PROPERTIES OF POLYNOMIALS, ORTHOGONAL ON A CIRCUMFERENCE, WITH RANDOM PARAMETERS

A. V. Teplyaev

UDC 519.2

A method for the construction of orthogonal polynomials in terms of circular parameters is known. In this paper it is shown that if for the latter one takes some independent random variables, then the spectral measure will be almost always absolutely continuous. One obtains estimates for the density of this measure.

Assume that there is given a sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$, satisfying the condition

$$|a_n| < 1, \quad n = 0, 1, 2, \ldots. \tag{1}$$

Then one can construct a sequence of polynomials $\{\phi_n(z)\}_{n=0}^{\infty}$ according to the recurrence relations

$$\phi_0(z) \equiv 1, \ \phi_{n+1}(z) = z\phi_n(z) - \overline{a_n}\,\phi_n^*(z), \ \phi_{n+1}^*(z) = \phi_n^*(z) - a_n z \phi_n(z), \tag{2}$$

where for any polynomial p(z) of degree n we have set $p^*(z) = z^n\,\overline{p(\overline{z}^{-1})}$. The constructed polynomials are orthogonal on the circumference $T = \{\,|z| = 1\}$ with respect to some measure σ, depending on the numbers $\{a_n\}_{n=0}^{\infty}$, which are called circular parameters.

It is known that if $\sum_{n=0}^{\infty} |a_n| < \infty$, then the measure σ is absolutely continuous with respect to the Lebesgue measure and the derivative σ', which exists almost everywhere, is uniformly separated from zero and from infinity. At the same time, the condition

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty \tag{3}$$

is equivalent to the Szegö condition $\ln \sigma' \in L_1(T)$, which admits an essential arbitrary singular component (see [1]).

In [2], E. M. Nikishin has formulated and solved the problem on the behavior of the random measures σ. He has shown that if an arbitrary sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$, satisfying only the conditions (1) and (3), is provided with independent random signs, then the measure σ is almost always absolutely continuous.

We mention that in Nikishin's paper there is some inaccuracy: without justification he considers independent the variables $\Psi_k(t) = r_k(t)\dfrac{\phi_k(z,t)}{\phi_k^*(z,t)}$; in addition, in the formulation of the theorem there is a misprint: instead of $\exp \gamma(\sigma')^2 \in L_1(T)$ one should have $\exp(\gamma \ln^2\sigma') \in L_1(T)$ (see below).

In this paper we wish to show that the basic steps of E. M. Nikishin's reasoning are valid also in the more general situation of arbitrary independent random variables.

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova Akademii Nauk SSSR, Vol. 177, pp. 157-162, 1989.

0090-4104/92/6101-1931$12.50

THEOREM. Assume that the circular parameters represent a sequence of independent random variables $\xi_0, \xi_1, \ldots$, satisfying the following conditions

$$|\xi_n| < 1, \quad \sum_{n=0}^{\infty} E|\xi_n|^2 < \infty, \quad \sum_{n=0}^{\infty} |E\xi_n| < \infty. \tag{4}$$

Then with probability 1:

a) the measure σ is absolutely continuous;

b) for some $\gamma > 0$ we have $\exp\{\gamma \mid \ln \sigma' \mid \ln \mid \ln \sigma' \mid\} \in L_1(T)$.

Proof. We consider the sequence of events $A_\nu = \{ \mid \xi_k \mid < 1 - 1/\nu, k = 0, 1, \ldots\}$, $\nu = 3, 4, 5, \ldots$ and the sequence of sections of random variables

$$\xi_k^\nu = \begin{cases} \xi_k, & \text{if } |\xi_k| < 1 - \frac{1}{\nu}, \\ 0, & \text{if } |\xi_k| \geqslant 1 - \frac{1}{\nu} \end{cases} \tag{5}$$

Since the series $\sum_{k=0}^{\infty} \xi_k$ converges with probability 1, we have $P\{\bigcup_{\nu=3}^{\infty} A_\nu\} = 1$. Consequently, almost always, starting with some (random) index ν_0, we have $\xi_k = \xi_k^\nu$, $k = 0, 1, \ldots$ for any $\nu > \nu_0$.

We fix ν and we consider the sequence of polynomials $\{\Phi_n^\nu(z)\}_{n=0}^{\infty}$, constructed according to formulas (2) with parameters $\{\xi_n^\nu\}_{n=0}^{\infty}$. Everywhere below we shall assume $\mid z \mid = 1$. Under this condition, we have $|\Phi_n(z)| = |\Phi_n^*(z)|$.

There exists a constant $C > 1$, depending on ν, such that $\mid 1 + x \mid^p \leq 1 + p\Re x + C^{\mid p \mid} \mid x \mid^2$ for any real number p and for any complex number x, $\mid x \mid < 1 - 1/\nu$. We note that $\phi_n(z)$ depends only on $\xi_0, \ldots, \xi_{n-1}$ and it does not depend on ξ_n. Therefore,

$$E\{|1 - \xi_n^\nu z \frac{\Phi_n^\nu(z)}{\Phi_n^{\nu*}(z)}|^p \mid \xi_0^\nu, \ldots, \xi_{n-1}^\nu\} \leqslant$$
$$\leqslant E\{1 - p\mathrm{Re}[\xi_n^\nu z \frac{\Phi_n^\nu(z)}{\Phi_n^{\nu*}(z)}] + C^{|p|}|\xi_n^\nu|^2 \Big| \xi_0^\nu, \ldots, \xi_{n-1}^\nu\} \leqslant$$
$$\leqslant 1 + |pE\xi_n^\nu| + C^{|p|} E|\xi_n^\nu|^2.$$

Directly from relations (2) there follows that

$$\Phi_n^*(z) = \prod_{k=0}^{n-1} \{1 - \xi_k z \frac{\Phi_k(z)}{\Phi_k^*(z)}\}. \tag{6}$$

We obtain by induction that

$$E|\Phi_n^{\nu*}(z)|^p \leqslant \prod_{k=0}^{n-1} \{1 + |pE\xi_k^\nu| + C^{|p|} E|\xi_n^\nu|^2\}, \quad n = 1, 2, \ldots$$

for any real p. Consequently, there exists a constant B such that

$$E|\Phi_n^{\nu*}(z)|^p \leqslant \exp(BC^{|p|}), \quad n = 0, 1, \ldots \tag{7}$$

From this it follows, in particular, that

$$E(\sup_n |\Phi_n^{\nu*}(z)|^{-2}) < \mathrm{const} < \infty. \tag{8}$$

Indeed, according to formula (6) we have:

$$E\{|\Phi_{n+1}^{\nu*}(z)|^{-1} | \xi_0, \ldots, \xi_{n-1}\} =$$
$$= |\Phi_n^{\nu*}(z)|^{-1} E\{|1 - \xi_n^\nu z \frac{\Phi_n^\nu(z)}{\Phi_n^{\nu*}(z)}|^{-1} | \xi_0, \ldots, \xi_{n-1}\} \geqslant$$
$$\geqslant (1 + |E\xi_n^\nu|)^{-1} |\Phi_n^{\nu*}(z)|.$$

Consequently, the sequence $\{|\Phi_n^{\nu*}(z)|^{-1}\prod_{k=0}^{n-1}(1+|E\xi_n^{\nu}|)\}_{n=1}^{\infty}$ forms a submartingale. This proves inequality (8) since

$$\prod_{k=0}^{\infty}(1+|E\xi_n^{\nu}|)<\infty, \quad E|\Phi_n^{\nu*}(z)|^{-2}\leqslant \exp(BC^2)<\infty.$$

It is known that the measure with density $|\phi_n^*(z)|^{-2}$ converges weakly to the measure σ under a suitable choice of the normalizing constant (see [1]). At the same time, with probability unity there exists a summable majorant for the justification of the limiting process, since almost always $\int_0^{2\pi}\sup_n|\Phi_n^{\nu*}(z)|^{-2}|dz|<\infty$. Consequently, with probability 1 the measure σ is absolutely continuous and $|\Phi_n^*(z)|^{-2}\longrightarrow\sigma'(z)$ almost everywhere with respect to the Lebesgue measure.

From inequalities (7) there follows that $E(\sigma')^p \leq \exp(DC^{|p|})$ ($D = \text{const}$) for any real number p. Setting $t = (\ln x)/(\ln c)$ $(x > 1)$, we obtain

$$\begin{aligned}P\{|\ln\sigma'|>x\}&=P\{(\sigma')^t>e^{tx}\}+P\{(\sigma')^{-t}>e^{tx}\}\leqslant\\ &\leqslant 2\exp\{DC^t-tx\}=2\exp\{Dx-\frac{x\ln x}{\ln c}\}.\end{aligned}$$

Making use of this estimate, we obtain

$$\begin{aligned}&E\exp\{\gamma|\ln\sigma'|\ln|\ln\sigma'|\}\leqslant\\ &\leqslant \text{const}+\int_1^{\infty}\gamma(\ln x+1)e^{\gamma x\ln x}P\{|\ln\sigma'|>x\}dx<\infty\end{aligned}\tag{9}$$

for some $\gamma > 0$, The theorem is proved.

Remark 1. If, under conditions (4), for any sufficiently large ν there exists a sequence of positive constants $\{b_n\}_{n=0}^{\infty}$ such that for any real number p, $|p| \geq 1$, and any complex number α, $|\alpha| = 1$, we have the inequalities

$$E|1+\alpha\xi_k^{\nu}|^p\leqslant e^{b_k p^2},\ k=0,1,\dots;\ \sum_{k=0}^{\infty}b_k<\infty,\tag{10}$$

then for any $\gamma > 0$ we have $\exp\{\gamma\ln^2\sigma'\} \in L_1(T)$ with probability 1.

Indeed, similarly to (7) one can show that

$$E\{\prod_{k=n}^{m}|1-\xi_n^{\nu}z\frac{\Phi_k^{\nu}(z)}{\Phi_k^{\nu*}(z)}|^p\}\leqslant e^{p^2\sum_{k=n}^{m}b_k},\quad |p|\geqslant 1.$$

We fix $\lambda > \gamma$. Then there exists n_0 such that $64\sum_{k=n_0}^{\infty}b_k<\frac{1}{\lambda}$. . We also have the inequality

$$E\{\prod_{k=0}^{\infty}|1-\xi_k^{\nu}z\frac{\Phi_k(z)}{\Phi_k^{\nu*}(z)}|^{2p}\}\leqslant \nu^{2n_0|p|}e^{4p^2\sum_{k=n_0}^{\infty}b_k},\quad |p|\geqslant 1.$$

Let $x_0 = 4n_0\ln\nu$. Then for any $x > x_0$, setting $t=\frac{x-2n_0\ln\nu}{8\sum_{k=n_0}^{\infty}b_k}\geqslant\frac{x}{16\sum_{k=n_0}^{\infty}b_k}$, we have $P\{|\ln\sigma'|>x\}\leqslant$

$$\leqslant 2\exp\{2tn_0\ln\nu+4t^2\sum_{k=0}^{\infty}b_k-tx\}\leqslant 2e^{-\lambda x^2}.$$

Consequently,

$$E\exp\{\gamma\ln^2\sigma'\}\leqslant\text{const}+\int_{x_0}^{\infty}\gamma 2xe^{\gamma x^2}P\{|\ln\sigma'|>x\}dx<\infty.\tag{11}$$

Remark 2. If there exist positive numbers $\{C_n\}_{n=0}^{\infty}$ such that $C_n<1$, $\sum_{n=0}^{\infty}C_n^2<\infty$ and the random independent circular parameters satisfy the conditions $|\xi_n|<C_n$, $|E\xi_n|<C_n^2$, then with probability 1 we have $\exp\{\gamma\ln^2\sigma'\} \in L_1(T)$ for any real number γ. We note that these conditions are satisfied by the random parameters considered in E. M. Nikishin's paper.

For the proof it is sufficient to verify the validity of the conditions (10). We have the following series of estimates ($|p| \geq 1$, $|\alpha| = 1$):

$$E|1+\alpha\xi_n|^p = E(1+2\operatorname{Re}\alpha\xi_n+|\xi_n|^2)^{p/2} \leq$$
$$\leq E\exp\{p\operatorname{Re}\alpha\xi_n+|p|O(c_n^2)\} \leq$$
$$\leq \exp\{|p|O(c_n^2)\}(1+|p|c_n^2+\sum_{k=2}^{\infty}\frac{|p|^k c_n^k}{k!}) \leq$$
$$\leq \exp\{|p|O(c_n^2)\}(e^{|p|c_n}-|p|c_n+|p|c_n^2) \leq e^{p^2 O(c_n^2)}.$$

Remark 3. Inequalities (9) and (11) cannot be improved in a certain sense. Namely: we define random variables ξ_0, ξ_1,... in the following manner:

$$\xi_n = \begin{cases} \frac{1}{2} & \text{with sequence } \frac{1}{2}n^{-1-\varepsilon} \\ -\frac{1}{2} & \text{with sequence } \frac{1}{2}n^{-1-\varepsilon} \\ 0 & \text{with sequence } 1-n^{-1-\varepsilon}, \ \varepsilon>0. \end{cases}$$

Then the sequence $\{\xi_n\}_{n=0}^{\infty}$ satisfies the conditions (4), but there exists $\gamma > 0$ such that $E\int_T \exp\{\gamma\,|\ln\sigma'|\ln|\ln\sigma'|\}\,|dz| = \infty$. If we set

$$\xi_0=\xi_1=0 \qquad \xi_n = \begin{cases} n^{-\frac{1}{2}-\varepsilon} & \text{with sequence } \frac{1}{2} \\ -n^{-\frac{1}{2}-\varepsilon} & \text{with sequence } \frac{1}{2} \quad 0<\varepsilon<\frac{1}{2}, \end{cases}$$

then the random variables $\{\xi_n\}_{n=0}^{\infty}$ satisfy the conditions (10) and $E\int_T \exp|\ln\sigma'|^{2+\delta}|dz| = \infty$ for $\delta > 4\varepsilon/(1-2\varepsilon)$.

We prove only the second assertion; the first one is proved in a similar manner. Indeed, according to (6) we have $|\phi_n^*(e^{i\theta})| \leq \prod_{k=0}^{n-1}(1+|\xi_k|)$, $\theta \in [0, 2\pi]$. We note that for real parameters we have $\phi_n(1) = \phi_n^*(1)$. Therefore, with probability not less than 2^{-n+2}, there occurs the event $\phi_n^*(1) = \prod_{k=0}^{n-1}(1+|\xi_k|)$ $n=2,3,\ldots$ By Bernshtein's theorem we have

$$\left|\frac{d}{d\theta}\phi_n^*(e^{i\theta})\right| \leq n\prod_{k=0}^{n-1}(1+|\xi_k|).$$

Consequently, with probability exceeding 2^{-n}, for any $\theta \in [-\frac{1}{2n}, \frac{1}{2n}]$ and for sufficiently large n we have the inequalities

$$|\phi_n^*(e^{i\theta})| \geq \frac{1}{2}\prod_{k=0}^{n-1}(1+|\xi_k|) \geq \frac{1}{2}e^{cn^{\frac{1}{2}-\varepsilon}} \quad (c>0).$$

Thus,

$$P\{\int_0^{2\pi}\exp|2\ln|\phi_n^*(e^{i\theta})||^{2+\delta}d\theta \geq \frac{1}{n}\exp(2cn^{\frac{1}{2}-\varepsilon}-2\ln 2)^{2+\delta}\} \geq 2^{-n}.$$

Setting $x=\exp(cn^{\frac{1}{2}-\varepsilon})^{2+\delta}$, we obtain that for sufficiently large x we have

$$P\{\sup_n\int_0^{2\pi}\exp|2\ln|\phi_n^*(e^{i\theta})||^{2+\delta}d\theta > x\} \geq \exp(-b\ln^{\alpha}x),$$

where $\alpha = \frac{1}{(\frac{1}{2}-\varepsilon)(2+\delta)} < 1$.

Function $f(x) = \exp|\ln x|^{2+\delta}$ is convex in domain $x > 0$. Therefore, sequence $\{\exp|\ln|\phi_n^*(e^{i\theta})||^{2+\delta}\}_{n=0}^{\infty}$ forms a submartingale for any fixed $\theta \in [0, 2\pi]$ since the sequence $\{\phi_n^*(e^{i\theta})\}_{n=0}^{\infty}$ forms a martingale. Thus,

$$\infty = E\{\sup \int_0^{2\pi} \exp|2\ln|\Phi_n^*(e^{i\theta})||^{2+\delta} d\theta\} \leqslant$$

$$\leqslant \int_0^{2\pi} E \sup_n \{\exp|2\ln|\Phi_n^*(e^{i\theta})||^{2+\delta} d\theta\} \leqslant$$

The proof is concluded.

LITERATURE CITED

1. E. M. Nikishin, "Random orthogonal polynomials on the circumference," Vestnik Moskov. Univ., Ser. I, Mat. Mekh., No. 1, 52-55 (1987).
2. Ya. L. Geronimus, Polynomials Orthogonal on a Circle and Interval, Pergamon Press, New York (1960).

DETECTION OF OUTLIERS BY THE CHAUVENET TEST IN OBSERVATIONS CONNECTED IN A HOMOGENEOUS MARKOV CHAIN

N. M. Khalfina

UDC 519.281

Let $x_1, \ldots, x_n$ be random variables, connected in a normal Markov chain. The paper investigates the asymptotic behavior when $n \to \infty$ of the distribution of the random variable N, equal to the number of outliers in the Chauvenet sense.

The statistical problem of detecting outliers in the results of observations has been considered for the first time by L. N. Bol'shev [1] and by L. N. Bol'shev and M. Ubaidullaeva [2] for functions of the normal distribution. Subsequently, the investigation has been carried out in two directions; in one of them the authors have obtained more precise results for particular distributions [3], [4], while in the other direction the authors have considered wider classes of distributions, up to most general ones [5] - [8]. But in all the above mentioned investigations one has assumed that one observes independent random variabl while in this paper the random variables will be assumed to be dependent, connected in a simple homogeneous Markov cha

Thus, assume that there is given a sequence of random variables $\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots, X_n$, connected a simple homogeneous Markov chain and, moreover, any pair of neighbors X_{m-1}, X_m has a nondegenerate two-dimensional normal distribution with parameters

$$E(X_{m-1}) = E(X_m) = a,\ \mathcal{D}(X_{m-1}) = \mathcal{D}(X_m) = \sigma^2,\ R(X_{m-1}, X_m) = \rho,\ |\rho| \neq 1.$$

Assume that on the random variables $X_1, \ldots, X_{n+1}$ one has performed n + 1 successive observations $x_1, \ldots, x_{n+1}$, and the parameters a, σ, ρ are known. The marginal probability density $f_1(x_i)$ of X_i and the conditional probability density

$$f(x_i | X_{i-1} = x_{i-1}), \quad i = 2, \ldots, n+1$$

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova Akademii Nauk SSSR, Vol. 177, pp. 163-169, 1989.