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ON ABSOLUTE CONTINUITY OF THE SPECTRUM OF RANDOM POLYNOMIALS ORTHOGONAL ON THE CIRCLE AND THEIR CONTINUAL VERSIONS

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In the paper, difference equations with random dependent coefficients are considered, for which the solutions are polynomials orthogonal on the circle. Analogous random canonical differential equations are considered, too. Necessary and sufficient conditions for absolute continuity of the spectral measure and estimations of its density are formulated.

1. Let a sequence of polynomials $\{\varphi_n(z) : n \in \mathbb{N}\}$ be constructed according to the following recurrence relation:

$$\varphi_{n+1}(z) = (1+|a_n|^2)^{-1/2}(z\varphi_n(z) - \overline{a_n}\varphi_n^*(z)),$$
$$\varphi_{n+1}^*(z) = (1-|a_n|^2)^{-1/2}(\varphi_n^*(z) - a_n z\varphi_n(z)), \qquad n=0,1,\ldots, \quad \varphi_0(z)=\varphi_0^*(z)=1,$$

where the complex parameters $\{a_n : n \in \mathbb{N}\}$ satisfy the condition

$$|a_n| < 1, \quad n = 0,1,\ldots. \tag{1}$$

It is well known that there exists a unique Borel probability measure σ on the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ such that the sequence $\{\varphi_n(z) : n \in \mathbb{N}\}$ is orthonormal in $L^2_{\sigma,\mathbb{T}}$ (see [1]).

E. M. Nikishin [3] formulated and solved the problem on the behavior of the spectral measure σ in the case of random, independent, symmetrically distributed parameters $\{a_n : n \in \mathbb{N}\}$ such that every a_n takes only two values and $\sum_{n=0}^{\infty}|a_n|^2 < \infty$. It is proved in [4] that, in fact, Nikishin's demonstration can be extended for the case of arbitrary independent random variables, provided their expectations and variances are summable and (1) holds a.s. It turns out that the method used in this paper is also applicable to the case of dependent variables.

Denote by σ' the density of an absolutely continuous component of a measure σ with respect to the Lebesgue measure μ on the circle $\mathbb{T}$.

Theorem 1. *Let random parameters $\{a_n : n \in \mathbb{N}\}$ satisfy a.s. the condition* (1) *and the following inequalities:*

$$\sum_{n=0}^{\infty} |\mathbf{E}\{a_n \mid a_{n-1},\ldots,a_0\}| < \infty, \qquad \sum_{n=0}^{\infty}|a_n|^2 < \infty.$$

Then with probability one we have:

a) *the measure σ is absolutely continuous with respect to the Lebesgue measure μ;*

b) *there exists an $\varepsilon > 0$ such that the function $\exp\{\varepsilon|\log\sigma'|\cdot\log|\log\sigma'|\}$ is integrable in μ.*

Theorem 2. *Let the conditions of Theorem 1 be satisfied, and let a sequence of positive numbers $\{c_n : n \in \mathbb{N}\}$ exist such that $\sum_{n=0}^{\infty} c_n^2 < \infty$ and $|a_n| < c_n$ a.s. for $n \in \mathbb{N}$. Then with probability one for any $\varepsilon > 0$ the function $\exp\{\varepsilon\log^2\sigma'\}$ is integrable in μ.*

2. M. G. Krein noted that the properties of solutions of the system of differential equations

$$\frac{d}{dt}p(t,\lambda) = i\lambda p(t,\lambda) - \overline{a(t)}p_*(t,\lambda),$$
$$\frac{d}{dt}p_*(t,\lambda) = -a(t)p(t,\lambda), \quad t \geqslant 0, \quad p(0,\lambda) = p_*(0,\lambda) = 1, \tag{2}$$

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where $a(\cdot)$ is an arbitrary complex-valued locally integrable function, are analogous to the properties of polynomials orthogonal on the circle [2].

There exists a unique Borel measure τ on the real line such that for any finite function $f \in L^2_{[0,\infty)}$ the following generalized Parseval formula holds:

$$\int_0^\infty |f(t)|^2\, dt = \int_{-\infty}^{+\infty} \left| \int_0^\infty f(t)p(t,\lambda)\, dt \right|^2 d\tau(\lambda).$$

Hence, the map $\mathcal{U}$ defined for such functions by the formula

$$(\mathcal{U})(\lambda) = \int_0^\infty f(t)p(t,\lambda)\, dt$$

can be extended to an isometry $L^2_{[0,\infty)} \to L^2_\tau$. The image of $\mathcal{U}$ coincides with the whole L^2_τ if and only if

$$\int_{-\infty}^{\infty} \frac{\log \tau'(\lambda)\, d\lambda}{1+\lambda^2} > -\infty,$$

where τ' is the density of the component of τ absolutely continuous in the Lebesgue measure. The properties of random differential equations (2) are similar to the properties of the random polynomials considered above.

Let $a(\cdot)$ be a measurable complex-valued random process with a.s. locally integrable trajectories. Consider a partition of the set $[0,\infty)$ into finite intervals defined by the sequence $\{t_n : n \in \mathbb{N}\}$, $0 = t_0 < t_1 < t_2 < \dots$. Define the random variables

$$\xi_n = \int_{t_n}^{t_{n+1}} a(t)\, dt \quad \text{and} \quad \eta_n = \int_{t_n}^{t_{n+1}} |a(t)|\, dt, \quad n = 0, 1, \dots.$$

Let $\mathcal{F}_n$ be the σ-algebra of events generated by the random variables ξ_m and η_m for $0 \leqslant m \leqslant n$.

Theorem 3. *Suppose that a.s.*

$$\sum_{n=0}^{\infty} |\mathbf{E}\{\xi_{n+1} \mid \mathcal{F}_n\}| < \infty, \qquad \sum_{n=0}^{\infty} \eta_n^2 < \infty, \qquad \sum_{n=0}^{\infty} (t_{n+1} - t_n)\eta_n < \infty.$$

Then with probability one

a) *the measure τ is absolutely continuous with respect to the Lebesgue measure;*

b) $\int_{-\infty}^{+\infty} \frac{\log \tau'(\lambda)\, d\lambda}{1+\lambda^2} \geqslant -\infty$;

c) *for some $\varepsilon > 0$*

$$\int_{-\infty}^{+\infty} \exp\{(\varepsilon|\log \tau'(\lambda)| - |\lambda|)_+ \log(\varepsilon|\log \tau'(\lambda)| - |\lambda|)_+\} \frac{d\lambda}{1+\lambda^2} < \infty,$$

where for any real x we set $(x)_+ = \max\{1, x\}$.

Theorem 4. *Let the hypotheses of Theorem 3 be satisfied and there exist a sequence of positive numbers $\{c_n : n \in \mathbb{N}\}$ such that $\sum_{n=0}^{\infty} c_n^2 < \infty$ and a.s. $\eta_n < c_n$, $n = 0, 1, \dots$. Then with probability one for any $\varepsilon > 0$*

$$\int_{-\infty}^{+\infty} \exp\{\varepsilon \log^2 \tau'(\lambda) - \lambda^2\}\, d\lambda < \infty.$$

These theorems are proved in [5] for the case where ξ_n and η_n do not depend on all of the variables ξ_m and η_m, $m \neq n$. However, this paper contains (in the part dealing with stochastic differential equations) additional arguments, which permit one to prove Theorem 3 and Theorem 4.

It is shown in [5] that for any nonnegative locally integrable function on $[0, \infty)$ there exists a random process $a(\cdot)$ satisfying the hypotheses of Theorem 4 such that a.s. $|a(\cdot)| = b(\cdot)$. We describe another method of constructing of a random function $a(\cdot)$ satisfying the hypotheses of Theorem 3 and Theorem 4.

Let $g(\cdot)$ be a measurable random process defined on $[0, \infty)$ such that $\sup_{0 \leqslant t < \infty} \mathbf{E}|g(t)|^2 < \infty$ and $\mathbf{E}\{ g(u) \mid \mathcal{F}_t \} = 0$ for any $u - 1 \geqslant t \geqslant 0$. (Here $\mathcal{F}_t$ is the σ-algebra generated by the values of $g(\cdot)$ on $[0, t)$.) Then for any $\varepsilon > 0$ the random function $a\colon t \mapsto g(t^{3+\varepsilon})$ satisfies the hypotheses of Theorem 3 (one should set $t_n = n^{1/(3+\varepsilon)}$, $n \in \mathbb{N}$). If, besides, the random process $g(\cdot)$ is bounded, then the given function satisfies the hypotheses of Theorem 4. In general, if $g(\cdot)$ is a measurable random process such that $\sup_{0 \leqslant t < \infty} \mathbf{E}|g(t)|^2 < \infty$, and for any $t \geqslant 0$ we have

$$\int_t^r \mathbf{E}|\mathbf{E}\{ g(u) \mid \mathcal{F}_t \}|\, du = o(r)_{r \to \infty},$$

then there exists an increasing function $s\colon [0, \infty) \to [0, \infty)$ such that the random process $a(\cdot) = g(s(\cdot))$ satisfies the hypotheses of Theorem 3.

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