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THE PURE POINT SPECTRUM OF RANDOM POLYNOMIALS ORTHOGONAL ON THE CIRCLE

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Let $\{a_n\}_0^\infty$ be a sequence of complex numbers satisfying the condition

$$|a_n| < 1, \qquad n = 0, 1, 2, \dots \tag{1}$$

We construct a sequence of polynomials with the help of the recursion relations

$$\begin{gathered}\varphi_{n+1}(z) = (1-|a_n|^2)^{-1/2}(z\varphi_n(z) - \overline{a}_n\varphi_n^*(z)), \\ \varphi_{n+1}^*(z) = (1-|a_n|^2)^{-1/2}(\varphi_n^*(z) - a_n z\varphi_n(z)), \qquad n = 0, 1, 2, \dots,\end{gathered} \tag{2}$$

with the initial conditions $\varphi_0(z) = \varphi_0^*(z) = 1$. It is well known that there exists a unique Borel probability measure σ on the circle $T = \{z \in \mathbf{C} : |z| = 1\}$ such that the sequence $\{\varphi_n\}_0^\infty$ is orthogonal in $L^2_{\sigma, T}$ (see [2]). We denote by μ and ν the standard Lebesgue measures on T and in the disk $|z| < 1$.

The problem of the properties of the spectral measure for random polynomials orthogonal on the circle was posed by Nikishin in [3] for the case when the parameters $\{a_n\}_0^\infty$ form a sequence of independent symmetrically distributed random variables, each of which can take only two values, and $\sum_0^\infty |a_n|^2 < \infty$. It was proved in [4] that the properties considered by Nikishin are enjoyed by the measure σ also under weaker restrictions on the random parameters. Namely, if the independent random variables are such that condition (1) holds almost surely, $\sum_0^\infty \mathbf{E}|a_n|^2 < \infty$, and $\sum_0^\infty |Ea_n| \infty$, then, with probability 1, (a) the measure σ is absolutely continuous (with respect to μ); and (b) there exists an $\varepsilon > 0$ such that

$$\exp\{\varepsilon|\log\sigma'|\cdot\log|\log\sigma'|\} \in L^1_{\mu, T}$$

(where σ' is the density of σ with respect to μ). In contrast to the situation described above, we consider here random parameters such that with probability 1 the measure σ does not have an absolutely continuous component.

The main result of this note is

Theorem 1. *Suppose that the parameters $\{a_n\}_0^\infty$ of the system of orthogonal polynomials form a sequence of independent identically distributed complex-valued random variables such that with probability* 1 *the condition* (1) *holds, the distribution of the random variable a_0 is absolutely continuous with respect to ν, and* $\mathbf{E}\log(1-|a_0|) > -\infty$. *Then with probability* 1 *the measure σ corresponding to this sequence of parameters is discrete.*

The proof is based, first, on the fact that the Lyapunov exponent γ, which will be defined by the relation (9) below, is greater than 0 with probability 1, and, second, on the criterion (which may be of independent interest) in Theorem 2 for the absence

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of an absolutely continuous component and a singularly continuous component of σ.

The results below are very similar to those of Simon and Wolff on discrete one-dimensional Schrödinger operators with a random ergodic potential [6] and on one-dimensional perturbations of selfadjoint operators [7].

We remark that in Theorem 1 the independence of the parameters $\{a_n\}_0^\infty$ is used only in the proof that the Lyapunov exponent is positive.

Theorem 2, like Lemmas 1, 2, and 3, is true also for a nonrandom sequence of parameters.

Assume that $\{a_n\}_1^\infty$ is a fixed sequence of complex numbers with $|a_n| < 1$ for all n. For any λ in the domain $|\lambda| < 1$ let σ_λ be the spectral measure corresponding to the sequence $\{\lambda, a_1, a_2, \dots\}$ of parameters. We define the functions

$$F_\lambda(z) = \int_T \frac{x+z}{x-z}\, d\sigma_\lambda(x)$$

($F_\lambda(z)$ is the Stieltjes transform of the measure σ_λ) and

$$G_\lambda(z) = \int_T \frac{1}{|x-z|^2}\, d\sigma_\lambda(x).$$

Theorem 2. *The following assertions are equivalent:*

(I) σ_λ *is discrete for almost all* λ *in the domain* $|\lambda| < 1$ (*with respect to* ν).

(II) $G_0(z) < \infty$ *for almost all* $z \in T$ (*with respect to* μ).

Theorem 2 follows easily from Lemmas 1 and 2, which we present below without proof. We next give the necessary definitions and sketch the proof of Lemma 3. After the formulation of this lemma comes a simple argument showing on the basis of Lemma 3 that condition (II) of Theorem 2 holds with probability 1 for the random parameters in Theorem 1.

Let $\sigma_\lambda^{\mathrm{pp}}$, $\sigma_\lambda^{\mathrm{sc}}$, and $\sigma_\lambda^{\mathrm{ac}}$ be the pure point component, the singularly continuous component, and the absolutely continuous component, respectively, of the measure σ_λ (with respect to μ).

Lemma 1. *Let* $A = \{z \in T : G_0(z) < \infty\} \cup \{z \in T : \sigma_0(\{z\} > 0)\}$ *and* $B = \{z \in T$: *the limit* $F_0(z) = \lim_{r\uparrow 1} F_0(rz)$ *exists, and* $\operatorname{Re} F_0(z) > 0\}$. *Then*

$$\sigma_\lambda^{\mathrm{pp}}(T\backslash A) = 0, \qquad \sigma_\lambda^{\mathrm{ac}}(T\backslash B) = 0, \qquad \sigma_\lambda^{\mathrm{sc}}(A \cup B) = 0$$

for any λ *in the domain* $|\lambda| < 1$.

Lemma 2. *Suppose that the measure* τ *is defined on* T *as follows*:

$$\tau(A) = \frac{1}{\pi} \int_{|\lambda|<1} \sigma_\lambda(A)\, d\nu(\lambda)$$

for any Borel set $A \subset T$. *Then the measures* τ *and* μ *are equivalent. The Stieltjes transform* $F_{(\tau)}(z)$ *of* τ *satisfies*

$$F_{(\tau)}(z) = 1 + \frac{F_0(z)-1}{F_0(z)+1}.$$

The proofs of these lemmas make essential use of a formula permitting us to express the Stieltjes transform $F_\lambda(z)$ of σ_λ in terms of the Stieltjes transform $F_0(z)$ of σ_0:

$$F_\lambda(z) = \frac{z(1+\lambda z)(F_0(z)+1) + (z+\bar\lambda)(F_0(z)-1)}{z(1-\lambda z)(F_0(z)+1) + (\bar\lambda - z)(F_0(z)-1)}.$$

This relation can be derived from the expansion (obtained by Geronimus in [1]) of $F_\lambda(z)$ as a continued fraction.

Below we do not have to consider the dependence of σ on the parameter with zero index. Therefore, we assume that a sequence $\{a_n\}_0^\infty$ satisfying condition (1) has been fixed.

In a sequence space we now define a unitary operator that, on the one hand, is connected in a natural way with the relations (2), and, on the other hand, corresponds to the operator of multiplication by x in the space $L^2_{\sigma,T}$.

Let $l^2_{[0,\infty)}$ be the Hilbert space of sequences, and let $e_0, e_1, \dots$ be the natural basis in it (i.e., $(e_n)_m = \delta_{nm}$). We define an isometric mapping $\mathscr{U}\colon l^2_{[0,\infty)} \to L^2_{\sigma,T}$ by setting $\mathscr{U}(e_n) = \overline{\varphi}_n$ for the basis vectors. Let H be the unitary operator of multiplication by the free variable in $L^2_{\sigma,T}$, i.e., $(H\zeta)(x) = x\zeta(x)$ for any $\zeta \in L^2_{\sigma,T}$. The operator $\mathscr{H}\colon l^2_{[0,\infty)} \to l^2_{[0,\infty)}$ needed is given by $\mathscr{H} = \mathscr{U}^* H \mathscr{U}$. With the help of relations established in [2] it can be shown that the elements of the matrix determining $\mathscr{H}$ are

$$\langle \mathscr{H} e_k, e_n\rangle = \begin{cases} 0, & k > n+1, \\ \sqrt{1-|a_n|^2}, & k = n+1, \\ -\overline{a}_n a_{k-1} \prod_{p=k}^{n-1} \sqrt{1-|a_p|^2}, & 1 \le k \le n, \\ \overline{a}_n \prod_{p=0}^{n-1} \sqrt{1-|a_p|^2}, & k = 0. \end{cases} \tag{3}$$

Hence, $\mathscr{H}$ is unitary if and only if

$$\sum_{n=0}^{\infty} |a_n|^2 = \infty. \tag{4}$$

Thus, another (direct) proof has been obtained for the well-known fact that the linear span of the functions $\{\varphi_n\}_0^\infty$ is dense in $L^2_{\sigma,T}$ if and only if (4) holds.

Below, up to Lemma 3, we assume that z is a fixed complex number, and $|z| = 1$. It follows from (3) that z is an eigenvalue of $\mathscr{H}$ if and only if there exists a sequence $\{\psi_n\}_0^\infty$ of complex numbers such that

$$0 < \sum_{n=0}^{\infty} |\psi_n|^2 < \infty, \tag{5}$$

$$\begin{aligned} z\psi_n = \sqrt{1-|a_n|^2}\,\psi_{n+1} - \overline{a}_n \sum_{k=1}^{n} a_{k-1} \prod_{p=k}^{n-1} \sqrt{1-|a_p|^2}\,\psi_k \\ + \overline{a}_n \prod_{p=0}^{n-1} \sqrt{1-|a_p|^2}\,\psi_0. \end{aligned} \tag{6}$$

It can be shown that if ψ_0 and ψ_0^* are complex numbers, and the sequence $\{\psi_n\}_0^\infty$ is constructed according to the recursion relations

$$\begin{aligned} \psi_{n+1} &= (1-|a_n|^2)^{-1/2}(z\psi_n - \overline{a}_n \psi_n^*), \\ \psi_{n+1}^* &= (1-|a_n|^2)^{-1/2}(\psi_n^* - a_n z \psi_n), \qquad n = 0, 1, 2, \dots, \end{aligned} \tag{7}$$

then

$$
(8)\quad \begin{aligned} z\psi_n = \sqrt{1-|a_n|^2}\psi_{n+1} - \bar{a}_n \sum_{k=1}^{n} a_{k-1} \prod_{p=k}^{n-1} \sqrt{1-|a_p|^2}\psi_k \\ + \bar{a}_n \prod_{p=0}^{n-1} \sqrt{1-|a_p|^2}\psi_0^* . \end{aligned}
$$

Thus, it follows from (6) and (8) that z is an eigenvalue of $\mathscr{H}$ if and only if (7) and (5) holds for $\psi_0 = \psi_0^*$, i.e., $\sum_0^\infty |\varphi_n(z)|^2 < \infty$.

We remark that under the conditions of Theorem 1 the parameters $\{a_n\}_0^\infty$ are such that (4) holds with probability 1. Thus, it can be assumed without loss of generality that (4) holds, and the range of the mapping $\mathscr{U}$ is equal to the whole of $L^2_{\sigma,T}$ (i.e., it is unitary).

Suppose that $G(z) < \infty$. Then, $\operatorname{Re} F(z) = 0$ and the function $f\colon x \mapsto x/(z-x)$ belongs to $L^2_{\sigma,T}$. Further, f satisfies the equation $zf = Hf + H\mathbf{1}$ (where $\mathbf{1} \in L^2_{\sigma,T}$ and $\mathbf{1}(x) \equiv 1$). Let $\{\psi_n\}_0^\infty$ be the sequence that is the image of f under the mapping $\mathscr{U}^*$. Then $\psi_0 = -1/2(1+F(z))$, and by (3) the equality (8) holds for $\psi_0^* = \psi_0 + 1 = 1/2 - 1/2F(z)$. Consequently, the recursion relations (7) are valid with the given initial conditions, and $\sum_0^\infty |\psi_n|^2 < \infty$. It is not hard to see that the converse is also true.

Lemma 3. *For any* $z \in T$ *the following assertions are equivalent:*

I. *Either* $G(z) < \infty$ *or the measure* σ *has an atom at* z.

II. *There exist complex numbers* ψ_0 *and* ψ_0^*, *not both zero, such that* $\sum_0^\infty |\psi_n|^2 < \infty$, *where* $\{\psi_n\}_0^\infty$ *is defined by* (7).

We define the matrices

$$
g_n(z) = (1-|a_n|^2)^{-1/2} \begin{pmatrix} z & -\bar{a}_n \\ -a_n z & 1 \end{pmatrix}, \qquad n = 0, 1, 2, \dots
$$

It follows from (7) that

$$
\begin{pmatrix} \psi_{n+1} \\ \psi_{n+1}^* \end{pmatrix} = g_n(z)\cdots g_0(z) \begin{pmatrix} \psi_0 \\ \psi_0^* \end{pmatrix}.
$$

According to the hypothesis of Theorem 1, these random matrices are such that for any $z \in T$ there exists a positive number γ (the Lyapunov exponent) such that with probability 1

$$
(9)\qquad \lim_{n\to\infty} \frac{1}{n} \log \|g_n(z)\cdots g_0(z)\| = \gamma
$$

(see [5]). This implies that almost surely there exist complex numbers ψ_0 and ψ_0^* such that

$$
\lim_{n\to\lambda} \frac{1}{n} \log \left\| g_n(z)\cdots g_0(z) \begin{pmatrix} \psi_0 \\ \psi_0^* \end{pmatrix} \right\| = -\gamma,
$$

and this implies condition II of Lemma 3. Since the matrix $g_0(z)$ is invertible, the satisfaction of this condition does not depend on the value of the parameter a_0, i.e., condition III of Theorem 2 is valid almost surely, which proves Theorem 1.

Remark. The positivity of the Lyapunov exponent is not of fundamental importance in this proof, since condition II of Lemma 3 can apparently be proved directly under weaker restrictions on the random parameters.

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