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Pure Point Spectrum of the Laplacians on Fractal Graphs

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We establish the pure point spectrum of the Laplacians on two point self-similar fractal graphs. All eigenvalues have infinite multiplicity and a countable system of orthonormal eigenfunctions with compact support is complete in the corresponding Hilbert space. © 1995 Academic Press, Inc.

INTRODUCTION

In the last decade, considerable attention has been paid by graph theorists to the study of spectra of the difference Laplacians on infinite graphs. We refer separately to the paper of Mohar and Woess [MW], which is an excellent survey of this theory. Explicit computational results about the spectrum of the Laplacians are known only when the graph under consideration satisfies certain kind of regularity property that leads to the existence of the absolutely continuous spectrum (see [MW, A]).

If we study fractal or disordered materials and the difference Laplacians are some discrete approximations, we should expect the spectrum to be pure point.

The first result of this type is the physics article [R] where the spectrum of the Laplacian on the Sierpinski lattice is considered. An application of the very interesting Renormalization Group method to this case was given by Bellissard in [B].

In this paper we study the spectrum of the Laplacians on so-called two-point self-similar fractal graphs (TPSG) (we mean the Laplacians which correspond to the adjacency matrix and the simple random walk). A good example of such a kind of graphs is the modified Koch graph which can be

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considered as the discrete approximation of the fractal set, namely the modified Koch curve [Ma1].

Roughly speaking, we will prove that if the TPSG has an infinite number of cycles and the length of these cycles approaches infinity, then the spectrum of the Laplacians is pure point.

The problem of the description of the spectrum as a set in $\mathbb{R}$ is not trivial as shown by the example of the modified Koch graph. The spectrum for this graph is the union of two sets. The first set is the Julia set of the rational function

$$R(z) = 9z(z-1)(z-\tfrac{4}{3})(z-\tfrac{5}{3})(z-\tfrac{3}{2})^{-1}.$$

This is a Cantor set of Lebesgue measure zero which may be obtained as a closure of a countable set of eigenvalues of the Laplacian with infinite multiplicity. The second set is a discrete countable set of eigenvalues with infinite multiplicity which has the limit points in the first set.

We note the new property of the eigenfunction of the Laplacians on TPSG: a countable system of orthonormal eigenfunction with compact support is complete in the Hilbert space where this operator is defined.

We consider in Theorem 5 the Anderson localization for the Schrödinger operator with Bernoulli potential on TPSG. It was proven that any eigenvalue of the Laplacian is an eigenvalue of infinite multiplicity of the Schrödinger operator for any coupling constant. Unfortunately, we cannot prove that the spectrum of such operator is pure point. However, we note that Aizenman and Molchanov [AM] proved the localization of the spectrum in the standard Anderson model for suffiently large disorders on general graphs.

The two-point self-similar fractal graphs can be considered as nested pre-fractals with two essential fixed points introduced by Lindstrøm [L]. We also note that some questions about the integrated density of states of the Laplacian on fractal graphs were studied in [Ma 2, FSh].

Some special examples of TPSG were considered in physical models of the percolation theory (see [S, BH]).

1. Notations and Auxiliary Results

1. Let $G=(V,E)$ be a connected infinite locally finite graph, with vertex set V and edge set E. We suppose that the degree d_x of all vertices $x \in V$ is finite.

Let $A=A(G)$ be the adjacency matrix of the graph G and $P=P(G)=(p_{u,v})$ $u, v \in V$ be the transition matrix, where

$$p_{u,v} = a_{u,v}/d_u$$

and $a_{u,v}$ is the number of edges between u and v.

Associated with each of the preceding two matrices are the difference Laplacians

$$\Delta_A = D(G) - A(G) \tag{1.1}$$

and

$$\Delta_p = I(G) - P(G), \tag{1.2}$$

where $D(G)$ is the diagonal matrix of d_x, $x \in V$, and $I(G)$ is the identity matrix over V.

Let us introduce the spaces of functions on V

$$l_2(V) = \left\{ f(x), x \in V, \sum_{x \in v} |f(x)|^2 < \infty \right\} \tag{1.3}$$

with the inner product

$$(g, f) = \sum_{x \in V} g(x) \bar{f}(x)$$

and

$$l_2^{\#}(V) = \left\{ f(x), x \in V; \sum_{x \in V} d_x \, |f(x)|^2 < \infty \right\} \tag{1.4}$$

with inner product

$$(g, f) = \sum_{x \in V} d_x g(x) \bar{f}(x).$$

We note that if the function $\deg(x) = d_x$, $x \in V$ is bounded, then the operators Δ_A and Δ_p are self-adjoint bounded operators in $l_2(V)$ and $l_2^{\#}(V)$, respectively.

2. Let us introduce so-called two point self-similar graphs.

Suppose $M = (V_M, E_M)$ and $G_0 = (V_0, E_0)$ are finite connected graphs and M is an ordered graph. We fix some $e_0 \in E_M$, which is not a loop, and vertices $\alpha, \beta \in V_M$ and $\alpha_0, \beta_0 \in V_0$, $\alpha \neq \beta$, $\alpha_0 \neq \beta_0$.

Informally speaking, the construction of a TPSG G is as follows: to get G_1 from M and G_0 we replace every edge $(a, b) \in E_M$, $a, b \in V_M$, by a copy of G_0 such that α_0 goes to a and β_0 to b. Then we take $\alpha_1 = \alpha$, $\beta_1 = \beta$ and proceed by induction. If a graph $G_n = (V_n, E_n)$ with fixed vertices α_n, $\beta_n \in V_n$ is defined then the graph G_{n+1} is obtained by replacement of every edge (a, b) of M by the copy of G_n such that α_n goes to a and β_n goes to b. The vertices α_{n+1}, β_{n+1} are the vertices α, β after this replacement.

We can assume that $G_n \subseteq G_{n+1}$ is the copy corresponding to e_0 and define infinite graph $G = \bigcup_{n=1}^{\infty} G_n$.

Let us give a more formal definition.

DEFINITION 1.1. A graph G is called TPSG with model graph M and initial graph G_0 if the following holds:

(i) There are finite subgraphs $G_0, G_1, G_2, \ldots$ such that $G_n \subseteq G_{n+1}$, $n \geqslant 0$, and $G = \bigcup_{n \geqslant 0} G_n$.

(ii) For any $n \geqslant 0$ and $e \in E_M$ there is a graph homomorphism $\Psi_n^e \colon G_n \to G_{n+1}$ such that $G_{n+1} = \bigcup_{e \in E_M} \Psi_n^e(G_n)$ and $\Psi_n^{e_0}$ is the inclusion of G_n to G_{n+1}.

(iii) For all $n \geqslant 0$ there are two vertices $\alpha_n, \beta_n \in V_n$ such that Ψ_n^e restricted to $G_n \backslash \{\alpha_n, \beta_n\}$ is a one-to-one mapping for every $e \in E_M$. Moreover $\Psi_n^{e_1}(V_n \backslash \{\alpha_n, \beta_n\}) \cap \Psi_n^{e_2}(V_n \backslash \{\alpha_n, \beta_n\}) = \varnothing$ if $e_1 \neq e_2$.

(iv) For $n \geqslant 1$, there is an injection $\kappa_n \colon V_M \to V_n$ such that $\alpha_n = \kappa_n(\alpha)$, $\beta_n = \kappa_n(\beta)$ and for every edge $e = (a, b) \in E_M$, $\Psi_{n-1}^e(\alpha_{n-1}) = \kappa_n(a)$, $\Psi_{n-1}^e(\beta_{n-1}) = \kappa_n(b)$.

We say that the vertices α_n, β_n are the boundary vertices of G_n, i.e., $\partial G_n = \{\alpha_n, \beta_n\}$ and $\operatorname{int} G_n = V_n \backslash \{\alpha_n, \beta_n\}$ are interior vertices of G_n.

Remark 1.1. One can see that G is defined uniquely if graphs M and G_0 along with $e_0 \in E_M$, $\alpha, \beta \in V_M$, $\alpha_0, \beta_0 \in V_0$ are given.

Suppose M does not have loops and G_0 is just two vertices and one edge. Then two-point self-similar graphs are in one-to-one correspondence to so-called post-critically finite (p.c.f.) self-similar sets with the post-critical set consisting of two points. Namely the graphs G_n are isomorphic to so-called pre-fractals for such p.c.f. sets. However, G is not a p.c.f. set since the limiting procedures in these two cases are different. The definition of a p.c.f. set can be found in [K] or [KL].

3. We need some auxiliary result on the structure of graph G.

DEFINITION 1.2. Two different vertices x and y of a graph Γ are equivalent if there is an automorphism φ of Γ such that $\varphi(x) = y$, $\varphi(y) = x$.

By induction it is easy to prove the following lemma.

LEMMA 1.1. *If the vertices $\alpha, \beta \in V_M$ and $\alpha_0, \beta_0 \in V_0$ are equivalent in M and G_0, respectively, then vertices α_n, β_n are equivalent in G_n for all n.*

Remark 1.2. We will suppose in what follows that M and G_0 satisfy assumptions of Lemma 1.1. We call such graph G symmetric. In this case the graph G does not depend on the orientation of M.

Although our results are valid for nonsymmetric graphs (with some additional assumptions on the orientation of M) we do not consider such graphs for the sake of simplicity.

Let us introduce the graph $\tilde{M}=(V_{\tilde{M}}, E_{\tilde{M}})$ which can be obtained in the same way as G_1 if we take the graph M instead of G_0 and the vertices α, β play the role of α_0, β_0.

We define the graph $\tilde{G}_{n+2}$ by replacement of every edge of $\tilde{M}$ by the copy of G_n such that for every edge $(a, b) \in E_{\tilde{M}}$, $a, b \in V_{\tilde{M}}$ we say α_n goes to a and β_n to b.

LEMMA 1.2. *The graphs $\tilde{G}_{n+2}$ and G_{n+2} are isomorphic.*

Proof. By definition $\tilde{G}_{n+2}$ can be written as

$$\tilde{G}_{n+2}=\bigcup_{e \in E_{\tilde{M}}} \tilde{\Psi}_n^e(G_n) \tag{1.5}$$

where the maps $\tilde{\Psi}_n^e$ have the same properties as Ψ_n^e in Definition 1.1. The proof follows by induction. ▮

Let us introduce the space $l_2(X)$ by $l_2(X)=\{f \in l_2(V): f(x)=0$ for $x \in V \backslash X\}$, where $X \subset V$. $l_2^{\#}(X)$ is defined analogously. By $\Delta_A(X)$, $\Delta_p(X)$ we denote the restriction of Δ_A, Δ_p to $l_2(X)$, $l_2^{\#}(X)$. More precisely, $\Delta_{A,p}(X)=P\Delta_{A,p}P$, where P is the orthogonal projector to $l_2(X)$ or $l_2^{\#}(X)$. We will call these operators the Laplacians with zero boundary conditions on $V \backslash X$. For simplicity, we denote the Laplacians with zero boundary conditions on ∂G_n by $\Delta_A(n)$ and $\Delta_p(n)$.

By Lemma 1.1 there is isomorphism $\varphi_n: G_n \to G_n$ such that $\varphi_n(\alpha_n)=\beta_n$, $\varphi_n(\beta_n)=\alpha_n$. This isomorphism induces unitary maps $U_n: l_2(G_n) \to l_2(G_n)$ and $U_n^{\#}: l_2^{\#}(G_n) \to l_2^{\#}(G_n)$ by formula $U_n^{\#} f=f \circ \varphi_n$.

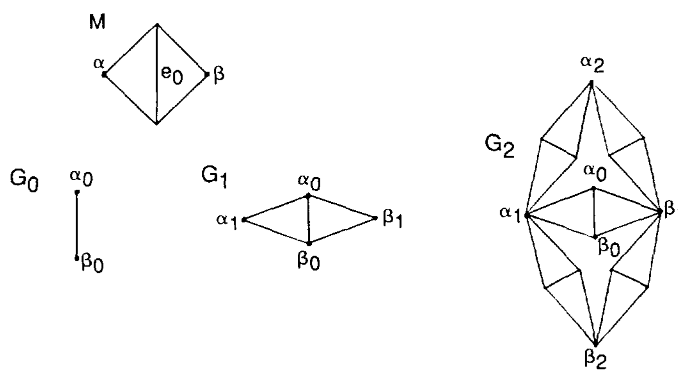

FIGURE 1

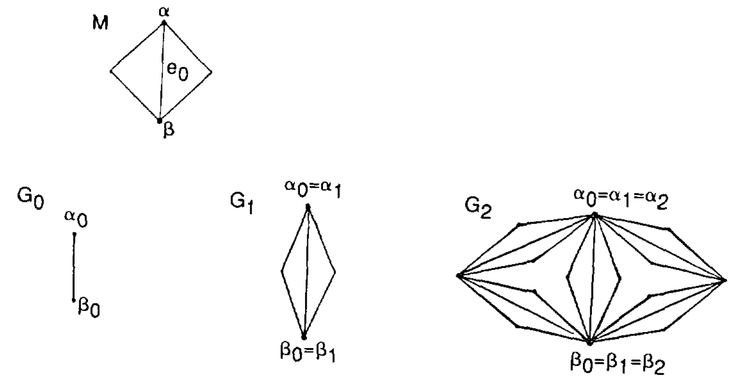

FIGURE 2

LEMMA 1.3. *$U_n(U_n^{\#})$ commutes with $\Delta_A(G_n)$ and $\Delta_A(n)$ ($\Delta_p(G_n)$ and $\Delta_p(n)$).*

Proof of this lemma immediately follows from the definition of Δ_A and Δ_p.

Let us consider the function $\deg(x) = d_x$. It can occur that the function $\deg(\cdot)$ is not bounded in general. Moreover, there can exist a point $x_0 \in V$ such that $\deg(x_0) = \infty$. The next lemma should be more clear from the following examples (see Figs. 1 and 2).

For an arbitrary graph $\tilde{G}$ let us denote by $d_x(\tilde{G})$ the degree of the vertex x in $\tilde{G}$.

LEMMA 1.4. (i) $d_{\alpha_n}(G_n) = d_{\alpha_0}(G_0) \cdot (d_\alpha(M))^n = d_{\alpha_{n-1}}(G_{n-1}) \cdot d_\alpha(M)$.

(ii) *If $x \in \operatorname{int} G_n$, then $\deg(x) = d_x(G_n) = d_x(G_{n+1})$ for every $n \geq 1$.*

(iii) *The function $\deg(x)$ is bounded if and only if $d_\alpha(M) = 1$.*

(iv) *If $x \in V$ and $x \neq \alpha_0$, β_0 then $\deg(x) < \infty$.*

(v) $\deg(\alpha_0) = \infty$ ($\deg(\beta_0) = \infty$) *if and only if α is indicent to e_0 and $d_\alpha(M) \geq 2$ (β is incident to e_0 and $d_\beta(M) \geq 2$).*

Proof. The first statement can be proved by induction. The second follows from (ii) and (iii) of Definition 1.1. Statement (iii) follows from (i) and equality $\max_{x \in G_{n+1}} d_x(G_{n+1}) = \max\{\max_{x \in G_n} d_x(G_n), d_{\alpha_n}(G_n) \cdot \max_{y \in M} d_y(M)\}$.

(iv) There exists $n_0 \in \mathbb{N}$ such that $x \in V_n$ for every $n \geq n_0$. If $x \in \operatorname{int} G_n$, the statement follows from (ii). Otherwise, $x \in \partial G_n$ for every $n \geq n_0$ and consequently x is equal to α_0 or β_0.

(v) By (iv), it follows that $\alpha_0 \in \partial G_n$ for any $n \geq n_0$, $n_0 \in \mathbb{N}$. If α is not incident to e_0, then α_0 is an interior point of G_{n_1} for some n_1. Let α be incident to e_0 and $d_\alpha(M) \geq 2$. Then statement (v) follows from (i). ∎

DEFINITION 1.3. We denote by

$$\partial G = \{x, \deg(x) = \infty\}$$

the boundary of the graph G. If $\partial G = \varnothing$, we say that G is a graph without boundary.

By Lemma 1.4 we obtain the following lemma.

LEMMA 1.5. (i) *$e_0 = (\alpha, \beta)$ and $d_\alpha(M) \geqslant 2$, if and only if $\partial G = \{\alpha_0, \beta_0\}$.*

(ii) *The boundary ∂G has only one point if and only if one of the points α or β is a vertex of e_0 and the degree of this vertex in M is not less than 2.*

(iii) *If conditions* (i), (ii) *are not satisfied for the graph G then $\partial G = \varnothing$.*

2. THE MAIN RESULTS

Let us introduce the main results of this paper. First, we consider the operator Δ_p. If the graph G is without boundary, then the operator is self-adjoint because it is a linear symmetric bounded operator.

If G has the boundary, we define the operator Δ_p with zero boundary conditions, i.e.,

$$\Delta_p^0 : l_2^\#(V^0) \to l_2^\#(V^0),$$

where

$$l_2^\#(V^0) = \{f \in l_2^\#(V), f(x) = 0, x \in \partial G\}.$$

The Δ_p^0 is a self-adjoint bounded operator, too.

THEOREM 1. *Suppose that the graph M has a cycle and the edge e_0 belongs to this cycle. Then the spectrum of the operator $\Delta_p(\Delta_p^0)$ is pure point. Moreover, a countable set of orthonormal eigenfunctions of $\Delta_p(\Delta_p^0)$ with compact support is complete in $l_2^\#(V)(l_2^\#(V^0))$ and every eigenvalue has infinite multiplicity.*

If e_0 does not belong to the cycle, we do not know the structure of the spectrum in general. However, there is the following theorem in a particular case.

THEOREM 2. *Suppose that the graph M has an odd cycle and there is an isomorphism φ: $M \to M$ such that $\varphi(\alpha) = \beta$, $\varphi(\beta) = \alpha$, and $\varphi(e_0) \neq e_0$. If*

(i) *the edge* e_0 *belongs to a path joining* α *and* β *or*

(ii) *the edge* e_0 *belongs to a path joining* α *(or* β*) with the cycle then the conclusions of Theorem* 1 *hold for* Δ_p *and* Δ_p^0.

Let us now consider the operator Δ_A. If the boundary of G is empty its action is well defined on all functions with compact support which form a dense subspace of $l^2(V)$. If $\partial G \neq \varnothing$ we define Δ_A^0 as an operator with zero boundary conditions (see above definition for Δ_p^0). This operator is symmetric and thus closable. We will denote its closure by the same symbol Δ_A (Δ_A^0).

THEOREM 1^0. *Suppose all conditions for the graph* G *in Theorem* 1 *hold. Then*:

(i) *The operator* Δ_A (Δ_A^0) *is self-adjoint.*

(ii) *All statements of Theorem* 1 *are true.*

THEOREM 2^0. *If all conditions of Theorem* 2 *are satisfied for the graph* G, *then the operator* Δ_A (Δ_A^0) *is self-adjoint and the statements of Theorem* 2 *hold for* Δ_A (Δ_A^0).

We note that the operator Δ_A is not self-adjoint in general. An example of a locally finite graph with no unique self-adjoint extension of Δ_A was given in [Mü].

The condition of the existence of a cycle in the graph M is not a necessary condition for the spectrum to be pure point. Moreover the graph G may be a tree in this case (see Fig. 3).

THEOREM 3. *Suppose there exist different vertices* $y_0, y_1, y_2 \in V(M)$ *such that there are edges* $(y_0, y_1), (y_1, y_2) \in E(M)$, $e_0 = (y_0, y_1)$, $d_{y_0}(M) = d_{y_2}(M) = 1$ *and the set* $\{y_0, y_2\}$ *does not coincide with the set* $\{\alpha, \beta\}$.

Then all results of Theorems 1 *and* 1^0 *hold.*

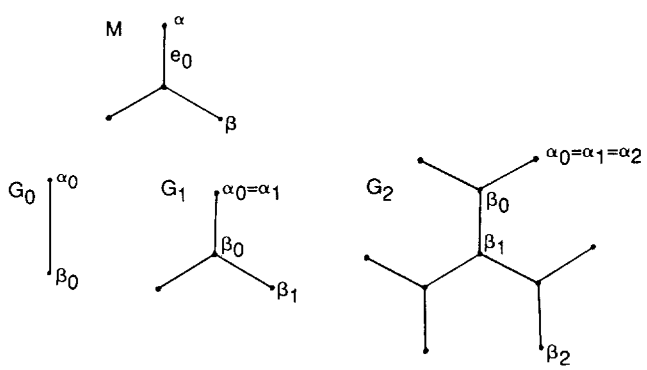

FIGURE 3

The simple example of a two-point self-similar graph such that the conditions of Theorems 1–3 are not satisfied is the lattice $\mathbb{Z}$. It is well known that the spectrum of the Laplacian in this case is absolutely continuous.

Condition (iv) in Definition 1.1 defines the structure of eigenfunctions of the Laplacians. It is easy to see that conditions (i)–(iii) of Definition 1.1 are satisfied for Sierpinsky lattice but Theorems 1–2^0 are not true in this case. By [B] it follows that there are such eigenvalues that if a function φ is an eigenfunction corresponding to one of them, then φ cannot have a compact support.

The problem of describing the spectrum as a set in $\mathbb{R}$ is hard enough as shown by the example of the operator Δ_p on the modified Koch graph in [Ma1].

Let us introduce functions $W: V \to \mathbb{R}$ which do not change the nature of the spectrum of the Laplacian; i.e., the spectrum of the Schrödinger operator

$$H = \Delta + W \tag{2.1}$$

will be pure point, too. Here we denote Δ_A and Δ_p by the same symbol Δ.

We note that periodic functions are potentials of this sort for the Schrödinger operator in $l_2(\mathbb{Z}^n)$ but only in the case of absolutely continuous spectrum.

Suppose that $W_0: V_{n_0} \to \mathbb{R}$ is a function such that $W_0(\varphi(x)) = W_0(x)$, where $\varphi: G_n \to G_n$ is an automorphism of G_n, $\varphi(\alpha_n) = \beta_n$, $\varphi(\beta_n) = \alpha_n$. Let us define the potential $W: V \to \mathbb{R}$ by induction. We denote by W_{m+1} the restriction of W on V_{n_0+m+1} and we suppose $W_{m+1}(x) = W_m(y)$, where $x = \Psi^e_{n_0+m}(y)$, $y \in V_{n_0+m}$, $e \in E_M$ for every $m \geqslant 0$.

THEOREM 4. *If the function W is defined as above, then all results of Theorems* 1–2^0, 3 *hold for the Schrödinger operator* (2.1).

Let us consider the so-called Bernoulli potential $\{W(x), x \in V\}$ made of a sequence of i.i.d. random variables taking only two values 0 and 1. We set

$$\mathbb{P}\{W(x) = 0\} = \mathbb{P}\{W(x) = 1\} = \tfrac{1}{2}, \qquad x \in V.$$

We are interested in the random Schrödinger operator

$$H_\beta = \Delta + \beta W$$

with a coupling constant $\beta > 0$.

THEOREM 5. *Let G satisfy conditions of one of the Theorems* 1–2^0, 3. *Then for any $\beta > 0$ with probability one, every eigenvalue of Δ is an eigenvalue of H_β of infinite multiplicity.*

3. The Auxiliary Theorem 3.1

Let $\mathscr{H}$ be a Hilbert space with the inner product (,) and $\mathscr{H}_n$, $n = 1, 2, \ldots$, be a sequence of finite dimensional subspaces of $\mathscr{H}$ such that $\mathscr{H}_n \subset \mathscr{H}_{n+1}$ and $\tilde{\mathscr{H}} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ is dense in $\mathscr{H}$.

We suppose that H is a closed symmetric operator on $\mathscr{H}$ such that $\tilde{\mathscr{H}}$ belongs to the domain of definition of the operator H and $H_n = P_n H P_n$, where P_n is the orthogonal projector on $\mathscr{H}_n$.

Then $H_n : \mathscr{H}_n \to \mathscr{H}_n$ and H_n is symmetric, too.

Let $\lambda_n^1, \ldots, \lambda_n^{K(n)}$ be all distinct eigenvalues of the operator H_n (restricted to $\mathscr{H}_n$).

Let $\tilde{F}_n^i$ be the eigenspace corresponding to λ_n^i and let F_n^i be an orthonormal basis of $\tilde{F}_n^i$.

THEOREM 3.1. *Let $m \in \mathbb{N}$, $\delta > 0$ and $c < \infty$ be fixed numbers and for every $n = 1, 2, \ldots$, there exists a linear operator $\Phi_n : \mathscr{H}_n \to \mathscr{H}_{n+m}$ such that $\|\Phi_n\| \leqslant c$, $(f, \Phi_n(f)) \geqslant \delta \|f\|^2$ for any $f \in \mathscr{H}_n$ and $H\Phi_n(f) = \lambda_n^i \Phi_n(f)$ for any $f \in \tilde{F}_n^i$, $i = 1, \ldots, K(n)$.*

Then the following statements hold:

(i) *The operator H has only pure point spectrum. The set of eigenvalues is $\bigcup_{n \geqslant 1} \bigcup_{1 \leqslant i \leqslant K(n)} \{\lambda_n^i\}$.*

(ii) *There is a countable set $S \subset \tilde{\mathscr{H}}$ of orthonormal eigenfunctions of the operator H which is complete in $\mathscr{H}$.*

(iii) *If $\Phi_n(f) \notin \mathscr{H}_n$ for any nonzero $f \in \mathscr{H}_n$ and every $n \geqslant 1$, then each eigenvalue of H has infinite multiplicity.*

(iv) *H is a self-adjoint operator in $\mathscr{H}$.*

Proof. At first we note from the definition of H_n that $\mathscr{H}_n = \bigoplus_{i=1}^{K(n)} \tilde{F}_n^i$. Let

$$S_n = \{f \in \mathscr{H}_n : Hf \in \mathscr{H}_n\}.$$

It is easy to see that $S_n \subset S_{n+1}$ for every $n \geqslant 1$.

We introduce the set S by the formula

$$S = \bigcup_{n \geqslant 1} \bigcup_{1 \leqslant i \leqslant K(n)} (F_n^i \cap S_n)$$

and we note that the set $S_n \cap F_n^i$ is not empty for $n \geqslant m+1$ because $\Phi_n(f) \in \mathscr{H}_{n+m}$ for every $f \in \mathscr{H}_n$ and

$$H_{n+m}\Phi_n(f) = P_{n+m} H P_{n+m} \Phi_n(f) = P_{n+m}(\lambda_n^i \Phi_n(f)) = \lambda_n^i \Phi_n(f), f \in F_n^i. \quad (3.1)$$

One can see from the conditions of Theorem 3.1 and (3.1) that if $\lambda \in \sigma(H_n)$ then λ is an eigenvalue of H. That gives us the inclusion

$$\bigcup_{n \geqslant 1} \bigcup_{1 \leqslant i \leqslant K(n)} \{\lambda_n^i\} \subset \sigma(H). \tag{3.2}$$

We will prove that the set S is complete in $\mathscr{H}$. Suppose that there exists $f \in \mathscr{H}$ such that $(f, g) = 0$ for any $g \in S$.

Let A be a subspace of $\mathscr{H}$ and P_A be the orthogonal projection to A. Then

$$\|P_A f\| \geqslant \frac{1}{\|g\|} |(g, f)| \tag{3.3}$$

for every $g \in A$, $g \neq 0$, and $f \in \mathscr{H}$. This follows from the expression

$$\begin{aligned} |\ \|g\|^{-1} (g, f)| &= \|g\|^{-1} |(P_A g, f)| = \|g\|^{-1} |(P_A^2 g, f)| \\ &= \|g\|^{-1} (g, P_A f)| \leqslant \|g\|^{-1} \|g\| \ \|P_A f\| \leqslant \|P_A f\|. \end{aligned}$$

Let us introduce the subspace A_n of $\mathscr{H}_n$ by the formula

$$A_n = \bigoplus_{i=1}^{K(n)} (\tilde{F}_n^i \cap S_n)$$

and let Q_n be the orthogonal projector to A_n.

If $f_n = P_n f$, $n = 1, 2, ...$, by (3.3) and the conditions of Theorem 3.1 we have

$$\begin{aligned} \|Q_{n+m} f_n\| &\geqslant |(\Phi_n(f_n), f_n| \ \|\Phi_n(f_n)\|^{-1} \\ &\geqslant (c \ \|f_n\|)^{-1} |(\Phi_n(f_n), f_n)| \geqslant c^{-1} \delta \ \|f_n\|. \end{aligned} \tag{3.4}$$

Since $A_{n+m} \subset \operatorname{Span} S$ we obtain $Q_{n+m} f = 0$. Hence

$$0 = \|Q_{n+m} f\| \geqslant \|Q_{n+m} f_n\| - \|f - f_n\| \geqslant c^{-1} \delta \ \|f_n\| - \|f - f_n\|.$$

This implies $f = 0$ since $\|f - f_n\| \to 0$ as $n \to \infty$. Therefore S is complete in $\mathscr{H}$ and (i), (ii) is proved.

(iii) For arbitrary eigenvalue λ of H there exists a corresponding eigenfunction $f \in S$ and consequently there are such n_0, i that $f \in F_{n_0}^i \cap S_{n_0}$. We denote $g_0 = \Phi_{n_0}(f)$ and $g_{k+1} = \Phi_{n_0 + km}(g_k)$. Then $\{g_k\}_{k=0}^{\infty}$ is a linearly independependent sequence of eigenfunctions of the operator H because, by the definition of Φ_n, $g_{k+1} \notin \mathscr{H}_{n_0 + km}$.

(iv) It is enough to prove that $\operatorname{Ran}(H \pm i)$ are complete sets in $\mathscr{H}$ (see [RS, Vol. 1, Theorem VIII.3) that follows from (ii) of our theorem. The theorem is proved. ▌

4. Proof of Theorems 1–5

Proof of Theorem 1 *and Theorem* 1^0

By Theorem 3.1 it is enough to construct the operator $\Phi_n\colon \mathscr{H}_n \to \mathscr{H}_{n+m}$, $m \geqslant 1$ with required properties. We will prove Theorem 1 only for the operator Δ_p because the case of the Δ_A is the same.

Let $\mathscr{H}_n = l_2^{\#}(\text{int}\, G_n)$. We suppose that the cycle in M is defined by the set of vertices $\{v_k\}_{k=0}^{l}$, $v_i \in V_M$, $v_0 = v_l$.

1. If $l = 2m$, $m \in \mathbb{N}$, we can introduce sets of edges

$$E^+ = \{(v_{2k}, v_{2k+1})\}_{k=0}^{m} \subset E_M,$$
$$E^- = \{(v_{2k-1}, v_{2k})\}_{k=1}^{m} \subset E_M.$$

We note that for any $x \in \Psi_n^e(V_n \backslash \partial G_n)$ there is a unique $y \in V_n \backslash \partial G_n$ such that $x = \Psi_n^e(y)$, $e \in E_M$.

By Remark 1.2 we may suppose that the maps Ψ_n^e, $e \in E^+ \cup E^-$ can be chosen such that if different edges e_1 and e_2 have a common vertex, then at least one of the following equalities holds

$$\Psi_n^{e_1}(\alpha_n) = \Psi_n^{e_2}(\alpha_n) \quad \text{or} \quad \Psi_n^{e_1}(\beta_n) = \Psi_n^{e_2}(\beta_n). \tag{4.1}$$

Let us define operators $\Phi_n^e\colon \mathscr{H}_n \to \mathscr{H}_{n+1}$ for any $e \in E_M$ as follows:

$$\Phi_n^e(f)(x) = \begin{cases} 0 & \text{if } x \notin \Psi_n^e(V_n \backslash \partial G_n) \\ f(y) & \text{if } x = \Psi_n^e(y), y \in V_n \backslash \partial G_n. \end{cases}$$

Then we define the operator

$$\Phi_n = \sum_{e \in E^+} \Phi_n^e - \sum_{e \in E^-} \Phi_n^e,$$

which maps $\mathscr{H}_n$ into $\mathscr{H}_{n+1}$. We will verify that it satisfies the conditions of Theorem 3.1.

We note that if $e_1, e_2 \in E_M$, and $e_1 \neq e_2$ then $\Phi_n^{e_1}(f)$ and $\Phi_n^{e_2}(f)$ have disjoint supports. Thus $\Phi_n^{e_1}(f)$ is orthogonal to $\Phi_n^{e_2}(f)$ and the bound $\|\Phi_n\| \leqslant c = l$ is obtained. By condition (ii) of Definition 1.1 we have $\Phi_n^{e_0}(f) = f$ and

$$(f, \Phi_n(f)) = \|f\|^2$$

for every $f \in \mathscr{H}_n$. Now if $f \in \tilde{F}_n^i$ then the equality

$$-\Delta_p \Phi_n(f) = \lambda_n^i \Phi_n(f)$$

follows from the definition of the operator Φ_n.

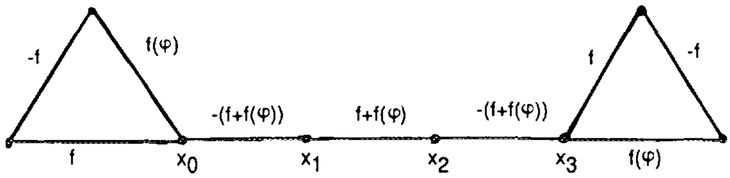

DIAGRAM 1

Since $\Phi_n(f)$ is an eigenfunction of the operator $-\Delta_p$ with compact support by the definition of the set S in the proof of Theorem 3.1, we find that S is a set of eigenfunctions with compact supports.

2. Let $l = 2m+1$, $m \geqslant 1$. The construction of the operator Φ_n in this case is more delicate. In graph $\tilde{M}$ (see Lemma 1.2) we have at least two cycles of length l, joining by a path, and e_0 belongs to one of these cycles.

Say these cycles are $\{v_k\}_{k=0}^{l}$, $\{u_k\}_{k=0}^{l}$, $v_0 = v_l$, $u_0 = u_l$ and they are joined by a path $v_0 = x_0, x_1, ..., x_r = u_0$.

Let $E_v^+ = \{(v_k, v_{k+1}), k \text{ is even}\}$, $E_v^- = \{(v_k, v_{k+1}), k \text{ is odd}\}$; E_u^+, E_u^-, E_x^+, E_x^- are defined similarly. Also, we define operators $\tilde{\Phi}_n^e$ analgously to Φ_n^e, using $\tilde{\Psi}_n^e$ instead of Ψ_n^e (see Lemma 1.2).

Then

$$\Phi_n = \sum_{e \in E_v^+} \tilde{\Phi}_n^e - \sum_{e \in E_v^-} \Phi_n^e - \sum_{e \in E_x^+} (\tilde{\Phi}_n^e + \tilde{\Phi}_n^e \circ U_n^{\#})$$
$$+ \sum_{e \in E_x^-} (\tilde{\Phi}_n^e + \tilde{\Phi}_n^e \circ U_n^{\#}) + (-1)^{r+1} \left(\sum_{e \in E_u^+} \tilde{\Phi}_n^e - \sum_{e \in E_u^-} \tilde{\Phi}_n^e \right).$$

We suppose that condition (4.1) is satisfied in this case, too. This construction is sketched in Diagram 1 if r is odd and on Diagram 2 if r is even.

We note that $\Phi_n \colon G_n \to G_{n+2}$ and this operator satisfies the conditions of Theorem 3.1 that can be proved analogously to case 1 using Lemmas 1.2 and 1.3. The theorem is proved.

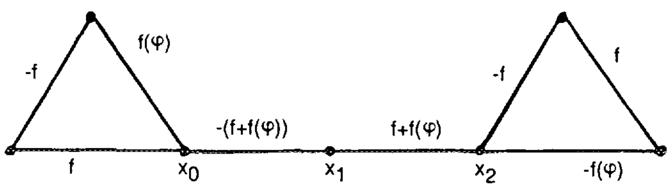

DIAGRAM 2

Proof of Theorem 2 *and Theorem* 2^0

We will consider only operator Δ_p because the case of Δ_A is the same. Also we assume that e_0 does not belong to a cycle, otherwise it is a special case of Theorem 1.

We define

$$\mathcal{H}_n = \{f \in l_2^{\#}(\mathrm{Int}\, G_n),\ \Delta_p f = \Delta_p(n) f \quad \text{or} \quad U_n^{\#} f = f\}.$$

We have $\mathcal{H}_n \subset \mathcal{H}_{n+1}$. Let us show that $\tilde{\mathcal{H}} = \bigcup_{n \geqslant 1} \mathcal{H}_n$ is complete in $\mathcal{H} = l_2^{\#}(V)$. For any $f \in \mathcal{H}$ there is such n that $\|f - f_n\| \leqslant \frac{1}{4}\|f\|$, where f_n is the restriction of f to V_n. Since $\varphi(e_0) \neq e_0$ we have $(U_{n+1}^{\#} f_n, f_n) = 0$ and so

$$\begin{aligned} |(f, f_n + U_{n+1}^{\#} f_n)| &\geqslant |(f_n, f_n + U_{n+1}^{\#} f_n)| - \|f - f_n\| \cdot \|f_n + U_{n+1}^{\#} f_n\| \\ &\geqslant \|f_n\|^2 - \frac{\sqrt{2}}{4}\|f_n\|^2 \geqslant \frac{3}{16}\|f\|^2 \end{aligned}$$

because $\|f_n\| \geqslant \frac{3}{4}\|f\|$ and $\|f_n + U_{n+1}^{\#} f_n\| = \sqrt{2}\|f_n\|$. This implies that $\tilde{\mathcal{H}}$ is complete since f is arbitrary and $f_n + U_{n+1}^{\#} f_n \in \tilde{\mathcal{H}}$.

Therefore we need only construct operator Φ_n which satisfies the conditions of Thoerem 3.1.

(i) One can see that the graph $\tilde{M}$ has two odd cycles joining by a path such that e_0 belongs to this path. In this case, Φ_n can be defined exactly the same way as in the proof of Theorem 1 for an odd cycle.

(ii) If, for example, α is incident to e_0, then there is a path $\alpha = x_0, x_1, \ldots, x_r = u_0$ and an odd cycle $\{u_n\}_{k=0}^{n}$, $u_0 = u_n$, where $e_0 = (x_0, x_1)$. Then Φ_n can be defined by

$$\begin{aligned} \Phi_n = \sum_{e \in E_x^+} (\Phi_n^e + \Phi_n^e \circ U_n^{\#}) - \sum_{e \in E_x^-} (\Phi_n^e + \Phi_n^e \circ U_n^{\#}) \\ + (-1)^r \left(\sum_{e \in E_u^+} \Phi_n^e - \sum_{e \in E_u^-} \Phi_n^e \right), \end{aligned}$$

where Φ_n^e, E_x^+, E_x^-, E_u^+, E_u^- are defined the same way as in the proof of Theorem 1.

If α is not incident with e_0 the proof is analogously (i). The theorem is proved. ▮

Proof of Theorem 3

At first we suppose that α, β are not from the set $\{y_0, y_2\}$. Without loss of generality we can assume that $d_{\alpha_n}(G_n) < d_{\beta_n}(G_{n+1})$ and $\Psi_n^{(y_1, y_2)}(\beta_n) = \beta_n$. Let us define

$$\mathcal{H}_n = \{f \in l_2^{\#}(G) : f(x) = 0 \text{ if } x \in V \backslash (V_n \backslash \beta_n)\}.$$

The operator $\Phi_n: \mathscr{H}_n \to \mathscr{H}_{n+1}$ can be given by the formula

$$\Phi_n(f)(x) = \begin{cases} f(x) & \text{if } \ x \in V_n \\ -f(x) & \text{if } \ x \in \Psi_n^{(y_1, y_2)}(y), y \in G_n. \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

If $\alpha = y_0$ the definition of the operator Φ_n is the same.

Let $\alpha = y_2$. Then we have to consider the graph $\tilde{M}$ (Lemma 1.2) instead of M which has the necessary properties to construct Φ_n by the formula (4.2). The theorem is proved. ▮

Proof of Theorem 4

The proof is one-to-one to the proof of Theorems 1–2^0, 3.

Proof of Theorem 5

It is easy to see that if Ψ is an eigenfunction of the operator Δ with compact support and supp $\Psi \cap$ supp $W = \varnothing$ then the function Ψ is an eigenfunction of the operator H_β.

Let Λ be a set of all eigenvalues of the Δ and let S be a countable set of orthonormal eigenfunctions of the Δ with compact support. For every $\lambda \in \Lambda$ there is an eigenfucnction $f \in S$ and the integer n_0 such that $\operatorname{supp} f \subset G_{n_0}$.

We note that graph G can be written as the union of copies of G_{n_0}. With probability one there is an infinity set of disjoint copies of G_{n_0} where W is zero. Consequently λ is an eigenvalue of the operator H_β of infinite multiplicity. The theorem is proved. ▮

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