JOURNAL OF FUNCTIONAL ANALYSIS **123**, 12–45 (1994)

A Difference Equation Arising from the Trigonometric Moment Problem Having Random Reflection Coefficients—An Operator Theoretic Approach

J. S. GERONIMO* AND A. TEPLYAEV

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332

Communicated by the Editors

Received August 1, 1992; revised December 15, 1992

Operators associated with polynomials orthogonal on the unit circle and their doubly infinite analogs are considered. We assume that the elements in the operators are obtained from a stationary stochastic process and we investigate the properties of the spectrum of these operators. Lyapunov exponents are introduced and we examine their relationship with the absolutely continuous components of the spectrum of the above operators. © 1994 Academic Press, Inc.

I. INTRODUCTION

Let $\{\alpha(i)\}_{i=-\infty}^{\infty}$ be a sequence of complex number such that $|\alpha(n)|<1$ for all $n\in\mathbb{Z}$. Consider the difference equation

$$\Psi(z,n)=T(z,n)\,\Psi(z,n-1), \tag{1.1}$$

where

$$T_n=a(n)\begin{pmatrix} z & \alpha(n) \\ \overline{\alpha(n)}z & 1\end{pmatrix}, \tag{1.2}$$

with $a(n)>0$ and

$$\frac{1}{a(n)^2}=1-|\alpha(n)|^2. \tag{1.3}$$

Also set

$$T_{n,L}(z)=T_L(z)\,T_{L-1}(z)\cdots T_n(z), \qquad n<L. \tag{1.4}$$

* Supported in part by NSF Grant DMS-9005944.

0022-1236/94 $6.00

It is well known (Szegö [22], Geronimus [11]) that the top component of the solution $\Phi(z, n)$, $n \geqslant 0$ of (1.1) with $\Phi(z, 0) = \binom{1}{1}$ is an orthonormal polynomial. That is, if

$$\Phi(z, n) = \begin{pmatrix} \phi(z, n) \\ \phi^*(z, n) \end{pmatrix} \tag{1.5}$$

then there exists a probability measure σ supported on the unit circle U for which

$$\int_U \phi(e^{i\theta}, n)\, \overline{\phi(e^{i\theta}, m)}\, d\sigma(\theta) = \delta_{n,m}. \tag{1.6}$$

The second component $\phi^*(z, n)$ is the so-called reciprocal polynomial, i.e.,

$$\phi^*(z, n) = z^n \bar{\phi}(1/z, n). \tag{1.7}$$

Finally we note that the Wronskian W of any two solutions of (1.1) obeys the relation

$$\begin{aligned} W[\Psi_1(z, n), \Psi_2(z, n)] &= \det(\Psi_1(z, n), \Psi_2(z, n)) \\ &= zW[\Psi_1(z, n-1), \Psi_2(z, n-1)]. \end{aligned} \tag{1.8}$$

Here we will consider the case when $\{\alpha(i)\}_{i=-\infty}^{\infty}$ is a stationary stochastic process which is also ergodic. In Geronimo [8] a similar half line problem is considered and we develop and extend these results. (See also [10].) This will be accomplished by associating to (1.1) an operator (Teplyaev [20, 21]) thereby allowing us to use the operator techniques developed by Carmona [4], Avron and Simon [2], Craig and Simon [5], Kotani [15], Simon [18], and others who were studying the discrete Schrödinger operator with random entries. The results we obtain apply to the non-random case when the coefficients in the difference equation are periodic, [8] or more generally almost periodic [18]. We begin in Section II by deriving the operator H associated with (1.1) and investigating its properties. In Section III we examine H_w when the coefficients in (1.1) form a stationary stochastic process. A non-random measure associated with the asymptotic trace per unit volume of H is introduced and it is shown that this measure is related to the eigenvalues distribution of distorted truncates of H. In Section IV we introduce the Lyapunov exponent and derive an integral relation between this exponent and the asymptotic trace per unit volume measure. m-Functions are then introduced in Section V and used to show relations between the absolutely continuous components of the spectral measures associated with H_w and the Lyapunov exponent. Finally in Section VI the half line problem is discussed.

II. Deterministic Preliminaries

Setting $\Psi(z,n)=\binom{\psi_1(z,n)}{\psi_2(z,n)}$ and writing (1.1) in component form we find the equations

$$\psi_1(z,n)=a(n)(z\psi_1(z,n-1)+\alpha(n)\,\psi_2(z,n-1)), \tag{2.1}$$

and

$$\psi_2(z,n)=a(n)(\psi_2(z,n-1)+\overline{\alpha(n)}\,z\psi_1(z,n-1)). \tag{2.2}$$

Inverting (1.1) we find

$$z\psi_1(z,n-1)=a(n)(\psi_1(z,n)-\alpha(n)\,\psi_2(z,n)), \tag{2.3}$$

and

$$\psi_2(z,n-1)=a(n)(\psi_2(z,n)-\overline{\alpha(n)}\,\psi_1(z,n)). \tag{2.4}$$

Iterating (2.4) down to $m<n$ we find

$$\psi_2(z,n)=\prod_{i=m+1}^{n}\frac{1}{a(i)}\psi_2(z,m)+\sum_{i=m+1}^{n}\overline{\alpha(i)}\prod_{j=i+1}^{n}\frac{1}{a(j)}\psi_1(z,i). \tag{2.5}$$

Now substituting this into (2.1) yields

$$\begin{aligned}z\psi_1(z,n-1)=&\frac{1}{a(n)}\psi_1(z,n)-\alpha(n)\prod_{i=m+1}^{n-1}\frac{1}{a(i)}\psi_2(z,m)\\&-\alpha(n)\sum_{i=m+1}^{n-1}\overline{\alpha(i)}\prod_{j=i+1}^{n-1}\frac{1}{a(j)}\psi_1(z,i).\end{aligned} \tag{2.6}$$

We use (2.6) to define a number of operators that will be useful in what follows (Teplyaev [20]). Let $l^2=\{\{\psi_n\}_{-\infty}^{\infty}:\sum|\psi_n|^2<\infty\}$, $\{e_n\}_{n=-\infty}^{\infty}$ be the standard basis in l^2, and $l^2(m,n)$ be the subspace of l^2 spanned by the vectors $\{e_i\}_{i=m}^{n}$. We set $l^2_+=l^2(0,\infty)$. Let $H_L\colon l^2\to l^2(-L,L)$ and $H_{L_+}\colon l^2(0,L)\to l^2(0,L)$ be defined by

$$\begin{aligned}(H_L\psi)_k=&\frac{1}{a(k+1)}\psi_{k+1}-\alpha(k+1)\sum_{i=-L+1}^{k}\overline{\alpha(i)}\prod_{j=i+1}^{k}\frac{\psi_i}{a(j)}\\&-\alpha(k+1)\prod_{j=-L+1}^{k}\frac{\psi_{-L}}{a(j)},\qquad -L\leqslant k<L\end{aligned} \tag{2.7}$$

$$(H_L\psi)_L=-\sum_{i=-L+1}^{L}\overline{\alpha(i)}\prod_{j=i+1}^{L}\frac{\psi_i}{a(j)}-\prod_{j=-L+1}^{L}\frac{\psi_{-L}}{a(j)}, \tag{2.8}$$

and

$$(H_{L_+}\psi)_K=\frac{1}{a(k+1)}\psi_{k+1}-\alpha(k+1)\sum_{i=1}^{k}\overline{\alpha(i)}\prod_{j=i+1}^{k}\frac{1}{a(j)}\psi_i$$
$$-\alpha(k+1)\prod_{j=1}^{k}\frac{1}{a(j)}\psi_0,\qquad 0\leqslant k<L \tag{2.9}$$

$$(H_{L_+}\psi)_L=-\sum_{i=1}^{L}\overline{\alpha(i)}\prod_{j=i+1}^{L}\frac{1}{a(j)}\psi_i-\prod_{j=1}^{L}\frac{1}{a(j)}\psi_0. \tag{2.10}$$

We also define $H\colon l^2\to l^2$ by

$$(H\psi)_n=\frac{1}{a(n+1)}\psi_{n+1}-\alpha(n+1)\sum_{i=-\infty}^{n}\overline{\alpha(i)}\prod_{j=i+1}^{n}a(j)\psi_i. \tag{2.11}$$

Let $\Phi(z,-L,L)=\binom{\phi(z,-L,L)}{\phi^*(z,-L,L)}=T_{-L+1,L}(z)\binom{1}{1}$ and $\Phi(z,L)=\binom{\phi(z,L)}{\phi^*(z,L)}=T_{1,L}(z)\binom{1}{1}$, where ϕ^* is given by (1.7). Let $k(-L,L)$ be the leading coefficient of $\phi(z,-L,L)$ and $k(L)$ the leading coefficient of $\phi(z,L)$. Now set

$$\hat{\phi}(z,-L,L+1)=\frac{z\phi(z,-L,L)+\phi^*(z,-L,L)}{k(-L,L)}, \tag{2.12}$$

and

$$\hat{\phi}(z,L+1)=\frac{z\phi(z,L)+\phi^*(z,L)}{k(L)}. \tag{2.13}$$

It is easy to check using the fact that $\phi(z,-L,L)$ and $\phi(z,L)$ have all their zeros inside the unit circle (Szegö [22], Geronimus [11]) that the zeros of $\hat{\phi}(z,-L,L+1)$ and $\hat{\phi}(z,L+1)$ are all located on the unit circle. In fact $\hat{\phi}(z,-L,L+1)$ and $\hat{\phi}(z,L+1)$ are just singular Szegö polynomials.

LEMMA 2.1. *For every $L>1$, H_L and H_{L_+} are unitary. Furthermore the eigenvalues of H_L are located at the zeros of $\hat{\phi}(z,-L,L+1)$ while the eigenvalues of H_{L_+} are located at the zeros of $\hat{\phi}(z,L)$.*

Remark. The above lemma is closely related to Theorem 7.2.2 in Atkinson's book [1].

Proof. We will prove the result for H_{L_+} as the same method will work for H_L. From (2.9) and (2.10) above we find

$$H_{L_+}e_k=\frac{e_{k-1}}{a(k)}-\overline{\alpha(k)}\sum_{i=k}^{L-1}\alpha(i+1)\prod_{j=k+1}^{i}\frac{e_i}{a(j)}$$
$$-\overline{\alpha(k)}\prod_{j=k+1}^{L}\frac{e_L}{a(j)},\qquad 0<k\leqslant L \tag{2.14}$$
$$H_{L_+}e_0=-\prod_{j=1}^{L}\frac{e_L}{a(j)}-\sum_{i=0}^{L-1}\alpha(i+1)\prod_{j=1}^{i}\frac{1}{a(j)}e_i,$$

while

$$H_{L_+}^* e_k = \frac{e_{k+1}}{a(k+1)} - \overline{\alpha(k+1)} \sum_{i=1}^{k} \alpha(i) \prod_{j=i+1}^{k} \frac{1}{a(j)} e_i$$

$$- \overline{\alpha(k+1)} \prod_{j=1}^{k} \frac{1}{a(j)} e_0, \qquad 0 \leqslant k < L, \tag{2.15}$$

and

$$H_{L_+}^* e_L = - \sum_{i=1}^{L} \alpha(i) \prod_{j=i+1}^{L} \frac{1}{a(j)} e_i - \prod_{j=1}^{L} \frac{1}{a(j)} e_0.$$

Consequently for $k < L$ we find

$$\langle H_{L_+} H_{L_+}^* e_k, e_L \rangle = -\overline{\alpha(k+1)} \prod_{j=k+1}^{L} \frac{1}{a(j)} [1 - A(k)],$$

$$\langle H_{L_+} H_{L_+}^* e_k, e_m \rangle = -\overline{\alpha(k+1)}\, \alpha(m+1) \prod_{j=k+1}^{m} \frac{1}{a(j)} [1 - A(k)],$$

$$k < m < L,$$

$$\langle H_{L_+} H_{L_+}^* e_k, e_k \rangle = \frac{1}{a(k+1)^2} + |\alpha(k+1)|^2 A(k),$$

$$\langle H_{L_+} H_{L_+}^* e_k, e_m \rangle = -\overline{\alpha(k+1)}\, \alpha(m+1) \prod_{j=m+1}^{k} \frac{1}{a(j)} [1 - A(m)],$$

$$0 \leqslant m < k,$$

and

$$\langle H_{L_+} H_{L_+}^* e_L, e_L \rangle = A(L),$$

where

$$A(p) = \sum_{i=1}^{p} |\alpha(i)|^2 \prod_{j=i+1}^{p} \frac{1}{a(j)^2} + \prod_{j=1}^{p} \frac{1}{a(j)^2}, \qquad p \geqslant 0.$$

Repeated use of (1.3) shows that $A(p) = 1$ for all $p > 0$. (We are using the conventions that the empty product equals one and the empty sum equals zero.) Therefore since $H_{L_+}^* H_{L_+}$ is Hermitian we see that $\langle H_{L_+} H_{L_+}^* e_k, e_m \rangle = \delta_{k,m}$. Analogous formulas can be derived to show that $\langle H_{L_+}^* H_{L_+} e_k, e_m \rangle = \delta_{k,m}$.

To see that the eigenvalues of H_{L_+} are at the zeros of $\hat{\phi}(z, \hat{L}+1)$ given by (2.13) note that (2.9) and (2.10) lead to the eigenvalue equation

$$z\psi_k = \frac{1}{a(k+1)}\psi_{k+1} - \alpha(k+1)\sum_{i=1}^{k}\overline{\alpha(i)}\prod_{j=i+1}^{k}\frac{\psi_i}{a(j)} - \alpha(k+1)\prod_{j=1}^{k}\frac{\psi_0}{a(j)}, \qquad 0 \leqslant k < L \tag{2.16}$$

and

$$z\psi_L = -\sum_{i=1}^{L}\overline{\alpha(i)}\prod_{j=i+1}^{L}\frac{\psi_i}{a(j)} - \prod_{j=1}^{L}\frac{\psi_0}{a(j)}. \tag{2.17}$$

If we set $\psi_0 = 1$ then ψ_k is a polynomial of degree k. Let $\hat{\psi}_i = \psi(i)/c(i)$ where $c(i)$ is leading coefficient of ψ_i. From Eq. (2.1) it is not difficult to see that $c(i)$ can be written as

$$c(i) = \prod_{j=i+1}^{k}\frac{1}{a(j)}\, c(k),$$

and we find that

$$z\hat{\psi}_k = \hat{\psi}_{k+1} - \alpha(k+1)\sum_{i=1}^{k}\overline{\alpha(i)}\prod_{j=i}^{k}\frac{\hat{\psi}_i}{a(j)^2} - \alpha(k+1)\prod_{j=1}^{k}\frac{1}{a(j)^2}, \qquad 0 \leqslant k < L,$$

and

$$z\hat{\psi}_L = -\sum_{i=1}^{L}\overline{\alpha(i)}\prod_{j=i+1}^{L}\frac{\hat{\psi}_i}{a(j)^2} - \prod_{j=1}^{L}\frac{1}{a(j)^2}.$$

The above formulas are exactly the same as those satisfied by $\hat{\phi}(z, L)$ (see (2.6) with $m = 0$ and the proper normalization), with $|\alpha(i)| < 1$, $i = 1 \cdots L$, $\alpha(L+1) = 1$ at a zero of $\hat{\phi}(z, L+1)$. Thus the eigenvalues of H_{L_+} are at the zeros of the singular polynomial $\hat{\phi}(z, L+1)$. ∎

LEMMA 2.2. *$\alpha(n) = 0$ for all $n \in \mathbb{Z}$ or*

$$\sum_{n=0}^{\infty} |\alpha(n)|^2 = \infty = \sum_{n=-1}^{-\infty} |\alpha(n)|^2, \tag{2.18}$$

if and only if H is unitary.

Proof. If $\alpha(n)=0$ for all n, H is just the shift operator on l^2 which is unitary. We find from (2.11) that when $\alpha(n)$ is not identically equal to zero

$$He_n=\frac{1}{a(n)}e_{n-1}-\overline{\alpha(n)}\sum_{i=n}^{\infty}\alpha(i+1)\prod_{j=n+1}^{i}\frac{1}{a(j)}e_i. \tag{2.19}$$

Thus

$$\begin{aligned}HH^*e_n=&\frac{1}{a(n+1)}\left(\frac{1}{a(n+1)}e_n-\overline{\alpha(n+1)}\sum_{i=n+1}^{\infty}\alpha(i+1)\prod_{j=n+2}^{i}\frac{1}{a(j)}e_i\right)\\&-\overline{\alpha(n+1)}\sum_{i=-\infty}^{n}\alpha(i)\prod_{j=i}^{n-1}\frac{1}{a(j+1)}\\&\times\left(\frac{1}{a(i)}e_{i-1}-\overline{\alpha(i)}\sum_{k=i}^{\infty}\alpha(k+1)\prod_{p=i+1}^{k}\frac{1}{a(p)}e_k\right),\end{aligned}$$

which leads to

$$\langle HH^*e_n,e_n\rangle=\frac{1}{a(n+1)^2}+|\alpha(n+1)|^2\sum_{i=-\infty}^{n}|\alpha(i)|^2\prod_{p=i+1}^{n}\frac{1}{a(p)^2}.$$

Now using (1.3) repeatedly and the fact, from (2.18), that $\prod_{-\infty}^{n}1/a(i)^2=0$ for all n, we find that

$$\langle HH^*e_n,e_n\rangle=1.$$

Also for $m>n$

$$\begin{aligned}\langle HH^*e_n,e_m\rangle=&\frac{-\overline{\alpha(n+1)}\,\alpha(m+1)}{a(n+1)}\prod_{j=n}^{m-1}\frac{1}{a(j+1)}\\&\times\left(-1+\sum_{i=-\infty}^{n}|\alpha(i)|^2\prod_{p=i+1}^{n}\frac{1}{a(p)^2}\right)=0,\end{aligned}$$

where (1.3) has been used repeatedly to obtain the last inequality. The same techniques can be used to show that $\langle H^*He_n,e_m\rangle=\delta_{n,m}$.

Let $B\colon l^2\to l^2$ be a unitary operator and χ_L be the projection onto the subspace spanned by $(e_{-L},e_{-L+1},...,e_L)$. If we look at $(1/2L+1)$ $\operatorname{tr}(\chi_L f(B))$ we see that it is a positive bounded linear functional on $C(U)$ which by the Riesz representation theorem has the representation

$$\frac{1}{2L+1}\operatorname{tr}(\chi_L f(B))=\int_U f(e^{i\theta})\,dk_L, \tag{2.20}$$

where k_L is a probability measure.

DEFINITION 2.1. B has an asymptotic trace per unit volume (ATUV) measure iff the limit

$$\Lambda(f) = \lim_{L \to \infty} \frac{1}{2L+1} \operatorname{tr}(\chi_L f(B)) \tag{2.21}$$

exists for all $f \in C(U)$.

Remark. If B is defined on l^2_+ then we say that B has an ATUV measure iff the limit

$$\Lambda(f) = \lim_{L \to \infty} \frac{1}{L+1} \operatorname{tr}(\chi_L f(N))$$

exists for all $f \in C(U)$. Here χ_L is the projection onto the subspace spanned by $\{e_0, e_1, ..., e_L\}$. Such measures have been extensively studied in the case when H is the Schrödinger operator, see Avron and Simon [2], Bellissard [3], Geronimo *et al.* [9], Johnson and Moser [13], and Simon [19].

LEMMA 2.3. *Let H_L be given by Eqs.* (2.7) *and* (2.8) *and H by Eq.* (2.11). *If* (2.18) *holds then*

$$\lim_{L \to \infty} \frac{1}{2L+1} (\operatorname{tr} f(H_L) - \operatorname{tr}(\chi_L f(H) \chi_L)) = 0, \tag{2.22}$$

for every $f \in C(U)$.

Proof. We will prove this by showing that (2.22) is true for any polynomial of z and z^*. Since H is unitary, products containing powers of both H and H^* reduce to powers of H or H^*. The same is also true for products containing powers of both H_L and H_L^*. Therefore the theorem will be true if we can show (2.22) holds for polynomials of the form z^n or z^{*n}. From Eqs. (2.7) and (2.8) we find that

$$H_L e_k = \begin{cases} 0, & |k| > L \\ \sum_{k-1}^{L} c_{k,i}^L e_i,\ c_{-L,-L-1} = 0, & |k| \leqslant L. \end{cases} \tag{2.23}$$

Furthermore from Eq. (2.11) we see that

$$\chi_L H \chi_L e_k = \begin{cases} 0, & |k| > L \\ \sum_{k-1}^{L} b_{k,i} e_i, & |k| < L \\ \sum_{i=-L}^{L} b_{k,i} e_i, & k = L, \end{cases} \tag{2.24}$$

where $b_{k,i} = c_{k,i}^L$, $k-1 \leqslant i \leqslant L-1$, $-L+1 \leqslant k \leqslant L$, and $-L \leqslant i \leqslant L-1$, $k = -L$. It follows from (2.23), (2.24), and (2.11) that $(H_L)^n e_k = \sum_{k-1}^{L} c_{k,i}^{L,n} e_i$ and $\chi_L H^n \chi_L e_k = \sum_{k-1}^{L} b_{k,i}^n e_i$ where $b_{k,i}^n = c_{k,i}^{L,n}$, $k-1 \leqslant i \leqslant L-n$, $-L+1 \leqslant k \leqslant L-n+1$, and $-L \leqslant i \leqslant L-n$, $k = -L$. Therefore

$(1/2L+1))\,\mathrm{Tr}[(H^L)^n - (\chi_L H^n \chi_L)] = (1/2L+1))\sum_{L-n+1}^{L}(c_{i,i}^{L,n} - b_{i,i}^{n})$. Since $(H_L)^n$ and H^n are unitary $|c_{k,i}^{L,n}|$ and $|b_{k,i}^n|$ are bounded above by one uniformly in L, i, and n. Thus $\lim_{L\to\infty}(1/2L+1))\,\mathrm{Tr}((H_L)^n - \chi_L H^n \chi_L) = 0$, $n = 1, 2, \ldots$. An analogous argument can be applied to show that $\lim_{L\to\infty}(1/2L+1))\,\mathrm{Tr}((H_L^*)^n - \chi_L H^{*n}\chi_L) = 0$ and the result follows. ∎

The proof of the above theorem also shows us

LEMMA 2.4. *Let H_{L_+} be defined by* (2.9) *and* (2.10). *If* (2.18) *holds then*

$$\lim_{L\to\infty}\frac{1}{L+1}\,\mathrm{tr}(f(H_{L_+}) - \chi_L^+ f(H)) = 0,$$

for all $f \in C(U)$. Here χ_L^+ is the projection onto the subspace of l^2 spanned by $\{e_0, e_1, \ldots, e_L\}$.

The above lemmas give the following theorem.

THEOREM 2.5. *If* (2.18) *holds then the $\lim_{L\to\infty}(1/(2L+1))\,\mathrm{tr}(f(H_L))$ converges for all $f \in C(U)$ if and only if $\lim_{L\to\infty}(1/(2L+1))\,\mathrm{tr}(\chi_L f(H))$ converges for all $f \in C(U)$. Likewise $\lim_{L\to\infty}(1/(L+1))\,\mathrm{tr}(f(H_{L_+}))$ converges for all $f \in C(U)$ if and only if $\lim_{L\to\infty}(1/(L+1))\,\mathrm{tr}(\chi_L^+ f(H))$ converges for all $f \in C(U)$.*

Since H is unitary, if (2.18) holds, we see that $(zI - H)^{-1}$ is a bounded linear operator on l^2 for $|z| \neq 1$. If we set $G(z, n, m) = \langle (zI - H)^{-1} e_n, e_m\rangle$ then the equation $(zI - H)G = I$ yields

$$zG(z, n, m) - \frac{1}{a(n+1)}G(z, n+1, m) + \alpha(n+1)$$
$$\times \sum_{i=-\infty}^{n} \overline{\alpha(i)} \prod_{j=i+1}^{n} \frac{G(z, i, m)}{a(j)} = \delta_{n,m}.$$

Setting $G_1(z, n, m) = \sum_{i=-\infty}^{n} \overline{\alpha(i)} \prod_{j=i+1}^{n} (G(z, i, m)/a(j))$, then substituting this into the above equation yields

$$zG(z, n, m) - \frac{1}{a(n+1)}G(z, n+1, m) + \alpha(n+1)\,G_1(z, n, m) = \delta_{n,m}. \tag{2.25}$$

Furthermore $G_1(z, n, m)$ satisfies the equation

$$G_1(z, n+1, m) = \overline{\alpha(n+1)}\,G(z, n+1, m) + \frac{1}{a(n+1)}G_1(z, n, m). \tag{2.26}$$

Set $\hat{G}(z, n, m) = \binom{G(z,n,m)}{G_1(z,n,m)}$. For $n \neq m$, (2.25) and (2.26) are equivalent to (1.1). Furthermore since $G(z, n, m)$ are matrix elements of a bounded linear operator for $|z| < 1$ we find that $\sum_{n=-\infty}^{\infty} |G(z, n, m)|^2 < \infty$ for all m and $|z| < 1$. This implies there exist two solutions to (1.1)

$$\Phi_+(z, n) = \begin{pmatrix} \phi_+^1(z, n) \\ \phi_+^2(z, n) \end{pmatrix}, \tag{2.27}$$

and

$$\Phi_-(z, n) = \begin{pmatrix} \phi_-^1(z, n) \\ \phi_-^2(z, n) \end{pmatrix}, \tag{2.28}$$

such that $\sum_{n=0}^{\infty} |\phi_+^1(z, n)|^2 < \infty$ and $\sum_{n=0}^{-\infty} |\phi_-^1(z, n)|^2 < \infty$ for $0 \leqslant |z| < 1$. To compute $\hat{G}$ we set

$$\hat{G}(z, n, m) = \begin{cases} C(m)\, \Phi_+(z, n), & n > m \\ D(m)\, \Phi_-(z, n), & n \leqslant m. \end{cases}$$

Since $\Phi_+(z, n)$ and $\Phi_-(z, n)$ are solutions to (1.1), $\hat{G}(z, n, m)$ satisfies (2.25) and (2.26) for $m \neq n$. With $m = n$ in (2.25) and (2.26) and using the fact that $\Phi_+(z, n)$ and $\Phi_-(z, n)$ satisfy (1.1) we arrive at the following equations for $C(m)$ and $D(m)$

$$\frac{D(m)\, \phi_-^1(z, m+1)}{a(m+1)} - \frac{C(m)\, \phi_+^1(z, m+1)}{a(m+1)} = 1, \tag{2.29}$$

and

$$-\frac{D(m)\, \phi_-^2(z, m)}{a(m+1)} + \frac{C(m)\, \phi_+^2(z, m)}{a(m+1)} = 0. \tag{2.30}$$

If $\Phi_+(z, n)$ and $\Phi_-(z, n)$ are linearly independent we can solve (2.29) and (2.30) to find

$$\hat{G}(z, n, m) = \begin{cases} \dfrac{z^{-m-1} \phi_-^2(z, m)\, \Phi_+(z, n)}{W[\Phi_-(z, 0), \Phi_+(z, 0)]}, & n > m, \\[2ex] \dfrac{z^{-m-1} \phi_+^2(z, m)\, \Phi_-(z, n)}{W[\Phi_-(z, 0), \Phi_+(z, 0)]}, & n \leqslant m. \end{cases} \tag{2.31}$$

THEOREM 2.6. *For* $0 \leqslant |z| < 1$ *the sequence* $\{\Phi_+(z, n)\}$ *is the unique solution of* (1.1) *satisfying* $\sum_{n=0}^{\infty} \|\Phi_+(z, n)\|^2 < \infty$ *for* $0 \leqslant |z| < 1$. *Furthermore* $\phi_+^1(z, n)$ *is non-zero and* $|\phi_+^1(z, n)| > |\phi_+^2(z, n)|$ *for* $0 < |z| < 1$ *and all* $n \geqslant 0$.

If (2.18) *holds then* $\{\Phi_-(z,n)\}$ *is the unique solution of* (1.1) *bounded at* $-\infty$ *for* $0 \leqslant |z| < 1$; *i.e.*, $\lim_{n\to\infty} \|\Phi_-(z,n)\| < \infty$, $0 \leqslant |z| < 1$. *Also* $\phi^2_-(z,n)$ *is non-zero and* $|\phi^2_-(z,n)| > |\phi^1_-(z,n)|$ *for* $0 < |z| < 1$.

Proof. The results concerning $\Phi_+(z,n)$ have already been proved in [8, Theorem 2.4] where it is shown that for $n \geqslant 0$, $\Phi_+(z,n)$ has the representation (up to a constant)

$$\Phi_+(z,n) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{e^{i\phi}+z}{e^{i\phi}-z}\,\Phi(e^{i\phi},n)\,d\sigma + \frac{1}{2}\begin{pmatrix} \delta_{0,n} \\ -\prod_{i=1}^{n} \frac{1}{a(i)} \end{pmatrix},$$

where $\Phi(e^{i\phi},n)$ is given by (1.5) and σ by (1.6).

To prove the results for $\Phi_-(z,n)$ we note that replacing ψ_1 and ψ_2 in (2.1) and (2.2) by ϕ^1_- and ϕ^2_-, respectively, then taking square magnitudes, and subtracting yields

$$|\phi^1_-(z,n)|^2 - |\phi^2_-(z,n)|^2 = |z|^2\,|\phi^1_-(z,n-1)|^2 - |\phi^2_-(z,n-1)|^2. \tag{2.32}$$

Consequently,

$$|\phi^2_-(z,n)|^2 - |\phi^1_-(z,n)|^2$$
$$= |\phi^2_-(z,i)|^2 - |\phi^1_-(z,i)|^2 + (1-|z|^2)\sum_{j=i}^{n-1} |\phi^1_-(z,j)|^2, \qquad i<n. \tag{2.33}$$

Utilizing the fact that $\sum_{n=0}^{-\infty} |\phi^1_-(z,n)|^2 < \infty$ we find from (2.33) that $\{\phi^2_-(z,n)\}_{n=0}^{\infty}$ is a Cauchy sequence for each fixed z, $0 \leqslant |z| < 1$. From

$$\phi^1_-(z,n) - a(n)\,\phi^1_-(z,n-1) = a(n)\,\alpha(n)\,\phi^2_-(z,n-1),$$

we find that

$$\sum_{n=0}^{-\infty} \left|\frac{\phi^1_-(z,n) - a(n)\,\phi^1_-(z,n-1)}{a(n)}\right|^2 = \sum_{n=0}^{-\infty} |\alpha(n)\,\phi^2_-(z,n-1)|^2.$$

If for any fixed z, $0 \leqslant |z| < 1$, $\lim_{n\to\infty} |\phi^2_-(z,n)| = C > 0$ then there exists a constant $k > 0$ such that

$$\sum_{n=0}^{-\infty} \left|\frac{\phi^1_-(z,n) - a(n)\,\phi^1_-(z,n-1)}{a(n)}\right|^2 \geqslant k \sum_{n=1}^{-\infty} |\alpha(n)|^2.$$

However, since (2.18) holds and $1/a(n) < 1$ this violates the fact that $\sum_{n=0}^{-\infty} |\phi^1_-(z,n)|^2 < \infty$. Hence $\lim_{n=-\infty} |\phi^2_-(z,n)| = 0$ for $0 \leqslant |z| < 1$. Now letting $i \to -\infty$ in (2.33) yields

$$|\phi^2_-(z,n)|^2 - |\phi^1_-(z,n)|^2 = (1-|z|^2)\sum_{j=-\infty}^{n-1} |\phi^1_-(z,j)|^2. \tag{2.34}$$

Therefore $|\phi^2_-(z,n)| \geqslant |\phi^1_-(z,n)|$ for $|z|<1$. In fact $|\phi^2_-(z,n)| > |\phi^1_-(z,n)|$ for $0<|z|<1$. To see this note that if $|\phi^2_-(z_0,n_0)| = |\phi^1_-(z_0,n_0)|$ for a fixed $0<|z_0|<1$ and n_0 then (2.33) implies that $|\phi^1_-(z_0,j)|=0$ for $j \leqslant n_0$. Since (2.18) implies there exists an $\alpha(j_0) \neq 0$, $j_0 < n_0$ we see from (1.1) that $\Phi_-(z,j_1)=0$. The uniqueness of the initial value problem then shows us that $\Phi_-(z,j)=0$ for all j. This coupled with (2.29) and (2.30) implies that $\phi^2_+(z_0,m)=0$ for all m. Since there exists an $|\alpha(i_0)|>0$ for $i_0>0$ the vanishing of $\phi^2_+(z_0,i_0+1)$ and $\phi^2_+(z_0,i_0)$ implies via (1.1) that $\phi^1_+(z_0,n)=0$ which violates the fact that $\phi^1_+(z_0,n)$ was shown to be non-zero for $0<|z_0|<1$. Hence $|\phi^2_-(z,i)| > |\phi^1_-(z,i)|$ for all i. The uniqueness of $\Phi_-(z,n)$ follows from (1.8) since

$$W[\Psi_1(z,n),\Phi_-(z,n)] = z^{n-k}W[\Psi_1(z,k),\Phi_-(z,k)]$$

which tends to zero as k tends to $-\infty$ if $\Psi_1(z,k)$ is bounded. ∎

Define

$$m^1_+(z) = \frac{\phi^1_+(z,1)}{\phi^1_+(z,0)},$$

$$m^2_+(z) = \frac{\phi^2_+(z,0)}{\phi^1_+(z,0)},$$

$$m^1_-(z) = \frac{\phi^2_-(z,-1)}{\phi^2_-(z,0)},$$

and

$$m^2_-(z) = \frac{\phi^1_-(z,0)}{\phi^2_-(z,0)}. \tag{2.35}$$

THEOREM 2.7. *Suppose* (2.18) *holds then* $m^1_+(z)/z$, $m^2_+(z)/z$, $m^1_-(z)$, *and* $m^2_-(z)$ *are in* H_∞. *Furthermore* $\log(m^1_+(z)/z)$ and $\log m^1_-$ *are in* H_p, $p<\infty$.

Proof. We will prove the results for m^1_- and m^2_- as similar methods apply to m^1_+ and m^2_+ [8].

Theorem 2.6 shows us that for $|z|<1$, $|\phi^2_-(z,0)| \leqslant |\phi^1_-(z,0)|$ where the inequality is strict for $0<|z|<1$. Consequently $m^2_-(z)$ is well defined for $0<|z|<1$ and bounded above by one in this region. Thus $m^2_-(z)$ may be extended to the region $0 \leqslant |z| < 1$ by the removable singularity theorem. The uniform boundedness of $m^2_-(z)$ for $|z|<1$ implies that it is an element of H_∞. With $n=0$ in (2.4), and ψ_2 and ψ_1 replaced by ϕ^2_- and ϕ^1_-, respectively, we find

$$m^1_-(z) = a(0)(1-\overline{\alpha(0)}\,m^2_-(z)).$$

Since $|\alpha(0)| < 1$ this equation and the fact that $m_-^2(z) \in H_\infty$ implies that $m_-^1(z) \in H_\infty$. Since $|m_-^2(z)| \leqslant 1$ for $0 \leqslant |z| < 1$ and $|\alpha(0)| < 1$ we see that

$$\operatorname{Re} m_-^1(z) \geqslant a(0)(1 - |\alpha(0)|\, |m_-^2(z)|) > 0. \tag{2.36}$$

Hence $\log m_-^1(z)$ is well defined and $\log m_-^1(z) \in H_p$, $1 < p < \infty$. The Cauchy–Schwarz inequality allows us to extend this to $p \leqslant 1$. ∎

It follows from (2.31), (2.35), Lemma 2.2, and the spectral theorem for unitary operators that when (2.18) holds there exists a positive Borel v measure such that

$$\int_{-\pi}^{\pi} \frac{dv(\phi)}{z - e^{i\phi}} = G(z, 0, 0) = \frac{1}{z} \frac{m_+^2(z)\, m_-^2(z)}{m_+^2(z)\, m_-^2(z) - 1}. \tag{2.37}$$

III. Properties of the Spectrum

Most of the results in this section follow with minor variation from similar results obtained for the random Schrödinger equation (see Carmona [4], Cycon *et al.* [6], and the proofs are included for the reader's convenience.

Let τ be an ergodic automorphism on the probability space $(\Omega, \mathscr{A}, \mu)$. Let $\alpha_w(n) = f(\tau^{-n} w)$ where f is a complex valued measurable function with the property that $|f(\Omega)| \subset [0, 1)$. We assume that (2.18) holds for all w. Let H be given by (2.11), then we find that

$$H_{\tau w} = T H_w T^*, \tag{3.1}$$

where $(T\phi)(n) = \phi(n-1)$, $\phi \in l^2$. Since T is invertible on l^2 we find $H_{\tau^{-1} w} = T^* H_w T$.

We say that a family $\{A_w\}$ of bounded operators on a Hilbert space $\mathscr{H}$ is weakly measurable if the mapping $w \mapsto \langle A_w \psi, \phi\rangle$ is measurable for all ϕ and $\psi \in \mathscr{H}$.

Lemma 3.1. *The map $w \to (zI - H_w)^{-1}$ is weakly measurable for all $z \in \mathbb{C} \setminus U$ where U is the unit circle.*

Proof. Since $\{H_w^n\}$ and $\{H_w^{*n}\}$ are weakly measurable for $n = 0, 1, 2, \ldots$, the result follows because $(zI - H_w)^{-1}$ for $|z| > 1$ can be approximated in norm by polynomials in H_w with coefficients independent of w while the same holds for $(zI - H_w)^{-1} = (zH_w^* - I)^{-1} H_w^*$, $|z| < 1$ in terms of polynomials in H_w^*.

It follows from the spectral theorem that H_w has the representation

$$H_w = \int_U \lambda \, dE(\lambda, w), \tag{3.2}$$

where $E(\lambda, w)$ is a resolution of the identity.

We now show,

THEOREM 3.2. *For any Borel subset Δ of U, $\{E(\Delta, w)\}$ is a weakly measurable family and $E(\Delta, \tau w) = TE(\Delta, w) T^*$.*

Proof. Let Γ be an open subarc of the unit circle and consider the function

$$F_{r,\Gamma}(x) = \frac{1-r^2}{2\pi} \int_U \chi_\Gamma(y) \frac{dy}{|y/r - x|^2 r^2 y} = \frac{1}{2\pi} \int_U \chi_\Gamma(y) \operatorname{Re}\left(\frac{y/r + x}{y/r - x}\right) \frac{dy}{y},$$

where $|r| < 1$, $|x| = 1$, and $\chi_\Gamma(y)$ is the characteristic function for Γ. It is not difficult to show

$$\lim_{r \to 1} F_{r,\Gamma}(x) = \chi_\Gamma(x),$$

and $|F_{r,\Gamma}(x)| \leqslant 1$ for all $r < 1$. Replacing x by H_w we find using the functional calculus and Lemma 3.1 that $\{F_{r,\Gamma}(H_w)\}$ is a measurable family for all $r \leqslant 1$ and that

$$\|E(\Gamma, w) h\|_2 = \lim_{r \to 1} \|F_{r,\Gamma}(H_w) h\|_2, \tag{3.3}$$

for all $h \in l^2$ which implies, via the polarization identity, the measurability of the family $\{E(\Gamma, w)\}$ for any open subarc of the unit circle. That $\{E(\Delta, w)\}$ is weakly measurable for any Borel set Δ follows by standard manipulation using unions and intersections of open intervals. From Eq. (3.3) we see that

$$E(\Gamma, \tau w) = TE(\Gamma, w) T^*, \tag{3.4}$$

for any open subarc of U which implies that $E(\Delta, \tau w) = TE(\Delta, w) T^*$ for any Borel subset of Δ. ∎

Let $\Sigma(H_w)$ be the spectrum of H_w. Now we prove the following

THEOREM 3.3. *Let $\{\alpha_w(n)\}$ be generated as above then there exists a set $\Sigma \subset U$ such that $\Sigma(H_w) = \Sigma$, μ–a.s. Furthermore $\Sigma_{\text{dis}}(H_w) = \varnothing$ μ–a.s. where $\Sigma_{\text{dis}}(H)$ is the set of isolated eigenvalues of H of finite multiplicity.*

Proof. See [4, 6]. From Cycon *et al.* [6, Lemma 9.3] we find that for any fixed Borel measurable set $\Delta \subset U$ the dimension of the range of $E(\Delta, w)$, dim Ran $E(\Delta, w)$, is either zero or infinite μ–a.s. For each rational $0 \leqslant p, q < 2\pi$ let $\Gamma_{p,q}$ be the arc of circle (e^{ip}, e^{iq}) and $r_{p,q}$ be the almost sure value of dim Ran $E(\Gamma_{p,q}, w)$. Let $\Omega_{p,q} = \{w: \text{dim Ran } E(\Gamma_{p,q}, w) = r_{p,q}\}$ and $\Omega_0 = \bigcap_{p,q \in Q} \Omega_{p,q}$. Since $\mu(\Omega_{p,q}) = 1$ and the intersection over $p, q \in Q$ is countable, we have $\mu(\Omega_0) = 1$. ∎

We will now show that if w_1 and w_2 are elements of Ω_0, $\Sigma(H_{w_1}) = \Sigma(H_{w_2})$. To see this notice that if $\lambda \notin \Sigma(H_{w_1})$ then dim Ran $E(\Gamma_{\theta_1,\theta_2}, w_1) = 0$ for all sufficiently small arcs containing λ. Since w_1 and w_2 are in Ω_0, dim Ran $E(\Gamma_{p,q}, w_1)$ = dim Ran $E(\Gamma_{p,q}, w_2)$ which implies that dim Ran $E(\Gamma_{\theta_1,\theta_2}, w_2) = 0$ for sufficiently small arcs containing λ. Therefore $\lambda \notin \Sigma(H_{w_2})$. By interchanging w_1 and w_2 the first part of the theorem is proved. To show that $\Sigma_{\text{dis}}(H_w) = \phi$, suppose that $\lambda \in \Sigma_{\text{dis}}(H_{w_1})$, $w_1 \in \Omega_0$. Then by the definition of Σ_{dis}, $0 < \text{dim Ran } E(\Gamma_{p,q}, w) < \infty$ for $\Gamma_{p,q}$ a sufficiently small arc containing λ. But this contradicts the fact that $w_1 \in \Omega_{\text{p}}$ so $\Sigma_{\text{dis}}(H_{w_1}) = \phi$ μ–a.s. ∎

Recall that each Borel measure on U has a decomposition

$$\alpha = \alpha_{pp} + \alpha_{ac} + \alpha_{sc}, \tag{3.5}$$

where α_{pp} is purely atomic or point masses, α_{ac} is absolutely continuous with respect to Lebesgue measure, and α_{sc} is continuous and singular with respect to Lebesgue measure.

Given a unitary operator H with spectral measure E acting on a Hilbert space $\mathscr{H}$ we write $H_{pp}(E) = \{\psi \in \mathscr{H} : \langle E\psi, \psi \rangle \text{ is atomic}\}$, $H_{ac}(E) = \{\psi \in \mathscr{H} : \langle E\psi, \psi \rangle \text{ is absolutely continuous}\}$, and $H_{sc}(E) = \{\psi \in \mathscr{H} : \langle E\psi, \psi \rangle \text{ is singular continuous}\}$. Furthermore we decompose $E = E_{pp} + E_{as} + E_{sc}$ where $E_{pp}\mathscr{H} = \mathscr{H}_{pp}$, $E_{ac}\mathscr{H} = \mathscr{H}_{ac}$, and $E_{sc}\mathscr{H} = \mathscr{H}_{sc}$ are orthogonal projections. This induces a decomposition of H via (4.3) into H_{pp}, H_{ac}, and H_{sc} with spectrum $\Sigma_{pp}(H)$, $\Sigma_{ac}(H)$, and $\Sigma_{sc}(H)$, respectively. We are now ready to prove our next result.

THEOREM 3.4. *Let $\{\alpha_w(n)\}$ be as above then there exists Σ_{ac}, Σ_{sc}, and $\Sigma_{pp} \subset U$ such that $\Sigma_{ac}(H_w) = \Sigma_{ac}$ μ-a.s., $\Sigma_{sc}(H_w) = \Sigma_{sc}$ μ-a.s., and $\Sigma_{pp}(H_w) = \Sigma_{pp}$ μ-a.s.*

Before proving the above result we note that Carmona has shown [4] (see also 6, p. 171]) that if $\mathscr{I}$ is the family of finite unions of real open intervals each of which has rational endpoints, then

$$\mu_s(A) = \lim_{n \to \infty} \sup_{I \in \mathscr{I}, |I| < 1/n} \mu(A \cap I), \tag{3.6}$$

where A is any Borel set, and μ_s is the singular component of μ. Furthermore it is not difficult to see [4] that

$$\mu_c(A) = \lim_{s \to 0} \lim_{n \to \infty} \sum_{0 \leq k < 2^{-n}} \mu(I_{n,k}) \, 1_{[0,s]}(\mu(I_{n,k})), \tag{3.7}$$

where A in any half open interval $[a, b)$ and $I_{n,k} = [a + k2^{-n}(b-a), a + (k+1)\,2^{-n}(b-a))$.

Proof of Theorem 3.4. We first show that $\{E_{ac}(w)\}$, $\{E_{pp}(w)\}$ and $\{E_{sc}(w)\}$ are weakly measurable. Since $E_s = E_{sc} + E_{pp}$ and $E_c = E_{ac} + E_{sc}$ where E_{sc}, E_{pp}, and E_{ac} are mutually orthogonal projections and since $\{E(w)\}$ is a weakly measurable family it is sufficient to show that $\{E_s(w)\}$ and $\{E_c(w)\}$ are weakly measurable families. Letting $\phi(t) = e^{it}$ and $\tilde{E}(A, w) = E(\phi(A), w)$ for A a Borel subset of $[0, 2\pi)$ we find from (3.6) that for any $\psi \in l^2$

$$\langle \tilde{E}_s(A, w)\psi, \psi \rangle = \lim_{n \to \infty} \sup_{I \in \mathscr{I},\, |I| < 1/n} \langle \tilde{E}(A \cap I, w)\psi, \psi \rangle.$$

The measurability of $\langle \tilde{E}_s(\delta, w)\,\psi, \psi \rangle$ follows from the countability of $\mathscr{I}$ and the weak measurability of $\{E(\delta, w)\}$ for any Borel measurable subset δ of U. The polarization identity and the measurability of ϕ show that $\{E_s(w)\}$ is weakly measurable. Likewise the weak measurability of $\{E_c(\delta, w)\}$ can be proved using (3.7). Replacing $E(A, w)$ in Theorem 3.3 by $\{E_{sc}(A, w)\}$, $\{E_{ac}(A, w)\}$, or $\{E_{pp}(A, w)\}$ then using the argument in that theorem gives the result. ∎

We now prove,

THEOREM 3.5. *Let* $\{\alpha_w(n)\}$ *be as above then there exists a set* Ω *with* $\mu(\Omega) = 1$ *such that* $\lim_{L \to \infty}(1/(2L+1))\,\mathrm{tr}(\chi_L(f(H_w))) = \int_U f(e^{i\phi})\,dk$ *exists for every* $f \in C(U)$ *where the probability measure* k *is not random. That is, for almost every* w, H_w *has the ATUV measure* k. *Furthermore* $\operatorname{supp} k = \Sigma$.

Proof. The proof will closely follow that given by Cycon *et al.* [6]. From the definition of k_L, Eq. (2.20) we see that

$$\begin{aligned}\int_U f(e^{i\phi})\,dk_L &= \frac{1}{2L+1} \sum_{|n| \leq L} \langle f(H_w)e_n, e_n \rangle \\ &= \frac{1}{2L+1} \sum_{|n| \leq L} \langle f(H_{\tau^{-n}w})e_0, e_0 \rangle,\end{aligned}$$

where (3.1) has been used to obtain the last equality. The Birkhoff ergodic theorem now says there is a set Ω_f depending upon f with $\mu(\Omega_f)=1$ such that

$$\lim_{L\to\infty}\frac{1}{2L+1}\sum_{|n|\leq L}\langle f(H_{\tau^{-n}w})e_0, e_0\rangle = \mathbb{E}(\langle f(H_w)e_0, e_0\rangle). \tag{3.8}$$

However, since $C(U)$ is separable there exists a countable set of functions S dense in $C(U)$ satisfying (3.8) on a common set Ω of μ measure one. Consequently (3.8) holds for all $f\in C(U)$ on Ω. Thus

$$\lim_{L\to\infty}\frac{1}{2L+1}\operatorname{tr}(\chi_L f(H_w)) = \int_U f(e^{i\phi})\,dk$$

for all $f\in C(U)$. To show the supp $k=\Sigma$ suppose $e^{i\phi_0}\notin\Sigma$. Then there exists a positive continuous function on U such that $f(e^{i\phi_0})=1$ and $f=0$ on Σ. Therefore $f(H_w)=0$ μ–a.s. Hence $\int_U f(e^{i\phi_0})\,dk=\mathbb{E}(\langle f(H_w)e_0, e_0\rangle)=0$ implying that $e^{i\phi_0}\notin\operatorname{supp} k$.

If $e^{i\phi_0}\notin\operatorname{supp} k$ there exists a positive continuous function f such that $f(e^{i\phi_0})=1$ and $f=0$ on $\operatorname{supp}\beta$. Therefore $\mathbb{E}(\langle f(H_w)e_0, e_0\rangle)=0$ which implies that $\langle f(H_w)e_0, e_0\rangle=0$ for μ-almost all w. Since $\langle f(H_w)e_n, e_n\rangle = \langle f(H_{\tau_w^{-n}})e_0, e_0\rangle$ we find that for μ-almost all w and for all n, $\langle f(H_w)e_n, e_n\rangle=0$. Since f is positive, continuous, and $f(e^{i\phi_0})=1$ this implies that for μ-almost all w, $f(H_w)=0$. Therefore $e^{i\phi_0}\notin\Sigma$. ▮

THEOREM 3.6. *Let $\{\alpha_w(n)\}$ be given as above and suppose $\{\alpha_w(n)\}$ is not identically zero. Then*

$$\begin{aligned}\lim_{L\to\infty}\frac{1}{2L+1}\operatorname{tr} f(H_L) &= \int_U f(e^{i\phi})\,dk\\ &= \lim_{L\to\infty}\frac{1}{L+1}\operatorname{tr}(\chi_L^+ f(H_w))\\ &= \lim_{L\to\infty}\frac{1}{L+1}\operatorname{tr}(f(H_{L_+}))\end{aligned}$$

for all $f\in C(U)$ and μ-almost all w.

Proof. If $\{\alpha_w(n)\}$ is not identically 0 then $\Sigma\,|\alpha_w(n)|=\infty$, consequently the first equality follows from Theorem 3.5 and Lemma 2.3. The proof of the second equality follows the proof in Theorem 3.5 where $2L+1$ is replaced by $L+1$ and $|n|\leq L$ is replaced by $0\leq n\leq L$. The last inequality follows from Lemma 2.4. ▮

IV. LYAPUNOV EXPONENT

From now on besides (2.18) we will also assume that

$$\mathbb{E}(\log(1-|\alpha_w(1)|)) = \int_\Omega \log(1-|\alpha_w(1)|)\, d\mu(w) > -\infty, \tag{4.1}$$

which implies that $\log(1-|\alpha_w(1)|) \in L(\mu)$ since $|\alpha_w(1)| < 1$ for all $w \in \Omega$. This implies that $\log(1-|\alpha_w(n)|)/n$ tends to zero as n tends to infinity for μ-a.e. w by the ergodic theorem. For $i < k$ let

$$\gamma_n(w, z) = \frac{1}{n} \log \|T_{1,n}(w, z)\|, \tag{4.2}$$

and

$$\gamma_{-n}(w, z) = \frac{1}{n} \log \|T_{-n+1,0}(w, z)\|. \tag{4.3}$$

THEOREM 4.1. *Suppose* (4.1) *holds. Then*

$$\lim_{n\to\infty} \gamma_n(w, z) = \gamma(z) = \lim_{n\to\infty} \gamma_{-n}(w, z) \tag{4.4}$$

exists and is independent of w for each fixed $z \in \mathbb{C}$ and μ-almost every w. Furthermore $\gamma(z)$ is subharmonic, greater than or equal to zero, and for μ-almost every w and every z, $\overline{\lim}_{|n|\to\infty} \gamma_n(w, z) \leqslant \gamma(z)$.

Proof. Using the Hilbert–Schmidt norm we find that $\|T_{1,1}(w, z)\| = a_w(1)[(1+|z|^2)(1+|\alpha_w(1)|^2]^{1/2}$. Consequently (4.2) implies that $\int \log^+ \|T_{1,1}(w, z)\|\, dw < \infty$. Equation (4.4) follows from Kingman's subadditive ergodic theorem (Krengel [16], Ruelle [17]) since $\log \|T_{1,n}\|$ is a subadditive stochastic process. The rest of the theorem follows from [8, Theorem 3.1], which has its roots in the work of Craig and Simon [5] and Herman [12]. ∎

COROLLARY 4.2. *Let*

$$\gamma_n^-(w, z) = \frac{1}{n} \log \|T_{-n+1,0}^{-1}(w, z)\|.$$

Then for fixed z and μ-almost all w

$$\lim_{n\to\infty} \gamma_n^-(w, z) = \gamma^-(z) = -\log|z| + \gamma(z).$$

Proof. Write

$$T_{-n+1,0}(z)=\begin{bmatrix} a_{11}(-n+1) & a_{12}(-n+1) \\ a_{21}(-n+1) & a_{22}(-n+1)\end{bmatrix}.$$

Since det $T_n(z)=z$ we find that

$$\begin{aligned} T^{-1}_{-n+1,0}(z) &= T^{-1}(z,-n+1)\,T^{-1}(z,-n+2)\cdots T^{-1}(z,\ 0) \\ &= \frac{1}{z^n}\begin{bmatrix} a_{22}(-n+1) & -a_{12}(-n+1) \\ -a_{21}(-n+1) & a_{11}(-n+1)\end{bmatrix}. \end{aligned}$$

Consequently

$$\gamma_n^-(w,z) = -\log|z| + \gamma_{-n}(w,z),$$

and the result follows by letting n tend to infinity and using Theorem 4.1. ▮

We now develop an integral relation between γ and the measure k. It has been shown by Geronimo [8, Theorem 3.2] that if (4.1) holds then for μ-almost all w,

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{|\phi^*(z,n)+\phi(z,n)|}{2}=\gamma(z), \tag{4.5}$$

where the above convergence is uniform on subsets of $\mathbb{C}\backslash U$.

Now consider the singular polynomial $\phi^*(z,n-1)+z\phi(z,n-1)$ where $\Phi(z,n-1)=\binom{\phi(z,n-1)}{\phi^*(z,n-1)}=T_{1,n-1}(z)\binom{1}{1}$. From Lemma 2.1 it follows that

$$\begin{aligned} &\frac{1}{n}\log|\phi^*(z,n-1)+z\phi(z,n-1)| \\ &\quad=\frac{1}{n}\log\prod_{i=1}^{n-1}a(i)+\frac{1}{n}\operatorname{tr}\ln|zI-H_{n_+}| \\ &\quad=\frac{1}{n}\log\prod_{i=1}^{n-1}a(i)+\int_{-\pi}^{\pi}\ln|z-e^{i\phi}|\,dk_{n_+}(\phi), \end{aligned} \tag{4.6}$$

where k_{n_+} is a probability measure placing mass $1/n$ at each eigenvalue of H_{n_+} of $\mathbb{C}/U$.

LEMMA 4.3. *For each compact subset K of $\mathbb{C}\backslash U$ there exists a d_1 and d_2 depending only on K such that*

$$d_1(1-|\alpha(n)|)<\left|\frac{\phi^*(z,n)+\phi(z,n)}{\phi^*(z,n)-\phi(z,n)}\right|<d_2(1-|\alpha(n)|)^{-1},$$

and

$$d_1 \sqrt{\frac{1-|\alpha(n)|}{1+|\alpha(n)|}} < \left| \frac{\phi^*(z,n)-\phi(z,n)}{\phi^*(z,n-1)+z\phi(z,n-1)} \right| < d_2 \sqrt{\frac{1-|\alpha(n)|}{1+|\alpha(n)|}}.$$

Proof. Let $\hat{K}$ be the part of K inside the unit circle. Since

$$S_n(z) = \frac{\phi^*(z,n)+\phi(z,n)}{\phi^*(z,n)-\phi(z,n)},$$

is a C-function, i.e., $\operatorname{Re} S_n(z) \geqslant 0$ for $|z|<1$, it has the representation

$$S_n(z) = w_n + \int_{-\pi}^{\pi} \frac{e^{i\phi}+z}{e^{i\phi}-z}\, d\rho_n(\theta), \qquad |z|<1$$

where ρ_n is a finite positive measure. With z equal to zero we find

$$S_n(0) = \frac{1+\alpha(n)}{1-\alpha(n)} = iv_n + c_n,$$

where $c_n = \int_{-\pi}^{\pi} d\rho_n(\theta)$. Therefore

$$c_n = \frac{1-|\alpha(n)|^2}{|1-\alpha(n)|^2} \leqslant \frac{1+|\alpha(n)|}{1-|\alpha(n)|}$$

and

$$|v_n| = \frac{|\overline{\alpha(n)}-\alpha(n)|}{|1+\alpha(n)|^2} \leqslant \frac{2}{1-|\alpha(n)|}.$$

Thus we find

$$|S_n(z)| \leqslant |v_n| + \int_{-\pi}^{\pi} \left| \frac{e^{i\phi}+z}{e^{i\phi}-z} \right| d\rho_n(\theta) \leqslant |v_n| + \hat{d}c_n,$$

where $\hat{d}$ depends only on the set $\hat{K}$. Combining the above inequalities we find $|S_n(z)| \leqslant c(1-|\alpha(n)|)^{-1}$. If we now apply the same analysis to $(\phi^*(z,n)-\phi(z,n))/(\phi(z,n)+\phi(z,n))$ we find that

$$\left| \frac{\phi^*(z,n)-\phi(z,n)}{\phi^*(z,n)+\phi(z,n)} \right| \leqslant c(1-|\alpha(n)|)^{-1}$$

or $|S_n(z)| \geqslant c_1(1-|\alpha(n)|)$. For the part of K exterior to the unit disk, $\hat{K}_1$, $-S_n(z)$ is a C-function and has the representation

$$-S_n(z) = i\tilde{v}_n + \int_{-\pi}^{\pi} \frac{e^{i\phi}+z}{e^{i\phi}-z}\, d_{\tilde{\rho}_n}(\theta), \qquad |z|>1.$$

Since $-S_n(\infty) = (1 + \overline{\alpha(n)})/(1 - \alpha(n))$ we find that $|S_n(z)| \leqslant \tilde{C}_1(1 - |\alpha(n)|)^{-1}$. Reapplying the above reasoning to $-(\phi^*(z,n) - \phi(z,n))/(\phi^*(z,n) + \phi(z,n))$ yields $|(\phi^*(z,n) - \phi(z,n))/(\phi^*(z,n) + \phi(z,n))| \geqslant \tilde{C}_2(1 - |\alpha(n)|)$ for $|z| > 1$. Therefore $\tilde{C}_3(1 - |\alpha(n)|) \leqslant |S_n(n)| \leqslant \tilde{C}_1(1 - |\alpha(n)|)^{-1}$ for $z \in K$ which yields the first set of inequalities in the lemma. Repeating the same analysis on $g_n(z) = (\phi^*(z,n) - \phi(z,n))/(\phi^*(z,n-1) + z\phi(z,n-1))$ yields the second set of inequalities. ▮

LEMMA 4.4. *If* (4.1) *holds, then for* μ*-almost all* w

$$\lim_{n \to \infty} \frac{1}{n} \log |\phi^*(z, n-1) + z\phi(z, n-1)| = \gamma(z) \tag{4.7}$$

uniformly on compact subsets of $\mathbb{C} \backslash U$.

Proof. The result follows from Lemma 4.3, (4.5), and the fact that (4.1) implies that for μ-almost all w, $(1/n) \log |1 - |\alpha_w(n)||$ tends to zero as n tends to infinity. ▮

Adopting the convention that $\int_{-\pi}^{\pi} \log |z - e^{i\phi}| \, dk(\phi)$ is equal to $-\infty$ if the integral diverges to $-\infty$ we establish,

THEOREM 4.5. *If* (4.1) *and* (2.18) *hold then*

$$\gamma(z) = R + \int_{-\pi}^{\pi} \log |z - e^{i\phi}| \, dk(\phi),$$

for all z *where* $R = \lim_{n \to \infty} (1/n) \log \prod_{i=1}^{n} a_w(i) < \infty$ *for* μ*-almost all* w.

Proof. From (4.6) and Lemma 4.4 we see that for μ-almost all w

$$\lim_{n \to \infty} \left(\frac{1}{n} \log \prod_{i=1}^{n-1} a_w(i) + \int_{-\pi}^{\pi} \log |z - e^{i\phi}| \, dk_{n_+} \right) = \gamma(z),$$

uniformly on compact subsets of $\mathbb{C} \backslash U$. With $z = 0$ we find that

$$R = \lim_{n \to \infty} \frac{1}{n-1} \log \prod_{i=1}^{n-1} a_w(i) < \infty.$$

The equality for $z \in \mathbb{C} \backslash U$ now follows from Theorem 2.5. The fact that the equality holds for all z follows from the fact that $\gamma(z)$ and $R + \int_{-\pi}^{\pi} \log(z - e^{i\phi}) \, dk(\phi)$ are both subharmonic functions. ▮

As a first application of the above results we show that k is a continuous measure.

THEOREM 4.6. *Let θ_1 and θ_2 be real and $|e^{i\theta_1}-e^{i\theta_2}|<c<1$. Then*

$$|k(\theta_1)-k(\theta_2)| \leqslant \frac{(\log 2+R)}{\log |e^{i\theta_1}-e^{i\theta_2}|}.$$

Remark. The proof is essentially due to Craig and Simon [5].

Proof. Suppose without loss of generality $\theta_2>\theta_1$. Then

$$0 \leqslant \gamma(e^{i\theta_1}) = R + \int \log |e^{i\theta_1}-e^{i\theta}| \, dk(\theta)$$

$$= R + \int_{\theta_1}^{\theta_2} \log |e^{i\theta_1}-e^{i\theta}| \, dk(\theta) + \int_{\substack{|e^{i\theta_1}-e^{i\theta}|\leqslant 1 \\ \{\theta<\theta_1\}\cup\{\theta_2<\theta\}}} \log |e^{i\theta_1}-e^{i\theta}| \, dk(\theta)$$

$$+ \int_{1 \leqslant |e^{i\theta_1}-e^{i\theta}|} \log |e^{i\theta_1}-e^{i\theta}| \, dk(\theta).$$

Since the second integral on the RHS of the above equation is negative we see that

$$\begin{aligned} 0 &\leqslant -\log |e^{i\theta_2}-e^{i\theta_1}| \, (k(\theta_2)-k(\theta_1)) \\ &\leqslant R + \int_{1 \leqslant |e^{i\theta_1}-e^{i\theta_2}|} \log |e^{i\theta_1}-e^{i\theta}| \, dk(\theta), \\ &\leqslant R + \log 2. \end{aligned}$$

Thus the result is proved. ∎

We now connect the Lyapunov exponents with exponentially decaying solutions of (1.1).

LEMMA 4.7. *Suppose* (4.1) *and* (2.18) *hold. Then for* $|z|<1$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi_+(z, n, w)\| = \log |z| - \gamma(z), \tag{4.8}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi_-(z, -n, w)\| = -\gamma(z), \tag{4.9}$$

for μ-almost all w. Here Φ_+ and Φ_- are solutions of (1.1) *given by* (2.27) *and* (2.28), *respectively.*

Proof. Since $\det T_n(z)=z$ for all n, Eqs. (4.8) and (4.9) follow from the Osceledec ergodic theorem (Ruelle [17], Krengel [16]), and Theorem 4.1. ∎

V. m-FUNCTIONS

We now use the results of the previous section to obtain results on m^1_+, m^2_+, m^1_-, and m^2_- defined by Eq. (2.35).

Since $\Phi_+(z, n, \tau^{-m}w)$ and $\Phi_+(z, n+m, w)$ satisfy the same equation and both are l^2 at $+\infty$, it follows from Theorem 2.6 that $\Phi_+(z, n, \tau^{-m}w) = c(z)\,\Phi_+(z, n+m, w)$ where $c(z)$ is independent of n. This implies that

$$m^1_+(z, \tau^{-m}w) = \frac{\phi^1_+(z, m+1, w)}{\phi^1_+(z, m, w)}, \tag{5.1}$$

and

$$m^2_+(z, \tau^{-m}w) = \frac{\phi^2_+(z, m, w)}{\phi^1_+(z, m, w)}. \tag{5.2}$$

Likewise we find that

$$m^1_-(z, \tau^{-m}w) = \frac{\phi^2_-(z, m-1, w)}{\phi^2_-(z, m, w)}, \tag{5.3}$$

and

$$m^2_-(z, \tau^{-m}w) = \frac{\phi^1_-(z, m, w)}{\phi^2_-(z, m, w)}. \tag{5.4}$$

With this we prove,

THEOREM 5.1. *If* (2.18) *and* (4.1) *hold then* $g_+(z) = \mathbb{E}(\log(m^1_+(z, w)/z))$ *and* $g_-(z) = \mathbb{E}(\log m^1_-(z))$ *are in* H_p, $p < \infty$, *and* $\operatorname{Re} g_\pm(z) = -\gamma(z)$.

Proof. We show the result only for g_- as the proof for g_+ follows in a similar manner. We note that (4.1), (2.36), and the fact that $\operatorname{Re} m^1_-(z, w) > 0$, $|z| < 1$ show that $\mathbb{E}[|\log(m^1_-(z, w))|] < \infty$ which implies that $g_-(z)$ is analytic for $|z| < 1$. Since $\operatorname{Re} m^1_-(re^{i\phi}, w) > 0$ for $|r| < 1$ and for all w we find that $\operatorname{Im} g_- \in L_p$, $p < \infty$, hence $g_- \in H_p$ for $1 < p < \infty$. The result for $p \leqslant 1$ follows from the Cauchy–Schwarz inequality.

To show that $\operatorname{Re} g_-(z) = -\gamma(z)$ we begin with (4.9) and use the fact that $\phi^1_-(z, n)/\phi^2_-(z, n)$ is bounded for all $n \leqslant 0$ (Theorem 2.6) to find

$$\begin{aligned} -\gamma(z) &= \lim_{n\to\infty} \frac{1}{n} \log \|\Phi_-(z, -n)\| \\ &= \lim_{n\to\infty} \frac{1}{2n} \log\left(|\phi^1_-(z, -n)|^2 + |\phi^2_-(z, -n)|^2\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left|\frac{\phi^2_-(z, -n)}{\phi^2_-(z, 0)}\right|. \end{aligned}$$

Now using (5.3) yields

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\phi_-^2(z,-n)}{\phi_-^2(z,0)} \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |m_-^1(z, \tau^i w)|.$$

Since $\log |m_-^1(z,w)| \in L^1(\mu)$ the ergodic theorem implies that $\operatorname{Re} g_-(z) = -\gamma(z)$. ∎

LEMMA 5.2.

$$\gamma(z) - \log |z| = \mathbb{E}\left(\frac{1}{2} \log\left[1 + \frac{1-|z|^2}{|z|^2 - |m_+^2(z,w)|^2}\right]\right)$$
$$\geqslant \frac{1}{2}\,\mathbb{E}\left\{\frac{1-|z|^2}{1-|m_+^2(z,w)|^2}\right\}, \tag{5.5}$$

and

$$\gamma(z) - \log |z| = \frac{1}{2}\,\mathbb{E}\left[\log\left\{1 + \frac{1-|z|^2}{|z|^2(1-|m_-^2(z,w)|^2)}\right\}\right]$$
$$\geqslant \frac{1}{2}\,\mathbb{E}\left\{\frac{1-|z|^2}{1-|zm_-^2(z,w)|^2}\right\}. \tag{5.6}$$

Proof. We prove only (5.6) since (5.5) follows from Lemma 4.3 of [8]. From (2.32) with $n=0$ we find using the definition of m_-^1 and m_-^2, and (5.4)

$$|m_-^1(z,w)|^2 = \frac{1-|m_-^2(z,w)|^2}{1-|zm_-^2(z,\tau w)|^2}.$$

Now taking expectations and using the fact that τ is an automorphism we find

$$-\mathbb{E}\{\log |m_-^1(z,w)|\} = -\frac{1}{2}\,\mathbb{E}\left\{\log\left[\frac{1-|m_-^2(z,w)|^2}{1-|zm_-^2(z,w)|^2}\right]\right\}.$$

Subtracting $\log |z|$ from both sides of the above equation and using Theorem 5.1 yields

$$\gamma(z) - \log |z| = \frac{1}{2}\,\mathbb{E}\left[\log\left\{1 + \frac{1-|z|^2}{|z|^2(1-|m_-^2(z,w)|^2)}\right\}\right].$$

The inequality in (5.6) is arrived at by using the inequality $\log(1+x) \geqslant x/(1+x)$ for $x \geqslant 0$ in the above equation. ∎

LEMMA 5.3. *For* $0 \leqslant |z| < 1$ *and all* w,

$$\frac{1}{2}\frac{1}{1-|zm_-^2(z,w)|^2}+\frac{1}{2}\frac{1}{1-|m_+^2(z,w)|^2}+\operatorname{Re}\left(\frac{1}{m_+^2(z,w)\,m_+^2(z,w)-1}\right)$$
$$=\frac{1}{2}\frac{|m_+^2(z,w)-\overline{m_-^2(z,w)}|^2\,(1-|z|^2\,|m_+^2(z,w)|^2\,|m_-^2(z,w)|^2)}{(1-|z|^2\,|m_-^2(z,w)|^2)(1-|m_+^2(z,w)|^2)|1-m_+^2(z,w)\,m_-^2(z,w)|^2}$$
$$-\frac{1}{2}\frac{(1-|z|^2)(1-|m_+^2(z,w)|^2)\,|m_-^2(z,w)|^2}{(1-|zm_-^2(z,w)|^2)\mid(1-|m_+^2(z,w)\,m_-^2(z,w)|^2)}$$

Proof. Beginning with the LHS of the above equation and putting everything over a common denominator we find that the numerator N is equal to

$$N=\tfrac{1}{2}(|m_+^2m_-^2|^2-2\operatorname{Re}(m_+^2m_-^2)+1)[1-|m_+^2|^2+1-|zm_-^2|^2]$$
$$+(\operatorname{Re}(m_+^2m_-^2)-1)(1-|zm_-^2|^2)(1-|m_+^2|^2),$$

where we suppress for the time being the dependence of m_+^1 and m_-^2 on w and z. Since $\operatorname{Re} m_+^2m_-^2=-|m_+^2-\bar{m}_-^2|^2/2+(|m_+^2|^2+|m_-^2|^2)/2$ we find after simplification that

$$N=\frac{1}{2}|m_+^2-\bar{m}_-^2|^2\,(1-|zm_-^2|^2\,|m_+^2|^2)$$
$$+\frac{1}{2}(1-|m_+^2|^2)(1-|m_-^2|^2)[2-|m_+^2|^2-|zm_-^2|^2]$$
$$+\left[\frac{|m_+^2|^2+|m_-^2|^2}{2}-1\right](1-|zm_-^2|^2)(1-|m_+^2|^2).$$

Adding and subtracting $|m_-^2|^2$ in the bracketed portion of the second term of the above equation then combining terms yields

$$N=\frac{1}{2}|m_+^2-\bar{m}_-^2|^2\,(1-|zm_-^2|^2\,|m_+^2|^2)$$
$$+\frac{1}{2}(1-|z|^2)(1-|m_+^2|^2)(1-|m_-^2|^2)\,|m_-^2|^2$$
$$-(1-|z|^2)(1-|m_+^2|^2)\,|m_-^2|^2\left[1-\frac{|m_+^2|^2+|m_-^2|^2}{2}\right].$$

Combining the last two terms on the RHS of the above equation now yields the lemma. ∎

LEMMA 5.4. *For* $0 \leqslant |z| < 1$,

$$2\left(\frac{\gamma(z)-\log|z|}{1-|z|^2}\right)-1+r\frac{\partial\gamma}{\partial r}$$
$$\geqslant \mathbb{E}\left[\frac{1}{2}\frac{1}{1-|zm_-^2(z,w)|^2}+\frac{1}{2}\frac{1}{1-|m_+^2(z,w)|^2}+\operatorname{Re}\frac{1}{m_+^2(z,w)\,m_-^2(z,w)-1}\right].$$

Proof. From (2.31) we find that

$$G(z,0,0)=\frac{z^{-1}\phi_+^2(z,0)\,\phi_-^1(z,0)}{\phi_-^1(z,0)\,\phi_+^2(z,0)-\phi_+^1(z,0)\,\phi_-^2(z,0)}=\frac{z^{-1}m_+^2(z,w)\,m_-^2(z,w)}{m_+^2(z,w)\,m_-^2(z,w)-1}.$$

Consequently from Theorem 3.5 with $f(x)=(z-x)^{-1}$ and Theorem 5.1 we find that

$$\begin{aligned} r\frac{\partial\gamma(z)}{\partial r} &= \mathbb{E}\left(\operatorname{Re}\left(\frac{m_+^2(z,w)\,m_-^2(z,w)}{(m_+^2(z,w)\,m_-^2(z,w)-1)}\right)\right)\\ &= \mathbb{E}\left(\operatorname{Re}\left(\frac{1}{(m_+^2(z,w)\,m_-^2(z,w)-1)}\right)+1\right). \end{aligned} \tag{5.7}$$

Equations (5.5) and (5.6) show us that

$$2\left(\frac{\gamma(z)-\log(z)}{1-|z|^2}\right)-1\geqslant \mathbb{E}\left(\frac{1}{1-|m_+^2(z,w)|^2}-1\right),$$

and

$$2\left(\frac{\gamma(z)-\log(z)}{1-|z|^2}\right)-1\geqslant \mathbb{E}\left(\frac{1}{1-|z|^2\,|m_-^2(z,w)|^2}-1\right).$$

Summing the above two equations, dividing by 2, then adding the result to (5.7) finishes the proof. ∎

LEMMA 5.5. *For every* w *and* $0 < |z| < 1$

$$\begin{aligned} \log\frac{1-|m_+^2(z,w)|^2}{|z|^2-|m_+^2(z,w)|^2} &\leqslant \log\left(\frac{1-|m_+^2(z,\tau^{-n}w)|^2}{|z|^2-|m_+^2(z,\tau^{-n}w)|^2}\right)\\ &\quad+\log\left[1+\frac{(1-|z|^2)(2/|z|^2)^n\prod_{i=1}^n(1-|\alpha_w(i)|)^{-1}}{(1-|m_+^2(z,\tau^{-n}w)|^2)}\right], \end{aligned} \tag{5.8}$$

and

$$\log \frac{1-|z|^2\,|m^2_-(z,w)|^2}{1-|m^2_-(z,w)|^2}$$
$$\leqslant \log\left(\frac{1-|z|^2\,|m^2_-(z,\tau^n w)|^2}{1-|m^2_+(z,\tau^n w)|^2}\right)$$
$$+\log\left[1+\frac{(1-|z|^2)\,2^n\prod_{i=0}^{n-1}(1-|\alpha_w(-i)|)^{-1}}{(1-|z|^2\,m^2_-(z,\tau^n w)|^2)}\right]. \qquad (5.9)$$

Proof. We only prove (5.9) since (5.8) has already been proved in an analogous manner in [8]. From (2.33) with $i=-n$ and $n=0$ we find

$$1-|m^2_-(z,w)|^2=\left|\frac{\phi^2_-(z,-n)}{\phi^2_-(z,0)}\right|^2$$
$$\times\left\{1-|m^2_-(z,\tau^n w)|^2+(1-|z|^2)\sum_{j=-1}^{-n}\left|\frac{\phi^1_-(z,-j)}{\phi^2_-(z,-n)}\right|^2\right\},$$

and

$$1-|z|^2\,|m^2_-(z,w)|^2$$
$$=\left|\frac{\phi^2_-(z,-n)}{\phi^2_-(z,0)}\right|^2$$
$$\times\left\{1-|z|^2\,|m^2_-(z,\tau^n w)|^2+(1-|z|^2)\sum_{i=0}^{-n+1}\left|\frac{\phi^1_-(z,i)}{\phi^2_-(z,-n)}\right|^2\right\}.$$

Consequently

$$\log \frac{1-|z|^2\,|m^2_-(z,w)|^2}{1-|m^2_-(z,w)|^2}$$
$$\leqslant \log \frac{1-|z|^2\,|m^2_-(z,\tau^n w)|^2}{1-|m^2_-(z,\tau^n w)|^2}$$
$$+\log\left[\frac{1+\dfrac{(1-|z|^2)\sum_{i=0}^{-n+1}\left|\dfrac{\phi^1_-(z,i)}{\varphi^1_-(z,-n)}\right|^2}{1-|z|^2\,|m^2_-(z,\tau^n w)|^2}}{1+\dfrac{(1-|z|^2)\sum_{i=-1}^{-n}\left|\dfrac{\phi^1_-(z,i)}{\phi^1_-(z,-n)}\right|^2}{1-|z|^2\,|m^2_-(z,\tau^n w)|^2}}\right].$$

The second term on the RHS of the above equation can be rewritten as

$$\log\left[\frac{1+\dfrac{(1-|z|^2)\sum_{i=0}^{-n+1}\left|\dfrac{\phi_-^1(z,-j)}{\varphi_-^1(z,-n)}\right|^2}{1-|z|^2\,|m_-^2(z,\tau^n w)|^2}}{1+\dfrac{(1-|z|^2)\sum_{i=-1}^{-n}\left|\dfrac{\phi_-^1(z,-j)}{\phi_-^1(z,-n)}\right|^2}{1-|z|^2\,|m_-^2(z,r^n w)|^2}}\right].$$

$$=\log\left[1+\frac{\dfrac{(1-|z|^2)\left(\left|\dfrac{\phi_-^1(z,0)}{\varphi_-^1(z,-n)}\right|^2-1\right)}{1-|z|^2\,|m_-^2(z,\tau^n w)|^2}}{1+\dfrac{(1-|z|^2)\sum_{i=1}^{-n}\left|\dfrac{\phi_-^1(z,-j)}{\phi_-^1(z,-n)}\right|^2}{1-|z|^2\,|m_-^2(z,r^n w)|^2}}\right].$$

$$\leqslant\log\left[1+\frac{(1-|z|^2)\left|\dfrac{\phi_-^1(z,0)}{\phi_-^1(z,-n)}\right|^2}{1-|z|^2\,|m_-^2(z,\tau^n w)|^2}\right]. \tag{5.10}$$

We now write

$$\frac{\phi_-^2(z,0)}{\phi_-^2(z,-n)}=\prod_{i=0}^{n-1}\frac{\phi_-^2(z,-i)}{\phi_-^2(z,-i-1)}.$$

From (2.2) with ψ_2 and ψ_1 replaced by ϕ_-^2 and ϕ_-^1, respectively, we find

$$\frac{\phi_-^2(z,-i)}{\phi_-^2(z,-i-1)}=a(-i)\left(1+\overline{\alpha(-i)}\,\frac{\phi_-^1(z,-i-1)}{\phi_-^2(z,-i-1)}\right).$$

Theorem 2.6 now implies that

$$\left|\frac{\phi_-^2(z,-i)}{\phi_-^2(z,-i-1)}\right|\leqslant a(-i)(1+|\alpha(-i)|)=\sqrt{\frac{1+|\alpha(-i)|}{1-|\alpha(-i)|}}<\frac{\sqrt{2}}{\sqrt{1-|\alpha(-i)|}}.$$

Consequently

$$\left|\frac{\phi_-^2(z,0)}{\phi_-^2(z,-n)}\right|^2\leqslant\frac{2^n}{\prod_{i=0}^{n-1}(1-|\alpha(-i)|)}.$$

Substituting the above inequality into the second term on the RHS of (5.10) yields (5.9). ∎

We now come to the main results of this section. These are analogs of the results of Kotani [15] and Simon [18] and draw upon their arguments.

THEOREM 5.6. *Suppose* (2.18) *and* (4.1) *hold, then for* μ*-almost every* w

$$\Sigma_{ac}(H_w)=\overline{\{e^{i\phi}:\gamma(e^{i\phi})=0\}},$$

up to sets $B\subset U$ *of Lebesgue measure zero.*

Proof. Since $\lim_{r\to 1}\int_{-\pi}^{\pi}(dk(\phi)/(z-e^{i\phi}))$, $z=re^{i\phi}$, $r<1$ exists for Lebesgue almost all e^{ϕ} [7] we find from Theorem 4.5 that $\partial\gamma/\partial r$ has radial limits for Lebesgue almost all $e^{i\phi}$. Suppose A is a Borel subset of U of positive Lebesgue measure and suppose $\gamma(e^{i\phi})=0$ for Lebesgue almost all $e^{i\phi}\in A$. Then for Lebesgue almost all $e^{i\phi}\in A$, $\lim_{r\to 1}(\gamma(re^{i\phi})/(1-r))=-\lim_{r\to 1}(\partial\gamma/\partial r)(re^{i\phi})$ is finite. Thus be Lemma 5.2

$$\limsup_{r\to 1}\mathbb{E}\left\{\frac{1}{1-|m_+^2(re^{i\phi},w)|^2}\right\}<\infty,$$

and

$$\limsup_{r\to 1}\mathbb{E}\left\{\frac{1}{1-r^2\,|m_-^2(re^{i\phi},w)|^2}\right\}<\infty.$$

Since $m_+^2(re^{i\phi},w)$ and $m_-^2(re^{i\phi},w)$ are in H_∞, $m_+^2(e^{i\phi},w)$ exists for every w and Lebesgue almost every $e^{i\phi}$ and the same is true for m_-^2. Hence for Lebesgue almost every $e^{i\phi}$, $m_+^2(e^{i\phi},w)$ and $m_-^2(e^{i\phi},w)$ exist for μ-almost every w. Therefore Fatou's Lemma implies that for Lebesgue almost every $e^{i\phi_0}\in A$

$$\mathbb{E}\left\{\frac{1}{1-|m_+^2(e^{i\phi_0},w)|^2}\right\}<\infty,$$

and

$$\mathbb{E}\left\{\frac{1}{1-|m_-^2(e^{i\phi_0},w)|^2}\right\}<\infty.$$

Consequently for μ-almost every w and Lebesgue almost every $e^{i\phi}\in A$, $1-|m_+^2(e^{i\phi},w)|^2>0$ and $1-|m_-^2(e^{i\phi},w)|^2>0$. If we set

$$F_w(z)=\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,dv(\theta,w), \tag{5.11}$$

then from (2.37) we find with $z=re^{i\phi}$,

$$\operatorname{Re}F_w(z)=\frac{1-|m_+^2(re^{i\phi},w)\,m_-^2(re^{i\phi},w)|^2}{|1-m_+^2(re^{i\phi},w)\,m_-^2(re^{i\phi},w)|^2}. \tag{5.12}$$

Since $dv_{ac}(\phi, w)/d\phi = \lim_{r\to 1} \operatorname{Re} F_w(re^{i\phi})$ Lebesgue almost everywhere [7] the above argument shows that for Lebesgue almost every $e^{i\phi} \in A$ and μ-almost every w,

$$\lim_{r\to 1} \operatorname{Re} F_w(re^{i\phi})$$

$$= \lim_{r\to 1} \frac{1 - |m_+^2(re^{i\phi}, w)\, m_-^2(re^{i\phi}, w)|^2}{|1 - m_+^2(re^{i\phi}, w)\, m_-^2(re^{i\phi}, w)|^2} > 0$$

which implies that $(dv_{ac}/d\phi)(\phi, w) > 0$ for μ-almost every w and Lebesgue almost every $e^{i\phi} \in A$.

To prove the converse suppose there exist a subset $A \subset U$ of positive Lebesgue measure such that for μ-almost every $w \in \Omega$ and Lebesgue almost every $e^{i\phi} \in A$, $dv_{ac}(\phi, w)/d\phi > 0$. Since $F_w(re^{i\phi})$ has a radial limit for all w and Lebesgue almost all $e^{i\phi}$, there exists a subset $B \subset A$ with $|B| = |A|$ such that $\lim_{r\to 1} \operatorname{Re} F_w(re^{i\phi})$ is positive and finite for every $e^{i\phi} \in B$ and μ-almost every w. Thus from (5.12) we see that for every $e^{i\phi} \in B$ and μ-almost every w, $m_+^2(e^{i\phi}, w)$ and $m_-^2(e^{i\phi}, w)$ cannot simultaneously equal one in magnitude. Consequently there must be a set E of positive μ measure where, say $m_+^2(e^{i\phi}, w)$ is not equal to one in magnitude. The ergodic theorem now tells us that there exists an N_0 such that for μ-almost every $w \in E^c$, $\sum_{i=1}^{N_0} \tau^i w \cap E \neq \emptyset$. This coupled with (5.8) shows that $|m_+^2(e^{i\phi}, w)| \neq 1$ on a set of μ-measure one. By Ergoroff's theorem there exists a set $E_1 \subset \Omega$ whose μ measure may be made arbitrarily close to one such that $\lim_{r\to 1} \log((1 - |m_+^2(re^{i\phi}, w)|^2)/(|r^2| - |m_+^2(re^{i\phi}, w)|^2)) = 0$ uniformly on E_1. Using the Ergodic Theorem as above with E^c replaced by E_1^c we find from (5.8) that for μ-almost every $w \in E_1^c$

$$\log \frac{1 - |m_+^2(z, w)|^2}{|z|^2 - |m_+^2(z, w)|^2}$$

$$\leqslant \log \frac{1 - |m_+^2(z, w_0)|^2}{|z|^2 - |m_+^2(z, w_0)|^2} - \log \prod_{i=0}^{N_0} (1 - |\alpha_w(i)|)$$

$$+ \log \left[\prod_{i=0}^{N_0} (1 - |\alpha_w(i)|) + \frac{(1 - |z|^2)(2/|z|^2)^{N_0}}{1 - |m_+^2(z, w_0)|^2} \right], \qquad (5.13)$$

where $w \in E_1^c$ and $w = \tau^{-i} w_0$, $w_0 \in E_1$ for some i, $1 \leqslant i \leqslant N_0$. Since the last term on the RHS of (5.13) is bounded above by

$$\log \left(1 + \frac{(1 - |z|^2)(2/|z|^2)^{N_0}}{1 - |m_+^2(z, w_0)|^2} \right);$$

which is integrable on E_1^c and $\log(1-|\alpha_w(1)|)\in L^1(\mu)$, we find for all $e^{i\phi}\in B$

$$\lim_{r\to 1}\gamma(re^{i\phi})=\lim_{r\to 1}\frac{1}{2}\int\log\frac{1-|m_+^2(re^{i\phi},w)|^2}{r^2-|m_+^2(re^{i\phi},w)|^2}\,d\mu(w)$$
$$=\lim_{r\to 1}\frac{1}{2}\int_{E_1}\log\frac{1-|m_+^2(re^{i\phi},w)|^2}{r^2-|m_+^2(re^{i\phi},w)|^2}\,d\mu(w)$$
$$+\lim_{r\to 1}\frac{1}{2}\int_{E_1^c}\log\frac{1-|m_+^2(re^{i\phi},w)|^2}{r^2-|m_+^2(re^{i\phi},w)|^2}\,d\mu(w).$$

That the right hand side of the above equation is equal to zero follows from the fact that the integrand in the first of the second equation converges uniformly to zero and (5.13). ∎

THEOREM 5.7. *Suppose* (2.18) *and* (4.1) *hold. If* $\gamma(e^{i\phi})=0$ *on an open arc* I *of* U *then for* μ*-almost every* w, v_w *is purely absolutely continuous on* I.

Proof. From Lemmas 5.3 and 5.4 and Fatou's lemma we find that for μ-almost every w and Lebesgue almost every $e^{i\phi}\in I$

$$m_+^2(e^{i\phi},w)=\overline{m_-^2(e^{i\phi},w)}. \tag{5.14}$$

Therefore (2.37) and (5.11) imply

$$\operatorname{Im}F_w(z)=\frac{\operatorname{Im}(m_+^2(z,w)\,m_-^2(z,w))}{|1-m_+^2(z,w)\,m_-^2(z,w)|^2},$$

and (5.14) says that for μ-almost every w and Lebesgue almost every $e^{i\phi}\in I$, $\lim_{r\to 1}\operatorname{Im}F_w(re^{i\phi})=0$. Since by (5.11) we see that $\operatorname{Re}F_w(z)>0$ for $|z|<1$ which implies that $\log F_w(z)\in H_2$. Consequently for μ-almost all w and Lebesgue almost all $e^{i\phi}\in I$, $\operatorname{Im}\log F_w(e^{i\phi})=0$ The Schwartz reflection principal which is valid for functions in H_2 [14, p. 87] tells us that $\log F_w(e^{i\phi})$ hence $F_w(e^{i\phi})$ is analytic on I for μ-almost all w and is non-zero for all $e^{i\phi}\in I$. This implies that for μ-almost all w, $\operatorname{Re}F_w(e^{i\phi})>0$ for all $e^{i\phi}\in I$ which gives the result. ∎

THEOREM 5.8. *Suppose* (2.18) *and* (4.1) *hold. If* $\gamma(e^{i\phi})=0$ *for* $e^{i\phi}\in A$ *where* A *has positive Lebesgue measure then for* μ*-almost every* w, $\alpha_w(n)$ *is a measurable function of* $\{\alpha_w(n)\}_{n\leqslant 1}$.

Proof. From $\{\alpha_w(n)\}_{n\leqslant 1}$ we can construct $\Phi(z,n,w)$ for $n\leqslant 1$ which exists by Theorem 2.6. This allows us to compute $m_-^1(z,w)$. From Lemma 5.4 we see that for μ-almost every w and Lebesgue almost every $e^{i\phi}\in A$, $m_+^2(e^{i\phi},w)=\overline{m_-^2(e^{i\phi},w)}$. But this completely determines m_+^2 since by Theorem 2.7 it is in H_∞. By the representation of $\Phi_+(z, 0, w)$ in terms of v_w (Theorem 2.6) we find that

$$\frac{1-m_+^2(z, w)}{1+m_+^2(z, w)} = \int_{-\pi}^{\pi} \frac{e^{i\phi}+z}{e^{i\phi}-z}\, dv_w(\theta)$$

which determines v_w. This in turn allows us to construct $\alpha_w(n)$, $n \geqslant 1$, which concludes the proof. ∎

COROLLARY 5.9. *If $\{\alpha_w(n)\}$ is an i.i.d. sequence of random variables such that* (4.1) *holds, then $\gamma(e^{i\phi})>0$ for almost every $e^{i\phi} \in U$. Therefore for μ-almost every w, $\Sigma_{ac}(w)=\phi$.*

VI. HALF-LINE OPERATOR

In Geronimo [8] various aspects of the case when τ is an endomorphism were considered. In particular the case when $\{\alpha_w(n)\}_{n=1}^{\infty}$ forms a stationary stochastic ergodic process was investigated. Many of these results can be recuperated from the results given above. Let $H_+ : l_+^2 \to l_+^2$ be given by

$$(H_+\psi)_k = \frac{\psi_{k+1}}{a(k+1)} - \alpha(k+1) \sum_{i=0}^{k} \overline{\alpha(i)} \prod_{j=i+1}^{k} \frac{\psi_i}{a(j)}$$
$$- \alpha(k+1) \prod_{j=1}^{k} \frac{\psi_0}{a(j)}, \qquad 0 \leqslant k. \tag{6.1}$$

Then we find from Teplyaev [21]

THEOREM 6.1. *H_+ is unitary if and only if $\sum_{i=1}^{\infty} |\alpha(i)|^2 = \infty$.*

The matrix elements $G_+(z, n, m)$ of the resolvent $G_+ = (zI - H_+)^{-1}$ satisfy the equations

$$zG_+(z, n, m) - \frac{1}{a(n+1)} G_+(z, n+1, m) + \alpha(n+1)\, G_+^1(z, n, m) = \delta_{n,m}, \tag{6.2}$$

where $G_+^1(z, n, m) = \sum_{i=1}^{n} \overline{\alpha(i)} \prod_{j=i+1}^{n} (G(z, i, m)/a(j)) - \prod_{j=1}^{n} (G(z, 0, m)/a(j))$. $G_+^1(z, n, m)$ satisfies the equation

$$G_+^1(z, n+1, m) = \overline{\alpha(n+1)}\, G_+(z, n+1, m) + \frac{1}{a(n+1)} G_1(z, n, m). \tag{6.3}$$

If we write

$$\hat{G}_+(z, n, m) = \begin{pmatrix} G_+(z, n, m) \\ G_+^1(z, n, m) \end{pmatrix}$$

it is not difficult to see, following the steps that led to (2.31), that

$$\hat{G}_+(z, n, m) = \begin{cases} \dfrac{\phi^*(z, m)\, \Phi_+(z, n)}{z^{m+1} W[\Phi_+(z, 0), \Phi(z, 0)]}, & n > m \\ \dfrac{\phi_+^2(z, m)\, \Phi_+(z, m)}{z^{m+1} W[\Phi_+(z, 0), \Phi(z, 0)]}, & n \leqslant m, \end{cases}$$

where $\Phi(z, n)$ is given by (1.5). Consequently

$$G(z, 0, 0) = \frac{\phi_+^2(z, 0)}{z[\phi_+^1(z, 0) - \phi_+^2(z, 0)]} = \frac{m_+^2(z, w)}{z(1 - m_+^2(z, w))}.$$

Since

$$\frac{1}{2} - zG(z, 0, 0) = \frac{1}{2} \frac{\phi_+^1(z, 0) + \phi_+^2(z, 0)}{\phi_+^1(z, 0) - \phi_+^2(z, 0)} = \frac{1}{2} \frac{1 + m_+^2(z, w)}{1 - m_+^2(z, w)}$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \frac{e^{i\phi} + z}{e^{i\phi} - z}\, d\sigma_w = F_w^\sigma(z), \tag{6.4}$$

from the representation for $\Phi_+(z, 0)$ (Theorem 2.6), we see that σ is the spectral measure associated with $G(z, 0, 0)$.

THEOREM 6.2. *Suppose (4.1) holds and H_+ is unitary for all w. Then for μ-almost all w, $\Sigma_{ac}(w) = \{e^{i\phi} : \gamma(e^{i\phi} = 0\}$ up to the set of Lebesgue measure zero. Furthermore if $\gamma(e^{i\phi}) = 0$ on an open subarc I of the unit circle then for μ-almost every w, σ_w is purely absolutely continuous on I. Finally if $\{\alpha_w(n)\}_{n \geqslant 1}$ is an i.i.d. sequence of random variables then for μ-almost every w, $\Sigma_{ac}(w) = \varnothing$.*

Proof. Since $\{\alpha_w(n)\}_{n=1}^{\infty}$ is a stationary stochastic process we may extend it to a bilateral stationary process by the Daniell–Kolmorgorov extension theorem [16, Theorem 4.8]. The theorem now follows from Theorems 5.6, 5.7, 5.8, and the fact that from (6.4)

$$\operatorname{Re} F_w^\sigma(z) = \frac{1}{2} \frac{1 - |m_+^2(z, w)|^2}{|1 - m_+^2(z, w)|^2}. \quad \blacksquare$$

ACKNOWLEDGMENT

The first author thanks William Green for many useful discussions on operator theory.

REFERENCES

1. F. V. ATKINSON, "Discrete and Continuous Boundary Value Problems," Academic Press, New York, 1964.
2. J. AVRON AND B. SIMON, Almost periodic Schrödinger operators. II. The integrated density of states, *Duke Math. J.* **50** (1983), 369–391.
3. J. BELLISSARD, Almost periodicity in solid state physics and C^* algebras, *in* "Proc. Harold Bohr Centenary" (C. Berg and B. Foglede, Eds.), Copenhagen, 1987.
4. R. CARMONA, "Random Schrödinger Operators," Lecture Notes in Math., Vol. 1180, Springer-Verlag, New York/Berlin, 1986.
5. W. CRAIG AND B. SIMON, Subharmonicity of the Lyapunov index, *Duke Math. J.* **50** (1983), 551–560.
6. H. L. CYCON, R. G. FROESE, W. KIRSCH, AND B. SIMON, Schrödinger operators with applications to quantum mechanics and global geometry, *in* "Texts and Monographs in Physics," Springer-Verlag, New York/Berlin, in press.
7. P. DURAN, "Theory of H_p Spaces," Academic Press, New York, 1970.
8. J. S. GERONIMO, Polynomials orthogonal on the unit circle with random recurrence coefficients, *in* "Proc. US–USSR Conference on Approximation Theory, St. Petersburg, Russia," Lecture Notes in Math., Vol. 1550, Springer-Verlag.
9. J. S. GERONIMO, E. M. HARRELL, AND W. VAN ASSCHE, On the asymptotic distribution of eigenvalues of banded matrices, *Constr. Approx.* **4** (1988), 403–417.
10. J. S. GERONIMO AND R. JOHNSON, Rotation numbers associated with difference equations satisfied by polynomials orthogonal on the unit circle, in preparation.
11. L. GERONIMUS, "Polynomials Orthogonal on a Circle and Interval," Pergamon, New York, 1960.
12. M. HERMAN, Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2, *Comment. Math. Helv.* **58** (1982), 453–502.
13. R. JOHNSON AND J. MOSER, The rotation number for almost periodic potentials, *Comm. Math. Phys.* **84** (1982), 403–438.
14. P. KOOSIS, "Introduction to H_p Spaces," Cambridge Univ. Press, London/New York, 1980.
15. S. KOTANI, Lyapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators, *in* "Stochastic Analysis" (K. Ito, Ed.), pp. 225–248, North-Holland, Amsterdam, 1984.
16. U. KRENGEL, "Ergodic Theorems," de Gruyter, New York, 1985.
17. D. RUELLE, Characteristic exponents and invariant manifolds in Hilbert space, *Ann. of Math.* **115** (1982), 243–290.
18. B. SIMON, Kotani theory for one dimensional stochastic Jacobi matrices, *Comm. Math. Phys.* **89** (1983), 227–234.
19. B. SIMON, Schrödinger semigroups, *Bull. Amer. Math. Soc.* (1982), 447–526.
20. A. V. TEPLYAEV, The pure point spectrum of random polynomials orthogonal on a circle, *Dokl. Akad. Nauk SSSR, Soviet Math. Dokl.* **44** (1992), 407–411.
21. A. V. TEPLYAEV, Orthogonal polynomials with random recurrence coefficients, *Trans. Sci. Sem. LOMI* **17** (1989), 157–162.
22. G. SZEGÖ, Orthogonal polynomials, *in* "Amer. Math. Soc. Colloq. Publ.," Vol. 23, Amer. Math. Soc., Providence, RI, 1978.