eigenspaces do not reveal any regularity, however. In the Dirichlet spectrum we often see $D-$ eigenspaces in the second subcycle interrupting the first quintet $(P+, P1, D-, P2)$ and in the third subcycle bracketing the $D^{(k)}$ eigenspace.

Figure 4.2 shows the approximation to the actual spectra $\sigma(D)$ and $\sigma(N)$ as subsets of the line, on three different scales differing by a factor of $5/r \approx 10.84$. Again we are using more data than is presented in the tables, with the associated inaccuracy. In particular, the tendency to overestimate eigenvalues is quite visible at the upper end of the spectra.

Conjecture 4.3: The sets $(\frac{5}{r})^m \sigma(D)$ and $(\frac{5}{r})^m \sigma(N)$ converge, locally in the Hausdorff metric, as $m \to \infty$ to the same Cantor set σ.

It would be interesting to know if σ can be identified as a rescaled limit of a Julia set of some sort of analytic function (we have checked that it does not appear to be related to Julia sets of quadratic polynomials).

Suppose $\lambda = 5r^{-1}x$ is a derived eigenvalue. Then $\rho(5r^{-1}x) \approx 5\rho(x)$ according to [KL]. Note that equality means $WR(5r^{-1}x) = WR(x)$. Our data indicates that this does happen.

Conjecture 4.4: a) Suppose $\lambda = 5r^{-1}x$ is a derived eigenvalue and x is a primitive eigenvalue of $P-$ type corresponding to the last $P-$ of either an SCN or SCD subcycle. Then $\rho(5r^{-1}x) = 5\rho(x)$ and $WR(5r^{-1}x) = WR(x)$.

b) Suppose x is a primitive Neumann eigenvalue of P1 or P2 type, and consider the derived sequence $(5r^{-1})^m x$ generated by Algorithm 2.2. Then $\rho((5r^{-1})^m x)$ $= 5\rho((5r^{-1})^{m-1}x) - 5$ for all $m \geq 1$

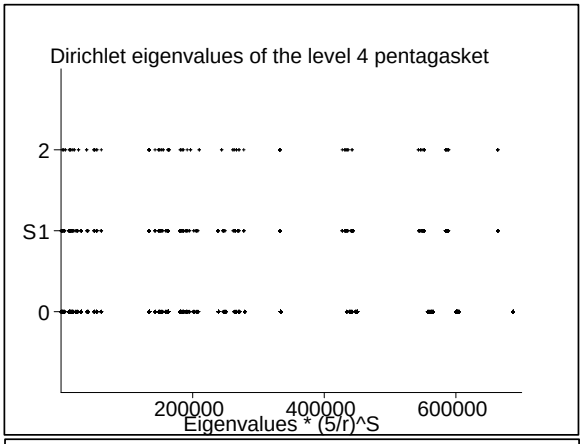
c) Suppose $\lambda = 5r^{-1}x$ is a derived eigenvalue for either Dirichlet or Neumann boundary conditions. Then

$$|\rho(5r^{-1}x) - 5\rho(x)| \leq 10.$$

5. Local symmetries

An eigenfunction corresponding to the trivial representation π^+ of D_5 is invariant under all the D_5 symmetries. In this section we will show that it also has further local symmetries around the center of the pentagasket, and in particular these symmetries force it to be constant along the Cantor set bordering the inner deleted pentagon. In [DSV] the analogous constancy was proved for D_3 symmetric eigenfunctions on SG (along the inner deleted triangle) by algebraic methods. In fact the same argument for local symmetries presented here is valid for SG, and

(a):



(b):

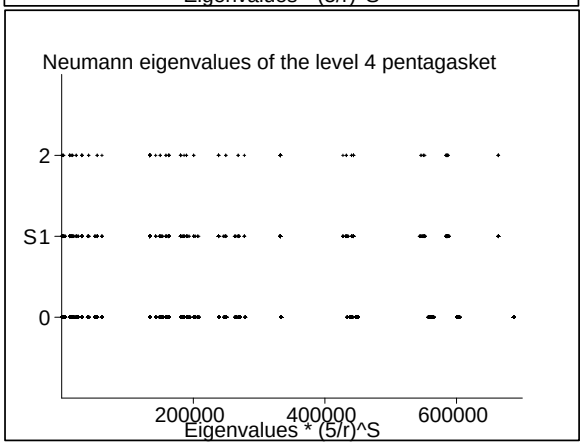


Figure 4.2: The approximate spectrum as a subset of the line on three different scales for (a) Dirichlet and (b) Neumann eigenvalues. The bottom line gives the approximate eigenvalues computed on level 4 (note that this includes considerably more eigenvalues than are reported in the Tables). The next two lines give a zoom of the lower portion of the spectrum enlarged by the factors $5/r$ and $(5/r)^2$. The apparent drift to the right (from the top down) is explained by the nature of the approximation error; the approximations are always overestimates and the relative error increases toward the upper end of the approximate spectrum.

other n–gaskets with n odd. Because these symmetries are only local, they do not form a group.

Consider first the case when the multiplicity of the corresponding eigenvalue is one. There will be an infinite sequence of local symmetries R_m for $m \geq 2$, defined on the union of $5 \cdot 2^{m-1}$ cells of level m that surround the central deleted pentagon. For x in such a cell, $R_m x$ is simply the reflection of x in the axis of symmetry that

is perpendicular to the side of the deleted pentagon where the cell is located. In Figure 5.1 the case $m = 2$ is illustrated. In the cell with vertices labeled a, b, c, d, e, the reflection R_2 fixes a, interchanges b and c, and interchanges d and e, and similarly in the other 9 cells. (Of course the vertices b and e belong to 2 cells, so R_2 is ambiguously defined there.)

Theorem 5.1. *Let u denote a Dirichlet eigenfunction with multiplicity one that is invariant under D_5. Then it is also invariant under all the local symmetries R_m, $m \geq 2$, meaning that $u(R_m x) = u(x)$ for any x in the domain of R_m.*

Proof: We give the proof for $m = 2$, but the same argument may be used inductively to show the result for general m. Note that the D_5 invariance means that $u(x) = u(x') = u(x'')$ for all the labeled vertices ($x = a, b, c, d, e$) in Figure 5.1 (in the induction argument the analogous statements would follow from the induction hypothesis).

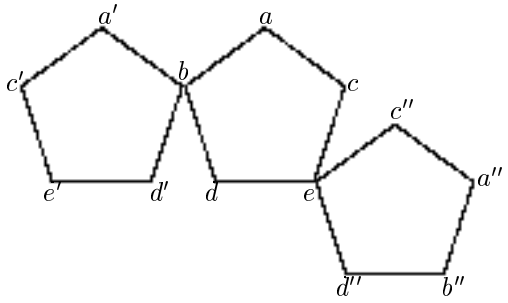


Figure 5.1: Labels on vertices of level 2 cells to help describe R_2.

Define a new function $\widetilde{u}$ by $\widetilde{u}(x) = u(R_2 x)$ if x is in the domain of R_2, and $\widetilde{u}(x) = u(x)$ if x is not in the domain. The above equalities mean that $\widetilde{u}$ is well defined even at the points of ambiguity for R_2, and moreover $\widetilde{u}$ is continuous. We claim that $\widetilde{u}$ satisfies the same eigenvalue equation as u. This is obvious at all vertex points other than those labeled b, c, d, e in Figure 5.1, because the reflection does not change the Laplacian. The claim is that the eigenvalue equation for u at b becomes the eigenvalue equation for $\widetilde{u}$ at c, and vice versa, while the same is true for d and e. For example the approximate Laplacian $\Delta u(b)$ at level 2 is

$$\frac{1}{2r^2 5^3}(2u(b) - p(u(a) + u(a') + u(d) + u(d')) - q(u(c) + u(c') + u(e) + u(e')),$$

while the approximate Laplacian $\Delta\widetilde{u}(c)$ at level 2 is

$$\frac{1}{r^2 5^3}(\widetilde{u}(c) - p(\widetilde{u}(a) + \widetilde{u}(e)) - q(\widetilde{u}(b) + \widetilde{u}(d))).$$

Both quantities equal

$$\frac{1}{r^2 5^3}(u(b) - p(u(a) + u(d)) - q(u(c) + u(e))).$$

A similar argument holds for the approximation at all levels. Thus $\Delta u(b) = \Delta\widetilde{u}(c)$, so $-\Delta u(b) = \lambda u(b)$ if and only if $-\Delta\widetilde{u}(c) = \lambda\widetilde{u}(c)$. A similar argument works switching b and c, and also for the d, e pair.

We have established the claim $-\Delta\widetilde{u} = \lambda\widetilde{u}$. Since the multiplicity of λ is one, we have $u = \widetilde{u}$ which means $u(x) = u(R_2 x)$ as desired. Q.E.D.

Corollary 5.2. *Such eigenfunctions are constant on the Cantor set of points bordering the central deleted pentagon.*

Proof: The invariance under R_m for all $m \leq k$ implies that u is constant on a set of $5 \cdot 2^{k-1}$ points equidistributed along the Cantor set. The result follows by continuity. Q.E.D.

If the eigenspace has multiplicity greater than one, the arguments only prove the existence of such eigenfunctions. Figure 5.2 shows the groundstate Dirichlet eigenfunction.

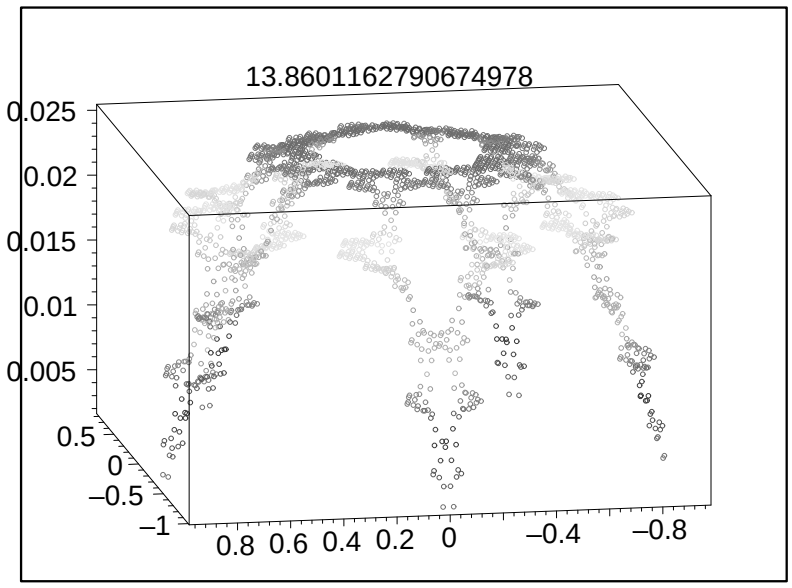


Figure 5.2: The groundstate Dirichlet eigenfunction.

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