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CONTINUAL ANALOGUES OF RANDOM POLYNOMIALS WHICH ARE ORTHOGONAL ON A CIRCLE*

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Abstract. The paper obtains conditions providing absolute continuity almost surely for the spectral measure of the corresponding random differential operators for a class of canonical systems of ordinary differential equations with random coefficients. Estimates for the densities of the spectral measures are given. Corollaries corresponding to the deterministic case are formulated. Systems of stochastic differential equations with similar properties are considered.

Key words. random ordinary differential operator, spectral measure, absolutely continuous spectrum, Krein's canonical differential system, stochastic differential equation

1. Introduction. Before we formulate the main results of the paper we list some facts concerning similar problems related with polynomials which are orthogonal on the unit circle.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. We specify a system of polynomials $\{\Phi_n(z)\}_{n=0}^{\infty}$ by the recurrence relations

$$\begin{aligned} \Phi_0(z) &= \Phi_0^*(z) = 1, \\ \Phi_{n+1}(z) &= z\,\Phi_n(z) - \overline{a}_n \Phi_n^*(z), \\ \Phi_{n+1}^*(z) &= \Phi_n^*(z) - a_n z\,\Phi_n(z). \end{aligned} \tag{1}$$

The fulfillment of the conditions

$$|a_n| < 1, \qquad n = 0, 1, 2, \dots, \tag{2}$$

is equivalent to the existence of a unique (up to a scaling) Borel measure τ on the interval $[0, 2\pi]$ (not concentrated in a finite number of points) such that

$$\int_0^{\pi} \Phi_n(e^{i\theta})\,\Phi_m(e^{i\theta})\,d\tau(\theta) = 0 \qquad \text{if} \quad n \neq m, \quad n \geqq 0, \quad m \geqq 0.$$

The same numbers $\{a_n\}_{n=0}^{\infty}$, called in our setting circular parameters, are Shur's parameters of a certain interpolation problem (see [2, Chap. II]).

The result we formulate below has been obtained in articles by Kolmogorov, Krein, and Geronimus (see [4]).

THEOREM 1. *The following statements are equivalent*:

(3) I) *The series* $\sum_{n=0}^{\infty} |a_n|^2$ *is convergent*;

II) $\int_0^{2\pi} \log \tau'(\theta)\,d\theta > -\infty$, *where* τ' *is the density of the absolutely continuous component of the measure* τ;

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III) *the closure of the linear subspace spanned by the functions* $\{\Phi_n(e^{i\theta})\}_{n=0}^{\infty}$ *does not coincide with* L^2_τ;

IV) *the series* $\sum_{n=0}^{\infty} |\Phi_n(z)|^2$ *converges at least in one point of the domain* $|z| < 1$;

V) *there exists a subsequence* $\{|\Phi_{n_\nu}(z)|\}_{\nu=0}^{\infty}$ *bounded at least in one point of the domain* $|z| < 1$;

VI) *there exists a limit* $\lim_{n\to\infty} \Phi_n^*(z) = \pi(z)$ *that is uniform on compact subsets of the domain* $|z| < 1$.

In the case when the series $\sum_{n=0}^{\infty} |a_n|$ is convergent, the measure τ is absolutely continuous with respect to Lebesgue's measure and its density is uniformly bounded from zero and infinity. Observe also that, under condition (3), τ may have an arbitrary singular component.

However, Nikishin [9] has shown that, if the sequence of numbers $\{a_n\}_{n=0}^{\infty}$ satisfying conditions (2) and (3) only, is provided by independent symmetrically distributed signs, then τ is absolutely continuous with probability 1. Although the proof given in [9] has a gap the principal points of Nikishin's reasoning remain valid in a more general setting of arbitrary independent random variables.

THEOREM 2. *Let the circular parameters be a sequence of independent identically distributed random variables* $\{\xi_n\}_{n=0}^{\infty}$ *satisfying the conditions*

$$|\xi_n| < 1, \quad n = 0, 1, 2, \ldots, \qquad \sum_{n=0}^{\infty} \mathbf{E}\,|\xi_n|^2 < \infty, \qquad \sum_{n=0}^{\infty} |\,\mathbf{E}\,\xi_n| < \infty.$$

Then the following statements hold with probability 1:

(a) τ *is absolutely continuous with respect to Lebesgue's measure*;

(b) *for some positive* $\gamma > 0$,

$$\exp\left\{\gamma\,|\log\tau'(\cdot)|\,\log\left(|\log\tau'(\cdot)|\right)\right\} \in L^1_{[0,2\pi]};$$

(c) *if, in addition, there are positive numbers* $\{C_n\}_{n=0}^{\infty}$ *such that* $\sum_{n=0}^{\infty} C_n^2 < \infty$ *and* $|\xi_n| < C_n$, $n = 0, 1, 2, \ldots$, *then* $\exp\{\gamma \log^2 \tau'(\cdot)\} \in L^1_{[0,2\pi]}$ *with probability* 1 *for any* $\gamma > 0$.

This theorem was proved in [10]. Notice, that using statement (b) and (c) one can deduce some estimates related to the polynomials $\Phi_n(z)$ as well as to the Fourier-Chebyshev expansion on a circle (see [4]).

Observe that the conclusion of item (b) of Theorem 2 remains valid under milder conditions on $\{\xi_n\}_{n=0}^{\infty}$. Indeed, let $\{\xi_n\}_{n=0}^{\infty}$ be a sequence of independent symmetrically distributed random variables such that $|\xi_n| < 1$, $n = 0, 1, 2, \ldots$, and $\sum_{n=0}^{\infty} \mathbf{E}\,|\xi_n|^2 < \infty$. Consider the conditional distribution of the random variables, assuming the sequence $\{|\xi_n|\}_{n=0}^{\infty}$ of their absolute values to be fixed. It is easily seen that the conditional distribution satisfies almost surely the condition of item (c) and, therefore, with probability 1, $\exp\{\gamma \log^2 \tau'(\cdot)\} \in L^1_{[0,2\pi]}$ for any $\gamma > 0$. Thus, the following statement is valid.

COROLLARY. *Assume the random circular parameters* $\{\xi_n\}_{n=0}^{\infty}$ *satisfy the condition* $|\xi_n| < 1$, $n = 0, 1, 2, \ldots$, *and there exist three sequences* $\{\zeta_n\}_{n=0}^{\infty}$, $\{\eta_n\}_{n=0}^{\infty}$, *and* $\{\theta_n\}_{n=0}^{\infty}$ *such that* $\xi_n = \zeta_n + \eta_n + \theta_n$, *the triples of random variables* $\{(\zeta_n, \eta_n, \theta_n)\}_{n=0}^{\infty}$ *are independent, the random variables* $\{\zeta_n\}_{n=0}^{\infty}$ *are symmetrically distributed,*

$$\sum_{n=0}^{\infty} \mathbf{E}\,|\zeta_n|^2 < \infty, \qquad \sum_{n=0}^{\infty} \mathbf{E}\,|\eta_n| < \infty, \qquad \sum_{n=0}^{\infty} |\,\mathbf{E}\,\theta_n| < \infty,$$

and $|\theta_n| < C_n$, $n = 0, 1, \ldots$, *for a sequence of constants satisfying* $\sum_{n=0}^{\infty} C_n^2 < \infty$. *Then, for any* $\gamma > 0$, $\exp\{\gamma \log^2 \tau'(\cdot)\} \in L^1_{[0,2\pi]}$ *a. s.*

In the present paper we consider two different continual analogues of propositions on random orthogonal polynomials on a circle. The first of them is related to the system of differential equations

$$\begin{aligned} &\frac{d}{dt} p(t,\lambda) = i\lambda\, p(t,\lambda) - \overline{a(t)}\, p^*(t,\lambda), \\ &\frac{d}{dt} p_*(t,\lambda) = -a(t)\, p(t,\lambda), \qquad p(0,\lambda) = p_*(0,\lambda) = 1 \end{aligned} \tag{4}$$

(here $a(\cdot)$ is a locally summable function). The properties of solutions of the system have been studied by Krein [8] and they turned out to be similar to those for polynomials which are orthogonal on a circle. This system is a canonical (in the sense of Krein) system of differential equation.

We study the case where $a(\cdot)$ is a measurable stochastic process almost all trajectories of which are locally summable. Theorem 4 is an analogue of Theorem 2. An analogue of a sequence of independent random variables is a process with a finite dependency interval. However, the derivative of a continuous process with independent increments can serve (in a sense) as a more appropriate analogue of the mentioned sequence and in this case, solutions of the system of Itô's stochastic differential equations

$$\begin{aligned} &dp(t,\lambda) = i\lambda\, p(t,\lambda) + \tfrac{1}{2}\,\left|a(t)\right|^2 p(t,\lambda)\, dt - \overline{a(t)}\, p_*(t,\lambda)\, dw(t), \\ &dp_*(t,\lambda) = \tfrac{1}{2}\,\left|a(t)\right|^2 p_*(t,\lambda)\, dt - a(t)\, p(t,\lambda)\, dw(t), \\ &p(0,\lambda) = p_*(0,\lambda) = 1, \end{aligned} \tag{5}$$

are analogues of polynomials which are orthogonal on a circle. Here the summands $\frac{1}{2}\left|a(t)\right|^2 p(t,\lambda)\, dt$ and $\frac{1}{2}\left|a(t)\right|^2 p_*(t,\lambda)\, dt$ are needed for the spectrum of the system to be real a. s. Without these summands the properties of solutions of systems (4) and (5) would be not so similar.

Section 4 of the article deals with systems of a more general form. We give the definition and some properties of self-conjugate boundary problems for systems of stochastic differential equations that are analogous to the self-conjugate boundary problems for canonical systems of differential equations. Their spectrum turns out to be real with probability 1, an expansion in a series by eigenfunctions exists, and so on. Theorem 6 is an analogue of Theorem 2.

2. Canonical system of differential equations. We consider the system of differential equations (4), where $a(\cdot)$ is a measurable locally summable function. One can show that there exists a unique spectral measure σ on the real line such that for any finite function $f(\cdot) \in L^2_{[0,\infty)}$ the following equality is valid:

$$\int_0^{\infty} \left|f(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|F(\lambda)\right|^2 d\sigma(\lambda), \qquad \text{where} \quad F(\lambda) = \int_0^{\infty} f(t)\, p(t,\lambda)\, dt.$$

Thus, the correspondence $f(\cdot) \longmapsto F(\cdot)$ generates a unitary mapping $\mathcal{U}$ of $L^2_{[0,\infty)}$ onto a part of L^2_{σ}.

THEOREM 3 [8]. *The following statements are equivalent*:

1) *the image of the mapping* $\mathcal{U}$ *does not coincide with* L^2_{σ};

II) $\int_{-\infty}^{\infty} (1+\lambda^2)^{-1} \log \sigma'(\lambda)\, d\lambda > -\infty$;

III) $\int_0^\infty |p(t,\lambda)|^2 dt < \infty$ *for at least one* λ $(\operatorname{Im}\lambda > 0)$;

IV) $\liminf_{t\to\infty} |p_*(t,\lambda)| < \infty$ *for at least one* λ $(\operatorname{Im}\lambda > 0)$;

V) *there exist a sequence* $\{t_n\}_{n=0}^\infty$ *of positive numbers* $(t_n \to \infty$ *as* $n \to \infty)$ *and a function* $\pi(\lambda)$ *analytic in the domain* $\operatorname{Im}\lambda > 0$ *such that* $\lim_{n\to\infty} p_*(t_n,\lambda) = \pi(\lambda)$ *and* $\pi(\lambda)\overline{\pi(\mu)} = -i(\lambda - \overline{\mu}) \int_0^\infty p(t,\lambda)\overline{p(t,\mu)}\,dt$ $(\operatorname{Im}\lambda > 0,\ \operatorname{Im}\mu > 0)$.

A part of the proof of the theorem is given in the Appendix to the article. See also the remark to Theorem 5.

It should be noted that Krein stated erroneously that the existence of the limit $\lim_{t\to\infty} p_*(t,\lambda) = \pi(\lambda)$ is equivalent to each of the properties I–III. One can construct examples in which $\liminf_{t\to\infty} |p_*(t,\lambda)| \neq \limsup_{t\to\infty} |p_*(t,\lambda)|$ (the second limit may be equal to infinity). However, if $a(\cdot) \in L^2_{[0,\infty)}$ or $a(\cdot) \in L^1_{[0,\infty)}$, the mentioned limit exists. Moreover, in the second case the measure σ is absolutely continuous and its density is bounded away from zero and infinity by the constants $(2\pi)^{-1}\exp\{\pm 2\int_0^\infty |a(t)|\,dt\}$.

The starting points of Krein's article [8] are a measure σ and the Hermitian kernel $H(t-s)$ generated by the measure; the functions $p(\cdot,\cdot)$, $p_*(\cdot,\cdot)$, and $a(\cdot)$, satisfying conditions (4) are specified by the resolvent of the kernel (see [3] and [6]).

Let now $a(\cdot)$ be a measurable complex-valued stochastic process whose trajectories are summable a. s. and let $0 = t_0 < t_1 < t_2 < \cdots$ be a partition of the ray $[0,\infty)$ into finite intervals. Put $\xi_n = \int_{t_n}^{t_{n+1}} a(t)\,dt$, $\zeta_n = \int_{t_n}^{t_{n+1}} |a(t)|dt$ and assume that the random variables ξ_m and ζ_m are independent of all the variables ξ_m and ζ_m for $n < m$.

THEOREM 4. *Let*

$$\sum_{n=0}^\infty \mathbf{E}\,\zeta_n^2 < \infty, \qquad \sum_{n=0}^\infty (t_{n+1} - t_n)\,\mathbf{E}\,\zeta_n < \infty, \qquad \text{and} \qquad \sum_{n=0}^\infty |\mathbf{E}\,\xi_n| < \infty.$$

Then, with probability 1:

(a) *the measure* σ *is absolutely continuous with respect to the Lebesgue measure and statements* I–V *of Theorem* 3 *are valid*;

(b) *for some* $\gamma > 0$,

$$\int_{-\infty}^\infty \exp\left\{\gamma \frac{|\log\sigma'(\lambda)|}{1+|\lambda|}\log\frac{|\log\sigma'(\lambda)|}{1+|\lambda|}\right\}\frac{d\lambda}{1+\lambda^2} < \infty;$$

(c) *if, in addition, there are numbers* $\{C_n\}_{n=0}^\infty$ *such that* $\zeta_n < C_n$, $n = 0,1,2,\ldots$, $\sum_{n=0}^\infty C_n^2 < \infty$, *and* $\sum_{n=0}^\infty (t_{n+1} - t_n)C_n < \infty$, *then, with probability* 1,

$$\int_{-\infty}^\infty \exp\{\gamma\log^2\sigma'(\lambda) - \lambda^2\}\,d\lambda < \infty$$

for each $\gamma > 0$.

Remark. A corollary of the theorem is valid which is similar to that of Theorem 2.

We give two examples of random functions $a(\cdot)$.

1) Let $f(\cdot)$ be a measurable complex-valued stochastic process such that $\mathbf{E}\,f(t) = 0$ for $t \geqq 0$ and $\sup_{0\leqq t<\infty} \mathbf{E}\,|f(t)|^2 < \infty$. Assume that the values of the process in the interval $[n, n+1)$ do not depend on the values of the process outside the interval $(n = 0,1,2,\ldots)$. If one takes $a(t) = f(t^\alpha)$ with $\alpha > 2$, then the stochastic process $a(\cdot)$ satisfies the conditions of Theorem 4. If, in addition, $f(\cdot)$ is the sum of a symmetric process and a uniformly bounded one, then item (c) of the theorem holds.

2) Let $g(\cdot)$ be an arbitrary complex-valued locally summable function specified on the ray $[0,\infty)$ and let $\{\eta_n\}_{n=1}^\infty$ be a sequence of independent random variables with

$|\mathbf{E}\,\eta_n| \leqq n^{-1+\varepsilon}$ $(0 \leqq \varepsilon < \frac{1}{2}, n = 1, 2, \ldots)$, and $\sup_{n\geqq 1} \mathbf{E}\,|\eta_n|^2 < \infty$. One can always find numbers $t_1, t_2, \ldots$ such that $0 = t_1 < t_2 < \cdots$, $t_n \to \infty$ as $n \to \infty$,

$$t_{n+1} - t_n \leqq n^{-1+\varepsilon} \quad \text{and} \quad \int_{t_n}^{t_{n+1}} |g(t)|\,dt \leqq n^{-1+\varepsilon} \quad (n = 1, 2, \ldots).$$

The stochastic process $a(\cdot) = \sum_{n=1}^{\infty} \eta_n g(\cdot) \mathbf{1}_{[t_n, t_{n+1})}(\cdot)$ satisfies the conditions of Theorem 4 and item (c) holds if there exists a sequence of independent uniformly bounded random variables $\{\theta_n\}_{n=1}^{\infty}$ such that the random variables $\eta_n - \theta_n$ are independent and symmetrically distributed.

3. Proof of Theorem 4. Assume for a moment that $a(\cdot)$ is a finite function. Then the measure σ is absolutely continuous with respect to Lebesgue's measure and one can find its density explicitly. Indeed, let $a(t) = 0$ for $t \geqq r$ and some $r \geqq 0$. Then, $p_*(t, \lambda) = p_*(r, \lambda)$ and $p(t, \lambda) = e^{i(t-r)\lambda} p(r, \lambda)$ for $t \geqq r$. It is easily seen that if $f(\cdot)$ is a finite function from $L^2_{[0,\infty)}$ and $f(t) = 0$ for $0 \leqq t < r$, then $(\mathcal{U}f)(\lambda) = p(r, \lambda) \int_0^{\infty} f(t - r) e^{it\lambda} dt$. Therefore, in this case,

$$d\sigma(\lambda) = \left(2\pi |p(r, \lambda)|^2\right)^{-1} d\lambda.$$

Let now $a(\cdot)$ be an arbitrary locally summable function and let σ be the spectral measure of the system of differential equations (4) corresponding to it. Denote by σ_r the spectral measure of the system of equations (4), where the function $a(\cdot)\mathbf{1}_{[0,r)} a(\cdot)$ is substituted for $a(\cdot)$. Clearly, the density of σ_r is equal to $(2\pi |p(r, \lambda)|^2)^{-1}$. Our aim is to show that $(1 + \lambda^2)^{-1} \sigma_r \longrightarrow (1 + \lambda^2)^{-1} \sigma$ as $r \to \infty$ in the sense of weak convergence of measures (the multiplier $(1 + \lambda^2)^{-1}$ is needed to provide finiteness of the corresponding measures).

Remark 1. The following statement follows from the investigation of nested circles given in [1, Chap. 9]. Fix a function $b(\cdot) \in L^1_{[0,r_0]}$ $(r_0 > 0)$. There exists a constant $M < \infty$, dependent on $b(\cdot)$, such that if $a(\cdot)$ is a locally summable function coinciding with $b(\cdot)$ in the interval $[0, r_0]$, then $\int_{-\infty}^{\infty} (1 + \lambda^2)^{-1} d\sigma(\lambda) < M$.

Remark 2. For each finite function $f(\cdot) \in L^2_{[0,\infty)}$ there exists a finite function $g(\cdot) \in L^2_{[0,\infty)}$ such that

$$\int_0^{\infty} f(t)\, p(t, \lambda)\, dt = \int_0^{\infty} g(t)\, e^{it\lambda} dt, \qquad \lambda \in \mathbf{C}. \tag{6}$$

Vice versa, for each finite function $g(\cdot) \in L^2_{[0,\infty)}$ there exists a finite function $f(\cdot) \in L^2_{[0,\infty)}$ satisfying (6). This statement is a corollary of the representation $p(t, \lambda) = e^{it\lambda}(1 - \int_0^t G(t, s) e^{-i\lambda s} ds)$, where $G(\cdot, \cdot)$ is a function measurable with respect to the pair of variables on the set $\{(t, s) \mid 0 \leqq s \leqq t < \infty\}$ and

$$\sup_{0 \leqq t \leqq r} \int_0^t |G(t, s)|\, ds < \infty \quad \text{and} \quad \sup_{0 \leqq s \leqq r} \int_s^r |G(t, t - s)|\, ds < \infty,$$

r any $r > 0$ (see [8]). One can easily obtain such a representation using the method successive approximations.

According to Remark 1, for each $r_0 > 0$ there exists $M < \infty$ such that $\int_{-\infty}^{\infty} (1 + \lambda^2)^{-1} d\sigma_r(\lambda) < M$ and $\int_{-\infty}^{\infty} (1 + \lambda^2)^{-1} d\sigma(\lambda) < M$. Thus, one can apply Helly's principle to the family of measures $(1+\lambda^2)^{-1}\sigma_r$ $(r \geqq r_0)$. The weak convergence $(1+\lambda^2)^{-1}\sigma_r \longrightarrow (1+\lambda^2)^{-1}\sigma$ as $r \to \infty$ follows from the fact that if $f(\cdot) \in L^2_{[0,\infty)}$ is an arbitrary function vanishing on the ray $[r,\infty)$, then

$$\int_{-\infty}^{\infty} |F(\lambda)|^2 d\sigma_r(\lambda) = \int_0^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(\lambda)|^2 d\sigma(\lambda),$$

where $F(\lambda) = \int_0^\infty p(t,\lambda) f(t)dt$. Indeed, the coincidence of the integrals of all the functions $|F(\cdot)|^2$ of such a kind with respect to both measures implies the coincidence of the measures themselves.

From (4) it follows, that

$$p_*(t,\lambda) = \exp\left\{ -\int_0^t \frac{a(s)p(s,\lambda)}{p_*(s,\lambda)}\, ds \right\},$$
$$\frac{d}{dt}\frac{p(t,\lambda)}{p_*(t,\lambda)} = \frac{i\lambda p(t,\lambda)p_*(t,\lambda) - \overline{a(t)}p(t,\lambda)p_*(t,\lambda) + a(t)p^2(t,\lambda)}{p_*^2(t,\lambda)}.$$

Since $|p_*(t,\lambda)|^2 - |p(t,\lambda)|^2 = 2\operatorname{Im}\lambda \int_0^t |p(s,\lambda)|^2 ds$ according to (4), we have

$$|p(t,\lambda)| = |p_*(t,\lambda)| \quad \text{and} \quad \left| \frac{d}{dt}\frac{p(t,\lambda)}{p_*(t,\lambda)} \right| \leqq |\lambda| + 2|a(t)|,$$

for real λ. Therefore, for any real x and λ,

$$\begin{aligned} |p_*(t_{n+1},\lambda)|^x &= |p_*(t_n,\lambda)|^x \exp\left\{ -x\operatorname{Re}\int_{t_n}^{t_{n+1}} a(s)\,\frac{p(s,\lambda)}{p_*(s,\lambda)}\,ds \right\} \\ &\leqq |p_*(t_n,\lambda)|^x \exp\left\{ -x\operatorname{Re}\left(\frac{p(t_n,\lambda)}{p_*(t_n,\lambda)} \int_{t_n}^{t_{n+1}} a(s)\,ds \right) \right. \\ &\qquad\qquad \left. + |x| \int_{t_n}^{t_{n+1}} |a(t)| \left(\int_{t_n}^{t} \left(|\lambda| + 2|a(s)|\right) ds \right) dt \right\} \\ &\leqq \prod_{k=0}^{n} \exp\left\{ -x\operatorname{Re}\left((\xi_k - \mathbf{E}\,\xi_k)\,\frac{p(t_k,\lambda)}{p_*(t_k,\lambda)} \right) \right. \\ &\qquad\qquad \left. + |x\,\mathbf{E}\,\xi_k| + |x|\,\zeta_k^2 + |\lambda x|\,(t_{k+1}-t_k)\zeta_k \right\}. \end{aligned} \tag{7}$$

Consider the following stochastic process:

$$a_N(\cdot) = \begin{cases} a(t), & \text{if } \ t_n \leqq t < t_{n+1}, \ \ \zeta_n < N \ \text{ and } \ (t_{n+1}-t_n)\zeta_n < N, \\ 0, & \text{if } \ t_n \leqq t < t_{n+1}, \ \text{ and} \\ & \text{either } \ \zeta_n \geqq N \ \text{ or } \ (t_{n+1}-t_n)\zeta_n \geqq N. \end{cases}$$

Clearly, $\mathbf{P}\{a_N(t) = a(t),\ t \in [0,\infty)\} \longrightarrow 1$ as $N \to \infty$, since $\sum_{n=0}^\infty \zeta_n^2 < \infty$ and $\sum_{n=0}^\infty (t_{n+1}-t_n)\zeta_n < \infty$ with probability 1. Thus, without restricting generality, one can assume that $\zeta_n < N$ and $(t_{n+1}-t_n)\zeta_n < N$, $n = 0,1,2,\ldots$, for some $N > 1$.

Let $\mathcal{F}_n$ be the σ-algebra of events generated by the random variables ζ_k and ξ_k for $0 \leqq k \leqq n$. Using an expansion in a Taylor series and the fact that $p(t_n,\lambda)$ and $p_*(t_n,\lambda)$ are measurable with respect to $\mathcal{F}_{n-1}$, while ζ_n and ξ_n are independent of

$\mathcal{F}_{n-1}$ $(n = 1, 2, \ldots)$, one can obtain the following estimate:

$$\mathbf{E}\left\{\exp\left(-x\mathrm{Re}\left((\xi_n - \mathbf{E}\,\xi_n)\,\frac{p(t_n,\lambda)}{p_*(t_n,\lambda)}\right)\right.\right.$$
$$\left.\left.+\,|x\,\mathbf{E}\,\xi_n| + |x|\,\zeta_n^2|\lambda x|\,(t_{n+1}-t_n)\,\zeta_n\right)\Big|\,\mathcal{F}_{n-1}\right\}$$
$$\leqq 1 + \left(\mathbf{E}\,\zeta_n^2 + (t_{n+1}-t_n)\,\mathbf{E}\,\zeta_n|\,\mathbf{E}\,\xi_n|\right)\,\exp\left\{\left(1+|\lambda|\right)|x|\,C\right\}, \tag{8}$$

where $C > 0$ is a nonrandom constant. Observe that the right-hand side of (7) is a submartingale adapted to the flow of σ-algebras $\{\mathcal{F}_n\}_{n\geqq 0}$. Therefore, the function $G(\cdot) = \mathbf{E}\,\sup_{t\geqq 0}|p_*(t,\cdot)|^{-2}$ is bounded on any finite interval. Hence it follows that, with probability 1, the measure σ is absolutely continuous and $\sigma'(\lambda) \leqq \sup_{t\geqq 0}(2\pi|p_*(t,\lambda)|^2)^{-1}$ because almost surely there exists a locally summable majorant for the densities of the measures σ_r, $r \geqq 0$. Inequality (8) yields

$$\mathbf{E}\,|\sigma'(\lambda)|^x \leqq \exp\left\{D\,\exp\left\{C\left(1+|\lambda|\right)|x|\right\}\right\}, \qquad x \in \mathbf{R},$$

for some constant D. This estimate suffices to conclude that

$$\sup_{-\infty<\lambda<\infty}\ \mathbf{E}\ \exp\left\{\gamma\,\frac{|\log\sigma'(\lambda)|}{1+|\lambda|}\,\log\frac{|\log\sigma'(\lambda)|}{1+|\lambda|}\right\} < \infty.$$

Hence item (b) of Theorem 4 follows. If, in addition, there exists a sequence $\{C_n\}_{n=0}^{\infty}$ satisfying the conditions of item (c), then the left-hand side of (8) does not exceed $\exp\{KC_n(x^2+|x|(2+|\lambda|))\}$ for some $K > 0$. Therefore, one can find positive constants α, β, and γ such that $\mathbf{E}\,|\sigma'(\lambda)|^x \leqq \exp\{\alpha x^2 + \beta|x| + \gamma|\lambda x|\}$ and, by increasing β, the parameter α can be made arbitrarily small. This implies the statement of point (b) of Theorem 4.

4. Self-conjugate boundary problems for stochastic differential equations. We consider the following systems of stochastic differential equations (in the Itô sense):

$$J\,dy(t) = \lambda\,A(t)\,y(t)\,dt - \tfrac{1}{2}\,B(t)\,J\,B(t)\,y(t)\,dt + B(t)\,y(t)\,dw(t), \tag{9}$$

where $w(\cdot)$ is a one-dimensional Wiener process adapted to the flow of σ-algebras $\{\mathcal{F}_t\}_{t\geqq 0}$, $y(\cdot)$ is an n-dimensional vector-function, $A(\cdot)$ and $B(\cdot)$ are measurable self-conjugate $n \times n$ matrix-functions adapted to $\{\mathcal{F}_t\}_{t\geqq 0}$ and uniformly bounded on any finite interval with probability 1, J is a constant $n \times n$ skew-Hermitian matrix and so, $J^* = J^{-1} = -J$, $A^*(\cdot) = A(\cdot)$ and $B^*(\cdot) = B(\cdot)$. The existence and uniqueness of a solution of (9) is established in [5].

By Itô's formula it is easy to calculate the differential

$$d\left[y^*(t,\mu)\,J\,y(t,\lambda)\right] = (\lambda - \overline{\mu})\,y^*(t,\mu)\,A(t)\,y(t,\lambda)\,dt.$$

The summand $\frac{1}{2}\,B(t)\,J\,B(t)\,y(t)\,dt$ in (9) is needed for the validity of the previous relation which is a basic property of solutions of ordinary (nonstochastic) differential equations and provides the self-conjugacy of the corresponding boundary problems (see [1, Chap. 9]).

Below we describe explicitly the solution of (9) with the initial condition $y(0) = u$ which is, with probability 1, continuous in t and λ and analytic in λ for any fixed $t \geqq 0$.

Let $V(\cdot)$ be a continuous solution of the matrix equation

$$J\,dV(t) = -\tfrac{1}{2}\,B(t)\,J\,B(t)\,V(t)\,dt + B(t)\,V(t)\,dw(t), \qquad V(0) = E.$$

It is easily seen that, according to Itô's formula,

$$d\big[V^*(t)\,J\,V(t)\big] = 0. \tag{10}$$

As for relation (9), the summand $-\tfrac{1}{2}\,B(t)\,J\,B(t)\,V(t)\,dt$ plays an important role for the previous equality to hold and gives a possibility to make the following substitution.

Denote by $\widetilde{Y}(\cdot,\lambda)$ the absolutely continuous solution of the ordinary differential equation (with a random Hermitian)

$$J\,\frac{d}{dt}\,\widetilde{Y}(t,\lambda) = \lambda\,V^*(t)\,A(t)\,V(t)\widetilde{Y}(t,\lambda), \qquad \widetilde{Y}(0,\lambda) = E. \tag{11}$$

It follows from (9)–(11) that the function $Y(\cdot,\lambda) = V(\cdot)\widetilde{Y}(\cdot,\lambda)$ is the principal matrix solution of equation (9), i. e.,

$$J\,dY(t,\lambda) = \lambda\,A(t)\,Y(t,\lambda)\,dt - \tfrac{1}{2}\,B(t)\,J\,B(t)\,Y(t,\lambda)\,dt + B(t)\,Y(t,\lambda)\,dw(t),$$

$$Y(0,\lambda) = E,$$

and with probability 1 each element of the matrix is a continuous function in the totality of variables t and λ and is an analytic function of exponential type in λ for any fixed $t \geqq 0$.

The function $y(\cdot) = Y(\cdot,\lambda)u$ is thought of as the solution of equation (9) with the initial condition $y(0) = u$.

Let $w(\cdot)$, $A(\cdot)$, $B(\cdot)$, and $V(\cdot)$ be defined on a certain probability space $(\Omega, \mathcal{F}, \mathbf{P})$. There exists a set of full measure $\Omega' \subset \Omega$ such that, for any $\omega \in \Omega'$,

(a) trajectories of the process $V(\cdot)$ are continuous;

(b) $V^*(t)JV(t) = J$ for any $t \geqq 0$.

It is easy to check that, for any $\omega \in \Omega'$,

$$Y^*(t,\mu)\,J\,Y(t,\lambda) - J = (\lambda - \overline{\mu}) \int_0^t Y^*(s,\mu)\,A(s)\,Y(s,\lambda)\,ds. \tag{12}$$

We specify the boundary conditions in the interval $[0,r]$ by square matrices M and N such that $M^*JM = N^*JN$ and for which the equalities $Nv = Mv = 0$ imply $v = 0$. The boundary problem in question is posed for each fixed $\omega \in \Omega'$ and consists in finding solutions of (9) such that $y(0) = Mv$ and $y(r) = Nv$ for some $v \neq 0$. According to this formulation the function $\widetilde{y}(t) = V^{-1}(t)y(t)$ is an absolutely continuous solution of the equation

$$J\,\frac{d}{dt}\,\widetilde{y}(t) = \lambda\,V^*(t)\,A(t)\,V(t)\widetilde{y}(t), \tag{13}$$

satisfying the boundary conditions $\widetilde{y}(0) = Mv$ and $\widetilde{y}(r) = \widetilde{N}v$ with $\widetilde{N} = V^{-1}(r)N$.

For this reason, for any $\omega \in \Omega'$, the eigenvalues of the boundary problem for equation (9) and the matrices M and N coincide with those of the boundary problem for equation (13) and the matrices M and $\widetilde{N}$ and the corresponding eigenfunctions are transformed into each other by the transformation $\widetilde{y}(t) = V^{-1}(t)y(t)$. The spectral functions of the problems also coincide.

In what follows we assume that the matrix $\int_0^r V^*(t)A(t)V(t)\,dt$ is positively defined for each $\omega \in \Omega'$. We formulate a number of statements valid for any $\omega \in \Omega'$.

(a) All the eigenvalues of our boundary problem are real and solve the equation $\det(N - Y(r,\lambda)M) = 0$, the multiplicity of an eigenvalue λ is equal to the number of linearly independent solutions of the equation $(N - Y(r,\lambda)M)v = 0$. Thus, one can renumber all the eigenvalues (taking into account their multiplicities) in such a way that $|\lambda_0| \leqq |\lambda_1| \leqq |\lambda_2| \leqq \dots$.

(b) The eigenfunctions of the problem in question can be scaled to satisfy

$$\int_0^r y_n^*(t)A(t)y_m(t)\,dt = \delta_{mn}.$$

(c) There exists a nondecreasing spectral matrix function $T(\cdot)$ (which is the weak limit of the spectral functions for the corresponding boundary problems on the intervals $[0,r]$ as $r \to \infty$).

Let $\varphi(\cdot)$ be a continuous finite vector function and let there exist a measurable function $\chi(\cdot)$ such that

$$\int_0^\infty \chi^*(t)\,A(t)\,\chi(t)\,dt < \infty, \qquad J\frac{d}{dt}\widetilde{\varphi}(t) = V^*(t)\,A(t)\,V(t)\,\chi(t), \qquad \widetilde{\varphi}(t) = V^{-1}(t)\,\varphi(t). \tag{14}$$

Then the following "generalized Parseval equality" is valid:

$$\int_0^\infty \varphi^*(t)\,A(t)\,\varphi(t)\,dt = \int_{-\infty}^\infty \psi^*(\lambda)\,dT(\lambda)\,\psi(\lambda), \tag{15}$$

where

$$\psi(\lambda) = \int_0^\infty Y^*(t,\lambda)\,A(t)\,\varphi(t)\,dt.$$

These statements are simple corollaries of the results proved in [1, Chap. 9].

5. Solutions of stochastic differential equations that are analogous to polynomials which are orthogonal on a circle. Equation (5) has the form (9) for $n = 2$ and the following values of the matrix coefficients:

$$A(\cdot) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad B(\cdot) = \begin{pmatrix} 0 & i\overline{a(\cdot)} \\ -ia(\cdot) & 0 \end{pmatrix},$$

where $a(\cdot)$ is a measurable locally bounded stochastic process adapted to $\{\mathcal{F}_t\}_{t\geqq 0}$. Let the boundary conditions for the problem be given by the matrices

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Assume also that all the elements of the matrix $T(\cdot)$ are equal to one and the same nondecreasing function $\sigma(\cdot)$. It easily seen that the matrix $V(\cdot)$ has the form

$$V(\cdot) = \begin{pmatrix} V_1(\cdot) & \overline{V_2(\cdot)} \\ V_2(\cdot) & \overline{V_1(\cdot)} \end{pmatrix}, \qquad |V_1(t)|^2 - |V_2(t)|^2 = 1, \tag{16}$$

for any $\omega \in \Omega'$ and $t \geqq 0$ (this fact follows from (12)).

Relations (14) and (15) allow us to state that, if

$$\varphi(\cdot) = \begin{pmatrix} \varphi_1(\cdot) \\ \varphi_2(\cdot) \end{pmatrix}, \qquad \chi(\cdot) = \begin{pmatrix} \chi_1(\cdot) \\ \chi_2(\cdot) \end{pmatrix},$$

χ_1 being a finite function from $L^2_{[0,\infty)}$,

$$\int_0^\infty \overline{V_1(s)}\,\chi_1(s)\,ds = \int_0^\infty V_2(s)\,\chi_1(s)\,ds = 0, \tag{17}$$

$$\varphi_1(t) = i\,V_1(t)\int_0^t V_1(s)\,\chi_1(s)\,ds - i\,\overline{V_2(t)}\int_0^t V_2(s)\,\chi_1(s)\,ds, \tag{18}$$

then $\int_0^\infty |\varphi_1(t)|^2 dt = \int_0^\infty |F(\lambda)|^2 d\sigma(\lambda)$, where $F(\lambda) = \int_0^\infty p(t,\lambda)\varphi_1(t)dt$. Relations (17) are needed to guarantee finiteness of $\varphi_1(t)$ while the subsequent equalities are simply the element-wise representation of the matrix relations (14) and (15) for the given particular case.

LEMMA. *The mapping $\varphi_1(\cdot) \longmapsto F(\cdot)$ can be continued with probability* 1 *to an unitary mapping $\mathcal{U}$ that maps $L^2_{[0,\infty)}$ onto some part L^2_σ.*

Proof. It suffices to show that the closure of the linear span of all the functions $\varphi_1(\cdot)$ specified by (17) and (18) coincides almost surely with $L^2_{[0,\infty)}$. Let $f(\cdot)$ be a random function such that $f(\cdot) \in L^2_{[0,\infty)}$ and $\int_0^\infty f(t)\varphi_1(t)dt = 0$ for any $\omega \in \Omega'$, where $\varphi_1(\cdot)$ runs through the set of all functions defined by (17) and (18). Combining the previous equality with (17) and (18) one can deduce by integrating by parts that

$$\overline{V_1(s)}\int_0^t f(s)V_1(s)\,ds - V_2(t)\int_0^t f(s)\overline{V_2(s)}\,ds = \alpha\overline{V_1(t)} + \beta\,V_2(t), \tag{19}$$

for all $t \geqq 0$ and $\omega \in \Omega'$ (here α and β are random constants). We claim that (19) implies, with probability 1, the equality $f(t) = 0$ almost surely with respect to Lebesgue's measure. Indeed, (19) is equivalent to the relation

$$\int_0^\infty f(s)\,V_1(s)\,ds - \alpha = \frac{V_2(t)}{\overline{V_1(t)}}\left(\int_0^t f(s)\,\overline{V_2(s)}\,ds + \beta\right).$$

According to (5) and (16) we also have

$$\frac{V_2(t)}{\overline{V_1(t)}} = 1 - \int_0^t \frac{V_1(s)V_2(s)a^2(s)}{(\overline{V_1(s)})^2}\,ds - \int_0^t \frac{a(s)}{(\overline{V_1(s)})^2}\,dw(s).$$

It follows from this representation that, with probability 1, the variation of the function $V_2(\cdot)/\overline{V_1(\cdot)}$ in any interval is equal either to zero or to infinity (i. e., is unbounded). Therefore, the ratio $V_2(\cdot)/\overline{V_1(\cdot)}$ must be constant on the open set of points t for which $\int_0^t f(s)V_1(s)\,ds \neq \alpha$ and for almost all t from the set $f(t)V_1(t) = |V_2(t)|^2 f(t)/\overline{V_1(t)}$; this in combination with (16) implies $f(t) = 0$.

THEOREM 5. *For solutions of the system of stochastic equations* (5), *statements* III–V *of Theorem* 3 *are equivalent with probability* 1.

In addition we give a proof of the equivalency of these statements which can be applied to solutions of system (4) as well as to solutions of system (5) for each $\omega \in \Omega'$ (in the last case the formula

$$p_*(t,\lambda)\,\overline{p_*(t,\mu)} - p(t,\lambda)\,\overline{p(t,\mu)} = -i(\lambda-\overline{\mu}) \int_0^t p(s,\lambda)\,\overline{p(s,\mu)}\,ds \tag{20}$$

follows from (12)).

Remark. The equivalence of statements I and II of Theorem 3 follows from Remark 2 (part 3) since, as it is proved in essence in [7], the closure in L^2_σ of Fourier transforms of all finite functions from $L^2_{[0,\infty)}$ does not coincide with L^2_σ if and only if $\int_{-\infty}^{\infty}(1+\lambda^2)^{-1}\log\sigma'(\lambda)d\lambda > -\infty$, where $\sigma'(\cdot)$ is the density of the absolutely continuous component of σ (it is assumed that $\int_{-\infty}^{\infty}(1+\lambda^2)^{-1}d\sigma(\lambda) < \infty$). In the case of stochastic differential equations, the structure of the images of finite functions from $L^2_{[0,\infty)}$ under the transformation $\mathcal{U}$ has not been fully investigated.

THEOREM 6. *If* $\int_0^\infty |a(t)|^2dt < \infty$ *a. s., then with probability* 1:

(a) σ *is absolutely continuous with respect to Lebesgue's measure*;

(b) $\int_{-\infty}^{\infty}\exp\{\gamma\log^2\sigma'(\lambda)\}(1+\lambda^2)^{-1}d\lambda < \infty$ *for some* $\gamma > 0$;

(c) *the image of the mapping* $\mathcal{U}$ *does not coincide with* L^2_σ.

Proof. Applying arguments similar to those used in proving Theorem 4 one can show that if $a(t) = 0$ for $t > r$, then $d\sigma(\lambda) = (2\pi|p_*(r,\lambda)|^2)^{-1}d\lambda$ with probability 1. It follows from (5) that

$$p_*(t,\lambda) = \exp\left\{\frac{1}{2}\int_0^t \left(|a(s)|^2 - \frac{a^2(s)p^2(s,\lambda)}{p_*^2(s,\lambda)}\right)ds - \int_0^t \frac{a(s)p(s,\lambda)}{p_*(s,\lambda)}\,dw(s)\right\} \quad \text{a. s.}$$

Thus, for almost all λ with respect to Lebesgue's measure the limit $\lim_{t\to\infty} p_*(t,\lambda) = \pi(\lambda)$ exists with probability 1. Recall that relation (20) holds for each $\omega \in \Omega'$. Thus, for $\lambda \in \mathbf{R}$, the equality $|p_*(t,\lambda)| = |p(t,\lambda)|$ is valid with probability 1. For any $x \in \mathbf{R}$, $t \geqq 0$, and $\lambda \in \mathbf{R}$, we have

$$\begin{aligned} |p_*(t,\lambda)|^x &\leqq \exp\left\{\left(|x| + \frac{x^2}{2}\right)\int_0^t |a(s)|^2 ds\right\} \\ &\times \exp\left\{-x\int_0^t \operatorname{Re}\frac{a(s)p(s,\lambda)}{p_*(s,\lambda)}\,dw(s) - \frac{x^2}{2}\int_0^t \left(\operatorname{Re}\frac{a(s)p(s,\lambda)}{p_*(s,\lambda)}\right)^2 ds\right\} \quad \text{a. s.} \end{aligned} \tag{21}$$

We added

$$\frac{x^2}{2}\int_0^t \left(|a(s)|^2 - \left(\operatorname{Re}\frac{a(s)p(s,\lambda)}{p_*(s,\lambda)}\right)^2\right)ds,$$

which is clearly non-negative, to the power of the exponent.

The first of the two exponents in (21) is a nondecreasing process, whereas the second is a martingale. Let us define a sequence of stopping times $N = 1, 2, \dots$, by the relation $\zeta_N = \sup\{t\colon \int_0^t a(s)|^2ds < N\}$. For any $x \in \mathbf{R}$, $t \geqq 0$, and $N \geqq 1$,

$$\mathbf{E}\left|p_*(\min\{t,\zeta_N\},\lambda)\right|^x \leqq \exp\left\{\left(|x| + \frac{x^2}{2}\right)N\right\},$$

and $\mathbf{P}\{\zeta_N < \infty\} \longrightarrow 0$ as $N \to \infty$.

We claim that the relations above provide the absolute continuity of σ. Indeed, since

$$\sup_{-\infty<\lambda<\infty} \mathbf{E}\left\{ \sup_{t\geqq 0} \left|p_*\big(\min\{t,\zeta_N\}\big)\right|^{-2}\right\} < \infty, \qquad N=1,2,\dots,$$

we have $\int_{-\infty}^{\infty}(1+\lambda^2)^{-1}\sup_{t\geqq 0}(|p_*(t,\lambda)|^{-2})d\lambda < \infty$ with probability 1. Therefore, the measures with densities $(2\pi|p_*(t,\lambda)|^2)^{-1}(1+\lambda^2)^{-1}$ converge a.s. in variation to a measure with density $(2\pi(1+\lambda^2)|\pi(\lambda)|^2)^{-1}$, for which relations (18) are, obviously, satisfied.

Inequality (b) is proved in the same way as a similar inequality in Theorem 4. From the proof we give in the Appendix it follows that only those function from L^2_σ can belong to the image of the mapping $\mathcal{U}$ which are boundary values of functions that are analytic in the upper half-plane. Thus, for example, the function $(\lambda-i)^{-1}$, lying in L^2_σ, does not belong to the image. This proves item (c).

6. Appendix. Here we establish the equivalence of statements III–V of Theorem 3 assuming only the following conditions:

(a) *p and p_* are continuous functions in the totality of variables t and λ and analytic in λ for each fixed $t \geqq 0$;*

(b) $p(t,\lambda)\to p(t,\lambda_0)$ as $\lambda\to\lambda_0$, $\lambda_0\in\mathbf{C}$, *uniformly in t from any finite interval*;

(c) *for $\lambda\in\mathbf{C}$, $\mu\in\mathbf{C}$, and $t\geqq 0$, the following relation is valid:*

$$p_*(t,\lambda)\,\overline{p_*(t,\mu)} - p(t,\lambda)\,\overline{p(t,\mu)} = -i(\lambda-\overline{\mu})\int_0^t p(s,\lambda)\,\overline{p(s,\mu)}\,ds. \tag{22}$$

It is not difficult to see that if $\lambda=\mu$, then

$$|p_*(t,\lambda)|^2 - |p(t,\lambda)|^2 = 2\operatorname{Im}\lambda\int_0^t |p(s,\lambda)|^2 ds. \tag{23}$$

Therefore, statements II and IV are equivalent and follow easily from V. Thus, it remains to deduce V from III.

Let $\int_0^\infty |p(s,\lambda)|^2 ds < \infty$ for some λ_0 ($\operatorname{Im}\lambda_0 > 0$). Clearly, there exists a sequence $\{t_n\}_{n=0}^\infty$ of positive numbers such that $t_n\to\infty$ and $p(t_n,\lambda_0)\to 0$ as $n\to\infty$ and a finite limit $\lim_{n\to\infty} p_*(t_n,\lambda_0)=\pi(\lambda_0)$ exists.

We shall prove that $\int_0^\infty |p(t,\lambda)|^2 dt < \infty$ for each λ ($\operatorname{Im}\lambda>0$). Indeed, in accordance with (23) and our choice of the sequence $\{t_n\}_{n=0}^\infty$, we have

$$\lim_{n\to\infty}|p_*(t_n,\lambda_0)|^2 = 2\operatorname{Im}\lambda_0\int_0^\infty |p(t,\lambda_0)|^2 dt. \tag{24}$$

Thus,

$$|p(t_n,\mu)\,p(t_n,\lambda_0)| = o\Big(|p_*(t_n,\mu)\,p_*(t_n,\lambda_0)|\Big) \qquad \text{as } n\to\infty, \quad \operatorname{Im}\mu>0. \tag{25}$$

It is easy to show that if $\int_0^\infty |p(t,\lambda)|^2 dt=\infty$ for some λ ($\operatorname{Im}\lambda>0$), then

$$\left|\int_0^t p(s,\lambda)\,\overline{p(s,\lambda_0)}\,ds\right|^2 = o\left(\int_0^t |p(s,\lambda)|^2 ds\right) \qquad \text{as } t\to\infty.$$

This estimate contradicts the following undoubtedly true estimates (they follow from (22) and (23)):

$$\begin{aligned}\left|p_*(t_n,\lambda)\right|^2 = |\lambda_0-\bar{\lambda}|^2|p_*(t_n,\lambda_0)|^{-2}\left|\int_0^{t_n} p(s,\lambda)\,\overline{p(s,\lambda_0)}\,ds\right| \\ +o\Big(\left|p_*(t_n,\lambda)\right|\Big),\quad n\to\infty, \\ \left|p_*(t_n,\lambda)\right|^2 \geqq 2\operatorname{Im}\lambda\int_0^{t_n}\left|p(s,\lambda)\right|^2 ds,\end{aligned}$$

since the right-hand side of the first of the relations is smaller in order than the right-hand side of the second, as $n\to\infty$.

Now it easily follows from (22) (for $\mu=\lambda_0$) that the limit $\lim_{n\to\infty} p_*(t_n,\lambda)=\pi(\lambda)$ exists for $\operatorname{Im}\lambda>0$. Thus, according to (23), there exists a limit $\lim_{n\to\infty}|p(t_n,\lambda)|<\infty$. Since the integrals $\int_0^\infty|p(t,\lambda)|^2dt$ and $\int_0^\infty|p(t,\lambda_0)|^2dt$ are convergent for $\operatorname{Im}\lambda>0$, there exists a sequence of positive numbers $\{s_n\}_{n=0}^\infty$ such that $s_n\to\infty$, $p(s_n,\lambda)\to 0$ and $p(s_n,\lambda_0)\to 0$ as $n\to\infty$. We specify a new sequence $\{\tilde{t}_n\}_{n=0}^\infty$ by the relations $\tilde{t}_{2k+1}=t_k$ and $\tilde{t}_{2k}=s_k$, $k=0,1,2,\ldots$. The arguments above show that a limit $\lim_{n\to\infty}|p(\tilde{t}_n,\lambda)|$ exists. Thus, $p(t_n,\lambda)\to 0$ as $n\to\infty$ for any λ ($\operatorname{Im}\lambda>0$). Hence, we have established that, for any λ and μ ($\operatorname{Im}\lambda>0$ and $\operatorname{Im}\mu>0$),

$$\pi(\lambda)\overline{\pi(\mu)} = -i(\lambda-\overline{\mu})\int_0^\infty p(t,\lambda)\,\overline{p(t,\mu)}\,dt. \tag{26}$$

It remains to prove that $p_*(t_n,\lambda)\longrightarrow\pi(\lambda)$ as $n\to\infty$ uniformly on the compact subsets of the open upper half-plane. To this end it suffices to show that the sequence $\{p_*(t_n,\lambda)\}_{n=0}^\infty$ is uniformly bounded on the compact subsets of the open upper half-plane. Assume the contrary. Then, without loss of generality, one can assume that a sequence of numbers $\{\lambda_n\}_{n=0}^\infty$ exists such that $\lim_{n\to\infty}\lambda_n=\lambda_\infty$, $\operatorname{Im}\lambda_\infty>0$, and $|p_*(t_n,\lambda_n)|\longrightarrow\infty$ as $n\to\infty$.

Formulas (22) and (23) imply the equalities

$$\begin{aligned}\left|p_*(t_n,\lambda_n)\right|^2 &= \left|p_*(t_n,\lambda_\infty)\right|^{-2} \\ &\times\left|(\lambda_n-\bar{\lambda}_\infty)\int_0^{t_n} p(s,\lambda_n)\,\overline{p(s,\lambda_\infty)}\,ds + p(t_n,\lambda_n)\,\overline{p(t_n,\lambda_\infty)}\right|^2, \\ \left|p_*(t_n,\lambda_n)\right|^2 &= 2\operatorname{Im}\lambda_n\int_0^{t_n}\left|p(s,\lambda_n)\right|^2ds + \left|p(t_n,\lambda_n)\right|^2.\end{aligned} \tag{27}$$

We claim that under our assumptions these equalities can not be true simultaneously. Indeed, if $\sup_{n\geqq 0}(\int_0^{t_n}|p(s,\lambda_n)|^2ds)<\infty$, then

$$\lim_{n\to\infty}\left|p(t_n,\lambda_n)\right|=\infty,\qquad \sup_{n\geqq 0}\left(\left|\int_0^{t_n} p(s,\lambda_n)\,\overline{p(s,\lambda_\infty)}\,ds\right|\right)<\infty$$

and

$$\left|p(t_n,\lambda_n)\,p(t_n,\lambda_\infty)\right| = o\Big(\left|p(t_n,\lambda_n)\right|^2\Big),\qquad n\to\infty.$$

However, this is a contradiction and so, one can assume, without loss of generality, that

$$\lim_{n\to\infty}\left|(\lambda_n-\overline{\lambda}_\infty)\int_0^{t_n} p(s,\lambda_n)\,\overline{p(s,\lambda_\infty)}\,ds\right| = \infty.$$

But in this case equalities (27) can not be true simultaneously, since the estimate

$$\left|\int_0^{t_n} p(s,\lambda_n)\,\overline{p(s,\lambda_\infty)}\,ds\right|^2 = o\left(\int_0^{t_n} \left|p(s,\lambda_n)\right|^2 ds\right), \qquad n\to\infty,$$

is valid. To justify the last relation we fix an arbitrary $r>0$. Then,

$$\left(\int_0^{t_n} \left|p(s,\lambda_n)\right|^2 ds\right)^{-1}\left|\int_0^{t_n} p(s,\lambda_n)\,\overline{p(s,\lambda_\infty)}\,ds\right|^2 \leqq \left(\int_0^{t_n} \left|p(s,\lambda_n)\right|^2 ds\right)^{-1}$$
$$\times\left(\int_0^r \left|p(s,\lambda_n)\,p(s,\lambda_\infty)\right| ds + \left(\int_r^{t_n} \left|p(s,\lambda_n)\right|^2 ds \int_r^{t_n} \left|p(s,\lambda_\infty)\right|^2 ds\right)^{1/2}\right)^2.$$

The right-hand side of the inequality can be made arbitrarily small if the numbers r and n are sufficiently large since, for any fixed r, the right-hand side of the inequality above tends to $\int_r^\infty |p(s,\lambda_\infty)|^2\,ds$ as $n\to\infty$ (note that

$$\int_0^r \left|p(s,\lambda_n)\,p(s,\lambda_\infty)\right| ds \longrightarrow \int_0^r \left|p(s,\lambda_\infty)\right|^2 ds$$

as $n\to\infty$ according to proposition (b)). This completes the proof of the theorem.

Observe, that equality (26), being in essence the definition of the function $\pi(\cdot)$, is still true if one takes $\alpha\pi(\cdot)$ for $\pi(\cdot)$, where $|\alpha|=1$. In fact, in many cases, $\pi(\cdot)$ is determined up to a factor whose absolute value is equal to 1 and which depends on the choice of the sequence $\{t_n\}_{n=0}^\infty$. In the case of orthogonal polynomials such a nonuniqueness does not occur.

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