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FRACTAL LAPLACIANS ON THE UNIT INTERVAL

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RÉSUMÉ. Nous étudions les valeurs/fonctions propres du laplacien sur $[0,1]$ définies par des mesures positives μ bornées, continues, supportées par $[0,1]$ et par la forme de Dirichlet classique sur $[0,1]$. Nous donnons des preuves simples d'existence, d'unicité, de concavité ainsi que des propriétés des zéros de ces fonctions propres. Par une récritures des équations définissant le laplacien comme une équation intégrale de Volterra-Stieltjes, nous étudions les comportements asymptotique des premières valeurs/fonctions propres de Neumann et de Dirichlet, lorsque la mesure μ varie. Nous étudions les bornes du domaine des valeurs propres, dès que μ possède une structure autosimilaire finie post-critique. Lorsque μ appartient à la classe des mesures autosimilaires sur $[0,1]$, nous décrivons à la fois, la méthode des éléments finis et la méthode des approximations par différences, afin d'obtenir des approximations numériques des valeurs/fonctions propres. Les fonctions propres en question, peuvent être considérées comme des analogues fractals du sinus et du cosinus de l'analyse de Fourier. Nous notons l'existence d'une sous-suite de fonctions propres à décroissance rapide indexées par les nombres de Fibonacci.

ABSTRACT. We study the eigenvalues and eigenfunctions of the Laplacians on $[0,1]$ which are defined by bounded continuous positive measures μ supported on $[0,1]$ and the usual Dirichlet form on $[0,1]$. We provide simple proofs of the existence, uniqueness, concavity, and properties of zeros of the eigenfunctions. By rewriting the equation defining the Laplacian as a Volterra-Stieltjes integral equation, we study asymptotic behaviors of the first Neumann and Dirichlet eigenvalues and eigenfunctions as the measure μ varies. For μ defined by a class of post critically finite self-similar structures, we also study asymptotic bounds of the eigenvalues. By restricting μ to a class of singular self-similar measures on $[0,1]$, we describe both the finite element and the difference approximation methods to approximate numerically the eigenvalues and eigenfunctions. These eigenfunctions can be considered fractal analogs of the classical Fourier sine and cosine functions. We note the existence of a subsequence of rapidly decaying eigenfunctions that are numbered by the Fibonacci numbers.

1. Introduction. Let μ be a continuous positive finite measure with support $\operatorname{supp}(\mu) = [0,1]$. In particular we are interested in the case when μ is a self-similar (fractal) measure. In this paper we study the eigenvalues λ and eigenfunctions u of the following equation:

$$\int_0^1 u'(x)v'(x)\,dx = \lambda \int_0^1 u(x)v(x)\,d\mu(x), \tag{1.1}$$

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where the equality holds for all $v \in C_0^\infty(0,1)$, the space of all infinitely differentiable functions with support contained in $(0,1)$. We impose either the Neumann boundary conditions

$$u'(0) = u'(1) = 0 \tag{1.2}$$

or the Dirichlet boundary conditions

$$u(0) = u(1) = 0. \tag{1.3}$$

The left side of (1.1) is the standard Dirichlet form

$$\mathcal{E}(u,v) := \int_0^1 u'(x)v'(x)\,dx.$$

In the Neumann case, the domain of $\mathcal{E}$, Dom$(\mathcal{E})$, is the Sobolev space $W^{1,2}(0,1)$ of functions u whose distributional derivative u' belongs to $L^2((0,1),dx)$. Such functions must be continuous and representable as

$$u(x) = c + \int_0^x g(y)\,dy, \quad x \in [0,1],$$

where $g \in L^2((0,1),dx)$ and $u' = g$. In the Dirichlet case,

$$\text{Dom}(\mathcal{E}) = W_0^{1,2}(0,1) := \big\{u \in W^{1,2}(0,1) : u(0) = u(1) = 0\big\}.$$

(See, e.g., [D].)

Throughout this paper we let $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the $L^2((0,1),\mu)$ and the μ-essential supremum norms, respectively.

Since $C_0^\infty(0,1) \subseteq W_0^{1,2} \subseteq W^{1,2} \subseteq L^2((0,1),\mu)$, Dom$(\mathcal{E})$ is dense in $L^2((0,1),\mu)$. Moreover, the embeddings $W_0^{1,2} \hookrightarrow L^2((0,1),\mu)$ and $W^{1,2} \hookrightarrow L^2((0,1),\mu)$ are compact. Therefore, the quadratic form $\mathcal{E}$ is closed. Hence, equation (1.1) defines a Laplacian $\Delta_\mu u$ as a distribution such that

$$\int_0^1 u'v'\,dx = \int_0^1 (-\Delta_\mu u)v\,d\mu$$

for all $v \in C_0^\infty(0,1)$. We can rewrite (1.1) as

$$-\Delta_\mu u = \lambda u.$$

In general, the Laplacian Δ_μ with domain Dom(Δ_μ) is defined as follows: for a continuous f and $u \in$ Dom(Δ_μ) we have

$$\Delta_\mu u = f \qquad \text{if and only if} \qquad u'' = f d\mu \tag{1.4}$$

in the distributional sense (Theorem 2.1). The domains of the Dirichlet Laplacian Δ_μ^D and Neumann Laplacian Δ_μ^N will be characterized in Section 3. We remark that Freiberg

[F1, F2, F3] has recently developed a theory for a general class of Laplace operators that includes the Laplacian Δ_μ.

One of the main motivations for studying equation (1.1) comes from the study of similar problems on the Sierpinski gasket. In [DSV], eigenfunctions are computed explicitly. However, because of the high multiplicities, no effective algorithm has been found for the expansion of an arbitrary function in terms of the eigenfunctions. The equation we consider here provides a simpler model on a self-similar set.

In Section 2 we give a concise summary of the fundamental properties of the eigenvalues and eigenfunctions. We show that u and u' are continuous and they have only isolated zeros (unless u is constant). Moreover, the Dirichlet and Neumann eigenvalues are simple. We also study the Sturm-Liouville theory and show that an nth Neumann eigenfunction has n zeros and its derivative has $n+1$ zeros, while the nth Dirichlet eigenfunction has $n+1$ zeros and its derivative has n zeros. Furthermore, for each eigenfunction u the zeros of u and u' alternate, and the zeros of the nth Dirichlet and Neumann eigenfunctions alternate. By converting equation (1.1) into an integral equation (see Section 3), results in this section can be derived from classical known results (see Atkinson [A]). For completeness, we include short proofs of the fundamental results.

Equation (1.1) can be written as a Volterra-Stieltjes integral equation (see Section 3):

$$u(x) = u(0) + u'(0)x - \lambda \int_0^x (x-y)u(y)\,d\mu(y) \quad \text{for all } 0 \le x \le 1. \tag{1.5}$$

We interpret $u'(0)$ and $u'(1)$ as the left-hand and right-hand derivatives, respectively. It is known (see [A]) that a solution u of (1.5) is differentiable and the derivative satisfies

$$u'(x) = u'(0) - \lambda \int_0^x u(y)\,d\mu(y) \quad \text{for all } 0 \le x \le 1. \tag{1.6}$$

Conversely, any solution of (1.6) is also a solution of (1.5) (see Theorem 3.1).

In view of equations (1.5) and (1.6), equation (1.1) is a generalization of the classical Sturm-Liouville equation

$$u''(x) = -\lambda g(x)u(x) \quad \text{for all } 0 \le x \le 1, \tag{1.7}$$

where u'' is assumed to exist at every $x \in [0,1]$ and $g \in L^1[0,1]$. If we define, for all Borel subsets $E \subseteq [0,1]$,

$$\mu(E) = \int_E g(x)\,dx,$$

then, since u' is differentiable at every $x \in [0,1]$ and $gu \in L^1[0,1]$, we have

$$u'(x) - u'(0) = \int_0^x u''(y)\,dy = -\lambda \int_0^x g(y)u(y)\,dy = -\lambda \int_0^x u(y)\,d\mu(y).$$

Hence, u satisfies (1.6). Conversely, if μ is absolutely continuous with Radon-Nikodym derivative $g \in L^1[0,1]$, then a solution u of (1.6) satisfies

$$u'(x) = u'(0) - \lambda \int_0^x g(y)u(y)\,dy$$

with $gu \in L^1[0,1]$. Hence $u''(x) = -\lambda g(x)u(x)$ for a.e. $x \in [0,1]$. Thus, u satisfies (1.7). In the special case $g(x) = 1$ on $[0,1]$, solutions of (1.7) are the classical Fourier sine and cosine functions. Thus, for μ singular, solutions of (1.1) are the eigenfunctions of a vibrating string with end-points at $0, 1$ and with a mass distribution given by the singular measure μ.

For the sine and cosine functions, the regions bounded by these functions and the interval on the x-axis between two successive zeros is convex. By using equations (1.5) and (1.6) we can easily obtain similar concavity results for the eigenfunctions of (1.1) (see Propositions 3.3 and 3.4).

We are interested in how eigenvalues λ and eigenfunctions u in equation (1.1) are affected by varying the measure μ. Assume $\{\mu_p : 1 < p < 1\}$ is a family of measures such that for each fixed $c \in (0,1)$, $\mu_p[0,c] \to 0$ as $p \to 1$. A typical example for such a class is provided by the iterated function system consisting of two similitudes

$$S_1(x) = r_1 x, \quad S_2(x) = r_2 x + (1 - r_2), \quad 0 < r_1, r_2 < 1,$$

and with $\mu = \mu_p$ defined by

$$\mu = p\mu \circ S_1^{-1} + (1-p)\mu \circ S_2^{-1}, \quad 0 < p < 1.$$

Let $z_1 = z_1(p)$ be the zero of the first Neumann eigenfunction u_p, and let λ_p be the first eigenvalue. In Section 4 we prove that $z_1(p) \to 1$ as $p \to 0$ and that $\lambda_p \to \infty$ as $p \to 0$. If we assume in addition that the first Neumann eigenfunctions u_p are normalized such that $\|u_p\|_2 = 1$ for all $p \in (0,1)$, then we show that $\|u_p'\|_\infty \to \infty$ as $p \to 0$. For the Dirichlet case, we show that the results $\lim_{p\to 0} \lambda_p = \infty$ and $\lim_{p\to 0} \|u_p'\|_\infty = \infty$ also hold for the first eigenvalue and (normalized) eigenfunction. However, if $z_2 = z_2(p) \in (0,1)$ is such that $u_p(z_2)$ is the maximum of the first Dirichlet eigenfunction, we show in Example 4.7 that the analog $\lim_{p\to 0} u_p(z_2) = 1$ need not hold.

Although many results concerning equations (1.2) and (1.3) have been obtained for general measures μ (see [A]), the restriction of μ to self-similar measures allows us to obtain good numerical approximations to the solutions and enables us to observe some interesting phenomena. To study the asymptotics of the eigenvalues, we restrict μ to be a self-similar measure defined by a *post-critically finite* (see [K1, K2]) self-similar structure. Let

$$S_1(x) = rx, \quad S_2(x) = rx + (1 - r), \quad 0 < r < 1. \tag{1.8}$$

Let μ be the self-similar measure satisfying the identity

$$\mu = p\mu \circ S_1^{-1} + (1-p)\mu \circ S_2^{-1}, \quad 0 < p < 1. \tag{1.9}$$

The *spectral dimension* (see [K1]), d_s, for this μ is given by

$$(pr)^{d_s/2} + \big((1-p)(1-r)\big)^{d_s/2} = 1.$$

Let $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ denote respectively the sets of reals, rationals, integers, and natural numbers. The *eigenvalue counting function* $\rho : \mathbb{R} \to \mathbb{N}$ is defined as

$$\rho(x) = \#\{\lambda : \lambda \text{ is an eigenvalue and } \lambda \le x\}.$$

Let λ_n be the nth Dirichlet or Neumann eigenvalue. In Section 5 we study some asymptotic properties of $\rho(\lambda_n)$ and λ_n.

In Section 6 we discuss numerical solutions to equation (1.1). To approximate the eigenvalues and eigenfunctions numerically, we further restrict μ to be a self-similar measure defined by the iterated function system

$$S_1(x) = \tfrac{1}{2}\,x, \quad S_2(x) = \tfrac{1}{2}\,x + \tfrac{1}{2}. \tag{1.10}$$

By making use of the identity

$$\int_0^1 f(x)\,d\mu = p\int_0^1 f(S_1x)\,d\mu + (1-p)\int_0^1 f(S_2x)\,d\mu,$$

we can solve equation (1.1) for $u, v \in S_m$, the space of bounded continuous piecewise linear functions with knots at $k/2^m$, $k = 0, 1, \dots, 2^m$, where m is any positive integer. This allows us to set up a generalized eigenvalue system

$$M_m\boldsymbol{u} = \lambda N_m\boldsymbol{u},$$

with the solutions $\boldsymbol{u}$ approximating the Neumann eigenfunctions u. In addition to the finite element method above, we also describe the difference approximation method (see Section 6). We show graphs of some approximate eigenfunctions. We observe that there exists a subsequence of rapidly decaying eigenfunctions that are numbered by the Fibonacci numbers.

2. Fundamental properties of solutions. In this section we summarize the basic existence and uniqueness results and study the oscillation properties of the eigenfunctions.

Theorem 2.1. *Let μ be a positive bounded continuous measure on $[0,1]$ such that* supp$(\mu) = [0,1]$*. Then the following hold.*

(a) *The initial value problem*

$$\begin{cases} -\Delta_\mu u = \lambda u \\ u(y) = \alpha \\ u'(y) = \beta \end{cases}$$

has a unique solution. In particular, if $u(y) = u'(y) = 0$ then $u(x) \equiv 0$.

(b) $-\Delta_\mu u = \lambda u$ *if and only if* $-u'' = \lambda\mu u$ *in the sense of generalized functions.*

(c) u *is continuously differentiable;* u' *is differentiable a.e. on* $[0,1]$.

(d) u *and* u' *have only isolated zeros (unless* $u \equiv$ *constant).*

Proof. (a) In Section 3 we will prove that this initial value problem is equivalent to an integral equation, for which the existence and uniqueness of solutions is known (see, for example, [A]).

There is a different approach that is more relevant to analysis on fractals (see [K2] for more details). We define Green's function for the interval $[0,1]$ by

$$g(x,y) = \begin{cases} x(1-y) & \text{if } x \le y \\ y(1-x) & \text{if } x \ge y \end{cases}$$

and Green's operator by

$$G_\mu f(x) = \int_0^1 g(x,y) f(y) d\mu(y).$$

For any continuous function f, we have that $G_\mu f$ is Lipschitz continuous, has zero boundary values and satisfies

$$-\Delta_\mu G_\mu f = f.$$

For any subinterval $[t, t+\delta]$ we can define Green's operator $G_{\mu,[t,t+\delta]}$ by

$$G_{\mu,[t,t+\delta]} f(x) = \delta^2 \int_t^{t+\delta} g(\tfrac{x-t}{\delta}, \tfrac{y-t}{\delta}) f(y) d\mu(y).$$

If δ is small enough (depending on λ), then u has a unique representation on $[t, t+\delta]$ as

$$u = \sum_{n=0}^{\infty} \left(\lambda G_{\mu,[t,t+\delta]}\right)^n h \tag{2.1}$$

where $h(x)$ is a linear function such that $h(t) = u(t),\ h(t+\delta) = u(t+\delta)$.

Let u_1 be the solution with boundary values $\{0,1\}$ on $[t, t+\delta]$. If δ is small enough (depending on λ), then $u_1'(t) \neq 0$. This implies that $u(t) = u'(t) = 0$ if and only if $u \equiv 0$ on $[t, t+\delta]$. Hence we have uniqueness of the initial value problem on $[t, t+\delta]$. It is easy to extend it to $[0,1]$ by dividing into small subintervals.

(b) We have that $-\Delta_\mu u = \lambda u$ if and only if $\int_0^1 u'v'dx = \lambda \int_0^1 uv d\mu$ for any test function, which means $-u'' = \lambda \mu u$ as generalized functions.

(c) The equation $u'' = z\mu u$ implies that u' is of bounded variation and hence u' is differentiable a.e. on $[0,1]$.

(d) If δ is small enough, then on an interval $[s, s+\delta]$ the Dirichlet and the non-zero Neumann eigenvalues are larger than $|\lambda|$ (to see this, estimate $G_{\mu,[t,t+\delta]}$). Hence u and u' can have at most one zero each in $[s, s+\delta]$. □

Proposition 2.2. *Let $-u'' = \lambda \mu u$. Then*

$$\left(\frac{u}{u'}\right)' = 1 + \lambda\mu \left(\frac{u}{u'}\right)^2 \tag{2.2}$$

$$\left(\frac{u'}{u}\right)' = -\lambda\mu - \left(\frac{u'}{u}\right)^2 \tag{2.3}$$

as generalized functions.

Proof. Direct computation. □

Proposition 2.3. (Comparison of solutions) *Let $-u_i'' = \lambda_i u_i$, $u_i \not\equiv 0$, $i = 1, 2$ and suppose that $a_2 \le a_1$, $0 \le \lambda_1 \le \lambda_2$,*

$$b_i = \min\{1, \{s > a_i : u_i(s) = 0\}\}$$

$$c_i = \min\{1, \{s > a_i : u_i'(s) = 0\}\}.$$

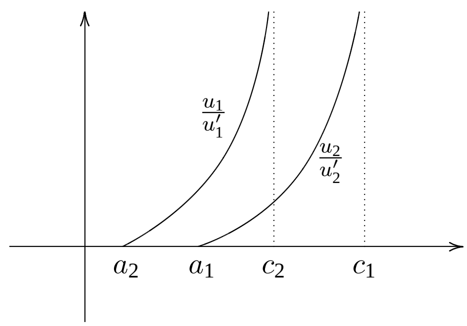
Then

(a) *If* $u_1(a_1) = u_2(a_2) = 0$, *then* $c_2 \le c_1$.
(b) *If* $u_1'(a_1) = u_2'(a_2) = 0$, *then* $b_2 \le b_1$.
(c) *If* $\lambda_1 = \lambda_2 > 0$, $a_1 = a_2 = 0$, $u_1(0) = 0$, $u_1'(0) \neq 0$, $u_2'(0) = 0$, $u_2(0) \neq 0$, *then* $b_2 \le b_1$, $c_2 \ge c_1$.

Proof. (a) According to (2.2), u_i/u_i' is an increasing function on $[a_i, c_i)$ and so

$$\left(\frac{u_1}{u_1'}\right)' \le \left(\frac{u_2}{u_2'}\right)'$$

on $[a_1, c_1) \cap [a_2, c_2)$, as on the illustration below.



Statements (b) and (c) are proved similarly by (2.2) and (2.3). □

Proposition 2.4. *Fix the initial conditions* $u(0), u'(0)$ *and let* $u'' = -\lambda\mu u$. *Then*

(a) *Zeros of* u *and* u' *move to the left as* λ *increases.*
(b) $f(\lambda) = \dfrac{u(1)}{u'(1)}$ *is an increasing function on each interval of* λ *where* $u'(1) \neq 0$.
(c) $g(\lambda) = \dfrac{u'(1)}{u(1)}$ *is a decreasing function on each interval of* λ *where* $u(1) \neq 0$.

Proof. By Proposition 2.3 (consider the intervals on which neither u nor u' have zeros. These intervals move to the left as λ increases). □

Theorem 2.5. *Assuming the same hypotheses as in Theorem 2.1, the following hold:*

(a) *The* n*th Neumann eigenfunction has* n *zeros and its derivative has* $n + 1$ *zeros.*
(b) *The* n*th Dirichlet eigenfunction has* $n + 1$ *zeros and its derivative has* n *zeros.*
(c) *For each eigenfunction* u, *zeros of* u *and* u' *alternate.*
(d) *Zeros of* n*th Dirichlet and Neumann eigenfunctions alternate.*
(e) *There exists a complete orthonormal basis consisting of the Dirichlet (Neumann) eigenfunctions. The Dirichlet (Neumann) eigenvalues* λ_n *are simple. Moreover,* $\lim_{n\to\infty} \lambda_n = \infty$.

Proof. Parts (a) to (d) follows from Propositions 2.3 and 2.4 (consider $u(1)/u'(1)$ and $u'(1)/u(1)$ as λ increases).

(e) The eigenfunctions form a complete orthonormal set because the Laplacian has a compact resolvent, which is Green's operator defined in the proof of Theorem 2.1. The eigenvalues tend to infinity by the same reason. Green's operator is compact because its kernel, Green's function $g(x, y)$ defined in the proof of Theorem 2.1, is continuous (see also [K2, Theorem B.1.13]). Another approach can be found in [A]. The simplicity of the eigenvalues follows from the uniqueness of the initial value problem in Theorem 2.1 (a). □

Remark. If the support of μ is smaller than $[0, 1]$, then the results of this section are still true with the following modification. A problem is that u is linear on the intervals that does not intersect the support of μ. The uniqueness implies that u cannot have an interval of zero values. However u' can have an interval of zero values, and in the results above such an interval should be taken as a single "zero".

3. Volterra-Stieltjes integral equation and concavity of solutions. Assuming that μ is a bounded continuous positive measure with $\operatorname{supp}(\mu) \subseteq [0, 1]$, we will show that (1.1) is equivalent to the following Volterra-Stieltjes integral equation (see [A, Chapter 11]):

$$u(x) = u(0) + u'(0)x - \lambda \int_0^x (x - y)u(y)\, d\mu(y), \quad 0 \le x \le 1. \tag{3.1}$$

It is known that a solution u of (3.1) is differentiable and the derivative satisfies

$$u'(x) = u'(0) - \lambda \int_0^x u(y)\, d\mu(y), \quad 0 \le x \le 1. \tag{3.2}$$

(See [A, Theorem 11.2.2].) The converse is also true. There is also a Green's function representation of u, essentially given in [S2], as follows

$$u(x) = u(0) + \big(u(1) - u(0)\big)x + \lambda \int_0^1 g(x, y)u(y)\, d\mu(y), \quad 0 \le x \le 1, \tag{3.3}$$

where $g(x, y)$ is as defined in Section 2. We summarize these in the following theorem and give a brief proof for completeness; details in a more general setting can be found in [F1] and [F2].

Theorem 3.1. *Let μ be a bounded continuous positive Borel measure with* $\operatorname{supp}(\mu) \subseteq [0, 1]$. *Then equations* (1.1), (3.1), (3.2) *and* (3.3) *are equivalent.*

Proof. Since the implication from (3.1) to (3.2) is known, we will show (3.2) implies (3.1). Let u be a solution of (3.2). Since u is continuous and μ is bounded, it follows from Fubini's Theorem that $u'(x) \in L^1[0, 1]$. By using Fubini's Theorem again,

$$\begin{aligned} u(x) - u(0) &= \int_0^x u'(s)\, ds = \int_0^x \Big(u'(0) - \lambda \int_0^s u(y)\, d\mu(y)\Big)\, ds \\ &= u'(0)x - \lambda \int_0^x \int_y^x u(y)\, ds d\mu(y) = u'(0)x - \lambda \int_0^x (x - y)u(y)\, d\mu(y). \end{aligned}$$

Hence u satisfies (3.1).

To see that (1.1) and (3.2) are equivalent, let u be a solution of (3.2). Then for any $v \in C_0^\infty(0,1)$, we have

$$\begin{aligned}\int_0^1 u'v'\,dx &= \int_0^1 \Big(u'(0) - \lambda\int_0^x u(y)\,d\mu(y)\Big)v'(x)\,dx \\ &= u'(0)\int_0^1 v'(x)\,dx - \lambda\int_0^1\int_0^x u(y)v'(x)\,d\mu(y)\,dx \\ &= -\lambda\int_0^1\int_y^1 u(y)v'(x)\,dx\,d\mu(y) \qquad \text{(Fubini's Theorem)} \\ &= \lambda\int_0^1 u(y)v(y)\,d\mu(y).\end{aligned}$$

Hence, $u(x)$ satisfies (1.1). The above also proves the converse.

Lastly, by using the definition of $g(x,y)$, we can rewrite equation (3.3) as

$$u(x) = u(0) + \big(u(1) - u(0)\big)x + x\int_0^1 (1-y)u(y)\,d\mu(y) - \lambda\int_0^x (x-y)u(y)\,d\mu(y).$$

It is easy to see that this is equivalent to (3.1). $\square$

The following corollary follows from the proof of Theorem 3.1.

Corollary 3.2. *Assume the same hypotheses on* μ *as in Theorem 3.1, and let* $u \in L^2([0,1],\mu)$. *Let* f *be a continuous function on* $[0,1]$. *Then the following conditions are equivalent.*

(a) *For all* $v \in C_0^\infty(0,1)$,

$$\int_0^1 u'v'\,dx = \int_0^1 vf\,d\mu.$$

(b) $u(x) = u(0) + u'(0)x + \int_0^x (x-y)f(y)\,d\mu(y)$, $0 \le x \le 1$.

(c) $u'(x) = u'(0) + \int_0^x f(y)\,d\mu(y), \quad 0 \le x \le 1.$

(d) $u(x) = u(0) + \big(u(1) - u(0)\big)x - \int_0^1 g(x,y)f(y)\,d\mu(y), \quad 0 \le x \le 1.$

Let Δ_μ^D and Δ_μ^N denote the Laplacian Δ_μ under the Dirichlet and Neumann boundary conditions respectively. Then their domains can be characterized, using Corollary 3.2, as follows. $u \in \mathrm{Dom}(\Delta_\mu^D)$ (resp. $u \in \mathrm{Dom}(\Delta_\mu^N)$) if and only if $u(0) = u(1) = 0$ (resp. $u'(0) = u'(1) = 0$) and there exists a continuous function f on $[0,1]$ ($f = \Delta_\mu^D(u)$ or $f = \Delta_\mu^N(u)$ respectively) satisfying any of the conditions in Corollary 3.2.

We now turn to concavity of the eigenfunctions. Although these results can be proved using the approach of Section 2, here we present another short proof, which is based in the use of integral equations.

Proposition 3.3. *Let μ be a bounded continuous positive Borel measure with* $\operatorname{supp}(\mu) \subseteq [0,1]$*, and let u be an nth Neumann eigenfunction of (3.1) satisfying $u(0) > 0$. Let $z_1 < z_2 < \cdots < z_n$ be the zeros of u and write $z_0 = 0$ and $z_{n+1} = 1$. Then u' is decreasing on $[z_i, z_{i+1}]$ for even i $(0 \le i < n+1)$ and u' is increasing on $[z_i, z_{i+1}]$ for odd i $(0 \le i < n+1)$. Consequently,*

(a) *If i $(0 \le i < n+1)$ is even, then u' has a local minimum at z_i and u is concave downward on $[z_i, z_{i+1}]$.*
(b) *If i $(0 < i < n+1)$ is odd, then u' has a local maximum at z_i and u is concave upward on $[z_i, z_{i+1}]$.*

Proof. Let i $(0 \le i < n+1)$ be even and let $z_i < x_1 < x_2 < z_{i+1}$. Since $u(x) > 0$ on (z_i, z_{i+1}), by using (3.2) we have

$$\begin{aligned} u'(x_1) &= -\lambda \int_0^{x_1} u(y)\, d\mu(y) = -\lambda \int_0^{z_i} u(y)\, d\mu(y) - \lambda \int_{z_i}^{x_1} u(y)\, d\mu(y) \\ &\ge -\lambda \int_0^{z_i} u(y)\, d\mu(y) - \lambda \int_{z_i}^{x_2} u(y)\, d\mu(y) = -\lambda \int_0^{x_2} u(y)\, d\mu(y) = u'(x_2). \end{aligned}$$

Hence u' is decreasing on $[z_i, z_{i+1}]$.

Now let i $(0 < i < n+1)$ be odd and let $z_i < x_1 < x_2 < z_{i+1}$. Since $u(y) < 0$ on (z_i, z_{i+1}), a similar argument shows that $u'(x_1) \le u'(x_2)$. Therefore u' is increasing on $[z_i, z_{i+1}]$. Consequences (a) and (b) follow directly from these results. □

Analogous results hold for Dirichlet eigenfunctions.

Proposition 3.4. *Let μ be a bounded continuous positive measure with* $\operatorname{supp}(\mu) \subseteq [0,1]$ *and let u be an nth Dirichlet eigenfunction of (3.1) satisfying $u'(0^+) > 0$. Let $0 = z_0 < z_1 < z_2 < \cdots < z_n = 1$ be the zeros of u. Then u' is increasing on $[z_i, z_{i+1}]$ for even i $(0 \le i < n)$ and u' is decreasing on $[z_i, z_{i+1}]$ for odd i $(0 \le i < n)$. Consequently,*

(a) *If i $(0 \le i < n)$ is even, then u' has a local maximum at z_i and u is concave downward on $[z_i, z_{i+1}]$.*
(b) *If i $(0 < i < n)$ is odd, then u' has a local minimum at z_i and u is concave upward on $[z_i, z_{i+1}]$.*

Proof. Use $u'(x) = u'(0) - \int_0^x u(y)\, dy$ instead of $u'(x) = -\int_0^x u(y)\, dy$ and use the same argument as in the proof of Proposition 3.3. □

4. The first eigenvalues and eigenfunctions. In this section we study behaviors of the eigenfunctions and eigenvalues as the measure varies. We will focus on properties of the first Dirichlet and (non-zero) Neumann eigenvalues and the associated eigenfunctions.

We will restrict our attention to families $\{\mu_p : 0 < p < 1\}$ of bounded continuous measures satisfying the condition that for any fix $c \in (0,1)$,

$$\mu_p[0,c] \to 0 \quad \text{as} \quad p \to 0.$$

An example of such a family is provided by self-similar measures defined by an iterated function system of two similitudes of the form

$$S_1(x) = r_1 x, \quad S_2(x) = r_2 x + (1 - r_2), \quad 0 < r_1, r_2 < 1.$$

Let μ be the self-similar measure satisfying the identity

$$\mu = p\mu \circ S_1^{-1} + (1-p)\mu \circ S_2^{-1}, \quad 0 < p < 1. \tag{4.1}$$

Note that μ depends on p.

Proposition 4.1. *Let* $\mu = \mu_p$ *be defined as in* (4.1). *Then for any fixed* $c \in (0,1)$, $\mu[0,c] \to 0$ *as* $p \to 0$.

Proof. Applying (4.1), we get

$$\begin{aligned}\mu[0,c] = p\mu[0, r_1^{-1}c] + (1-p)\mu[-(1-r_2)/r_2,\ c/r_2 - (1-r_2)/r_2]\\ \le p + (1-p)\mu[0, c/r_2 - (1-r_2)/r_2].\end{aligned}$$

Define $c_0 = c, c_1 = c_0/r_2 - (1-r_2)/r_2$, and in general define $c_{i+1} = c_i/r_2 - (1-r_2)/r_2$. Then by applying the above calculations repeatedly, we have

$$\mu[0,c] \le p + p(1-p) + p(1-p)^2 + \cdots + p(1-p)^i + (1-p)^{i+1}\mu[0, c_{i+1}].$$

It is easy to see that $c_i - c_{i+1} = (1-r_2)/r_2^{i+1}(1-c_0)$. In fact,

$$c_i - c_{i+1} = \frac{c_{i-1} - c_i}{r_2} = \cdots = \frac{c_0 - c_1}{r_2} = \frac{1 - r_2}{r_2^{i+1}}(1 - c_0).$$

Consequently there must exist some i_o such that $c_{i_o+1} < 0$ and therefore

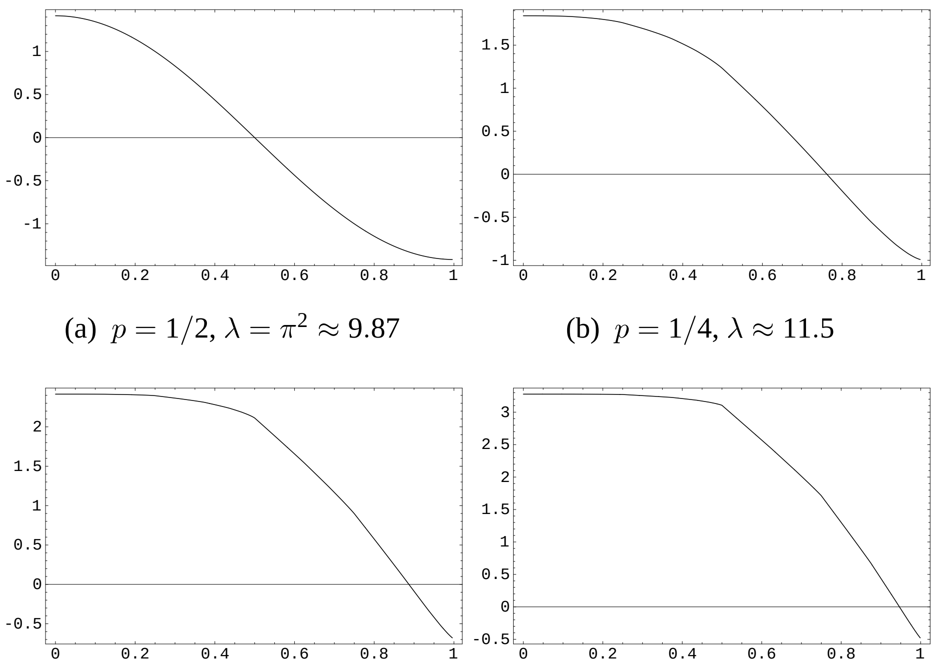
$$\mu[0,c] \le p\left(\frac{1-(1-p)^{i_o+1}}{1-(1-p)}\right) = 1 - (1-p)^{i_o+1},$$

which tends to 0 as $p \to 0$. □

Let λ denote the first Dirichlet or nonzero Neumann eigenvalue and let u denote a corresponding eigenfunction. According to Theorem 2.1 (e), λ is a simple eigenvalue and therefore all associated eigenfunctions are scalar multiples of each other. Moreover, according to Theorem 2.5 (a), the first Neumann eigenfunctions have a unique common zero in $(0,1)$, which we will denote by z_1. Figure 1 shows the behavior of the first Neumann eigenfunctions as the measure varies.

Note that equation (3.2) and the boundary condition $u'(1) = 0$ together imply that $\int_0^1 u\, d\mu = 0$ and consequently

$$\int_0^{z_1} u\, d\mu = \int_{z_1}^1 |u|\, d\mu. \tag{4.2}$$



(a) $p = 1/2$, $\lambda = \pi^2 \approx 9.87$ (b) $p = 1/4$, $\lambda \approx 11.5$

(c) $p = 1/8$, $\lambda \approx 16.5$ (d) $p = 1/16$, $\lambda \approx 27.2$

Figure 1. Approximate first $L^2(\mu)$-normalized Neumann eigenfunctions u and eigenvalues λ as p varies, where μ is defined by (4.1) with $r = 1/2$

Theorem 4.2. *Let $\{\mu_p : 0 < p < 1\}$ be a family of continuous probability measures with* $\operatorname{supp}(\mu_p) \subseteq [0, 1]$ *such that for any fixed $c \in (0, 1)$, $\mu_p[0, c] \to 0$ as $p \to 0$. Let $z_1 = z_1(p)$ be the common zero of the Neumann eigenfunctions associated to the first eigenvalue. Then $z_1 \to 1$ as $p \to 0$.*

Proof. For each p, we let $u = u_p$ be the first eigenfunction satisfying $u(0) = 1$ and hence

$$u(x) = 1 - \lambda \int_0^x (x - y) u(y)\, d\mu(y). \tag{4.3}$$

Suppose $z_1(p)$ does not tend to 0 as $p \to 0$. Then there would exist $0 < b < 1$ and a sequence $\{p_k\}$ such that $p_k \to 0$ and $z_1 = z_1(p_k) \le b$ for all k. We notice by assumptions that

$$\int_0^{z_1} u_{p_k}\, d\mu_{p_k} \le \int_0^b 1\, d\mu_{p_k} = \mu_{p_k}[0, b] \to 0 \quad \text{as} \quad k \to \infty.$$

Hence by (4.2),

$$\int_{z_1}^1 |u_{p_k}|\, d\mu_{p_k} = \int_0^{z_1} u_{p_k}\, d\mu_{p_k} \to 0 \quad \text{as} \quad k \to \infty. \tag{4.4}$$

Since $\mu_{p_k}[z_1, 1] \geq \mu_{p_k}[b, 1] \to 1$ as $k \to \infty$, (4.4) implies that for all $x \in [z_1, 1]$,

$$0 \geq u_{p_k}(x) \geq u_{p_k}(1) \to 0 \quad \text{as} \quad k \to \infty, \tag{4.5}$$

for otherwise, the concavity of u_{p_k} on $[z_1, 1]$ would force the integral on the left-hand side of (4.4) to be greater than some positive constant independent of k.

Let $c = (b+1)/2$. Then the concavity of u_{p_k} and (4.5) imply that

$$|u'_{p_k}(c)| \leq \frac{|u_{p_k}(c)|}{c - z_1} \to 0 \quad \text{as} \quad k \to \infty. \tag{4.6}$$

We will show that (4.6) is impossible. By Proposition 3.3, $|u'_{p_k}|$ is increasing on $[0, z_1]$. By applying the Mean-Value Theorem to this interval, we get

$$\left|u'_{p_k}(z_1)\right| \geq \frac{1}{z_1} \geq \frac{1}{b} > 0. \tag{4.7}$$

Also, by using (4.2),

$$\begin{aligned}\left|u'_{p_k}(z_1)\right| &= \lambda_{p_k} \int_0^{z_1} u_{p_k}(y)\, d\mu_{p_k}(y) = \lambda_{p_k} \int_{z_1}^1 |u_{p_k}(y)|\, d\mu_{p_k}(y) \\ &= \lambda_{p_k} \int_{z_1}^c |u_{p_k}(y)|\, d\mu_{p_k}(y) + \lambda_{p_k} \int_c^1 |u_{p_k}(y)|\, d\mu_{p_k}(y),\end{aligned}$$

with the second term dominating. In fact,

$$\frac{\int_{z_1}^c |u_{p_k}(y)|\, d\mu_{p_k}(y)}{\int_c^1 |u_{p_k}(y)|\, d\mu_{p_k}(y)} \leq \frac{|u_{p_k}(c)|\mu_{p_k}[z_1, c]}{|u_{p_k}(c)|\mu_{p_k}[c, 1]} \to 0 \quad \text{as} \quad k \to \infty.$$

Combining these observations with (4.7) leads to

$$\lim_{k\to\infty} \left|u'_{p_k}(c)\right| = \lim_{k\to\infty} \lambda_{p_k} \int_c^1 \left|u_{p_k}(y)\right| d\mu_{p_k}(y) = \lim_{k\to\infty} \left|u'_{p_k}(z_1)\right| \geq \frac{1}{b},$$

contradicting (4.6). The first equality is because for any p

$$\begin{aligned}u'_p(c) &= -\lambda_p \int_0^c u_p(y)\, d\mu_p(y) = -\lambda_p \int_0^{z_1} u_p(y)\, d\mu_p(y) - \lambda_p \int_{z_1}^c u_p(y)\, d\mu_p(y) \\ &= -\lambda_p \int_{z_1}^1 |u_p(y)|\, d\mu_p(y) - \lambda_p \int_{z_1}^c u_p(y)\, d\mu_p(y) = -\lambda_p \int_c^1 |u_p(y)|\, d\mu_p(y).\end{aligned}$$

This proves the result. □

Theorem 4.3. *Assume the same hypotheses as in Theorem 4.2 and let $\lambda = \lambda_p$ be the first Neumann eigenvalue of the corresponding equation (3.1). Then $\lambda \to \infty$ as $p \to 0$.*

Proof. We assume, as in the proof of Theorem 4.2, that for all p, u_p are chosen so that $u_p(0) = 1$ and (4.3) holds. We will prove the assertion by contradiction. Suppose there exists a sequence $\{p_k\}$ with $p_k \to 0$ such that λ_{p_k} is bounded by some $M > 0$. Then, by (3.2), for all $x \in [0, z_1]$,

$$|u'_{p_k}(x)| \leq M \cdot 1 \cdot \mu_{p_k}[0, x] \leq M. \tag{4.8}$$

(The first inequality is because $u(x) \leq u(0) = 1$ on $[0, z_1]$.) Now, fix any number N sufficiently large so that $N > 2M$ and $1 - 1/N > 0$. Then for any $x \in [0, 1 - 1/N]$,

$$\left|u'_{p_k}(x)\right| \leq M\mu_{p_k}[0, x] \leq M\mu_{p_k}\left[0, 1 - \frac{1}{N}\right] \to 0 \quad \text{as} \quad k \to \infty.$$

Hence, $u_{p_k} \to 1$ uniformly on $[0, 1 - 1/N]$ as $k \to \infty$. In particular, there exists some $k_o \in \mathbb{N}$ such that for all $k \geq k_o$,

$$u_{p_k}(x) > \frac{1}{2} \quad \text{on} \quad \left[0, 1 - \frac{1}{N}\right].$$

Since $u(z_1) = 0$ and $z_1 < 1$, the Mean-Value Theorem implies that there exists some $\xi_{p_k} \in [1 - 1/N, z_1]$ such that

$$\left|u'(\xi_{p_k})\right| \geq \frac{1/2}{1/N} = \frac{N}{2} > M,$$

contradicting (4.8). Thus $\lambda_p \to \infty$ as $p \to 0$. $\square$

Theorem 4.4. *Assume the same hypotheses as in Theorem 4.2 and assume in addition that* $\|u_p\|_2 = 1$. *Then*

$$\|u'_p\|_\infty = \left|u'_p(z_1)\right| = \lambda \int_0^{z_1} u_p(y)\, d\mu(y) = \lambda \int_{z_1}^1 \left|u_p(y)\right| d\mu_p(y) \to \infty \quad \textit{as} \quad p \to 0.$$

Proof. We prove by contradiction. Suppose there exists a subsequence $\{u_{p_k}\}$, with $p_k \to 0$ as $k \to \infty$, and a positive constant C such that $|u'_{p_k}(z_1)| \leq C$ for all k. Obviously, this can happen only if $u_{p_k}(1) \to 0$ as $k \to \infty$. Hence, for all k sufficiently large we have

$$\int_{z_1}^1 u_{p_k}(y)^2\, d\mu_{p_k}(y) \leq \int_{z_1}^1 \left|u_{p_k}(y)\right| d\mu_{p_k}(y) \to 0 \quad \text{as} \quad k \to \infty. \tag{4.9}$$

By assumption,

$$\int_0^{z_1} u_{p_k}(y)^2\, d\mu_{p_k}(y) + \int_{z_1}^1 u_{p_k}(y)^2\, d\mu_{p_k}(y) = 1. \tag{4.10}$$

Therefore by (4.2), (4.9), and (4.10) we have

$$\int_0^{z_1} u_{p_k}(y)\, d\mu_{p_k}(y) \to 0 \quad \text{as} \quad k \to \infty \tag{4.11}$$

$$\int_0^{z_1} u_{p_k}(y)^2\, d\mu_{p_k}(y) \to 1 \quad \text{as} \quad k \to \infty. \tag{4.12}$$

Let $x_1 = x_1(p_k) \in [0, z_1]$ be the unique number such that $u_{p_k}(x_1) = 1$. Such an x_1 exists; otherwise we would have $u_{p_k}(0) \le 1$ and therefore

$$\int_0^{z_1} u_{p_k}(y)^2 \, d\mu_{p_k}(y) \le \int_0^{z_1} u_{p_k}(y) \, d\mu_{p_k}(y) \to 0 \quad \text{as} \quad k \to \infty,$$

contradicting (4.12). Consider the following two cases.
Case 1. $x_1 \to z_1$ as $k \to \infty$. In this case we have $\lim_{k\to\infty} x_1(p_k) = \lim_{k\to\infty} z_1 = 1$ and therefore $\lim_{k\to\infty} |u'_{p_k}(z_1)| = \infty$, a contradiction.
Case 2. $x_1 \not\to z_1$ as $k \to \infty$. By taking a subsequence if necessary we may assume that $x_1(p_k) \le c < 1$ for some c and for all p_k. In this case,

$$\int_{x_1}^{z_1} u_{p_k}(y)^2 \, d\mu_{p_k}(y) \le \int_{x_1}^{z_1} u_{p_k}(y) \, d\mu_{p_k}(y) \to 0 \quad \text{as} \quad k \to \infty \qquad \text{(by (4.11))}.$$

This forces

$$\int_0^{x_1} u_{p_k}(y)^2 \, d\mu_{p_k}(y) \to 1 \quad \text{as} \quad k \to \infty \qquad \text{(by (4.12))}.$$

But $\mu_{p_k}[0, x_1] \le \mu_{p_k}[0, c] \to 0$ as $k \to \infty$, and therefore $u_{p_k}(0) \to \infty$, forcing $|u'_{p_k}(\xi_{p_k})| \to \infty$ for some $\xi_{p_k} \in [0, x_1]$. Consequently, $\lim_{k\to\infty} |u'_{p_k}(z_1)| = \infty$ because $u'_{p_k}(x)$ is decreasing on $[0, z_1]$. This again contradicts the assumption and the proof is complete. □

We now turn to properties of the first Dirichlet eigenvalues and eigenfunctions. Figure 2 shows rather striking behavior of the first Dirichlet eigenfunctions as the measure varies.

Theorem 4.5. *Let* $\{\mu_p : 0 < p < 1\}$ *be a family of continuous probability measures with* $\operatorname{supp}(\mu) \subseteq [0, 1]$ *such that for any fixed* $c \in (0, 1)$, $\mu_p[0, c] \to 0$ *as* $p \to 0$. *Let* λ_p *be the first Dirichlet eigenvalue. Then* $\lim_{p\to 0} \lambda_p = \infty$.

Proof. For $0 < p < 1$, let u_p be the first Dirichlet eigenfunction satisfying $u'_p(0) = 1$. Then the concavity of u_p forces

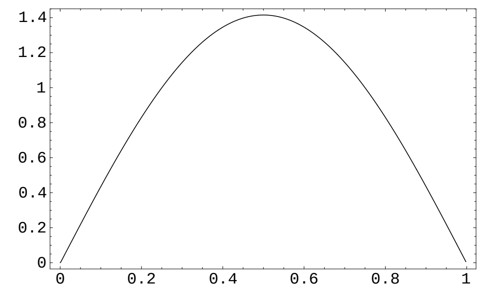
$$u_p(x) \le 1 \quad \text{for all} \quad 0 \le x \le 1. \tag{4.13}$$

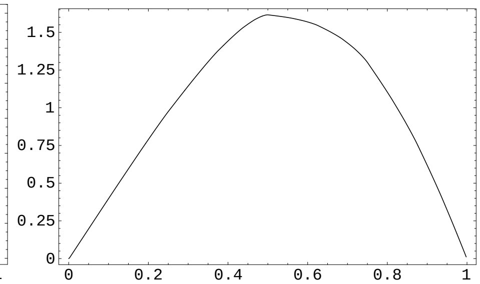
Also, by putting $x = 1$ into (3.1), we have

$$\lambda_p \int_0^1 (1 - y) u_p(y) \, d\mu_p(y) = 1. \tag{4.14}$$

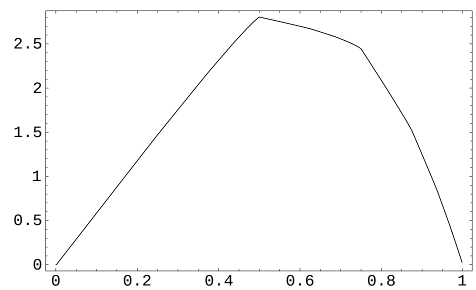
We prove the assertion by contradiction. Suppose there exists a constant $C > 0$ and a sequence $\{p_k\}$ such that $\lim_{k\to\infty} p_k = 0$ but $\lambda_{p_k} \le C$ for all k. By replacing with a subsequence if necessary, we can assume that there exists a sequence $\{c_{p_k}\}$ with $0 < c_{p_k} < 1$ such that

$$\lim_{k\to\infty} c_{p_k} = 1 \tag{4.15}$$





(a) $p = 1/2$, $\lambda = \pi^2 \approx 9.87$ (b) $p = 1/4$, $\lambda \approx 11.5$



(c) $p = 1/8$, $\lambda \approx 16.5$ (d) $p = 1/16$, $\lambda \approx 27.2$

Figure 2. Approximate first $L^2(\mu)$-normalized Dirichlet eigenfunctions u and eigenvalues λ as p varies, where μ is defined by (4.1) with $r = 1/2$

and

$$\mu_{p_k}[0, c_{p_k}] \le \frac{1}{2C}. \tag{4.16}$$

Then by (4.13), (4.14) and (4.16),

$$\begin{aligned}
1 &= \lambda_{p_k} \int_0^{c_{p_k}} (1-y) u_{p_k}(y)\, d\mu_{p_k}(y) + \lambda_{p_k} \int_{c_{p_k}}^1 (1-y) u_{p_k}(y)\, d\mu_{p_k}(y) \\
&\le \lambda_{p_k} \mu_{p_k}[0, c_{p_k}] + \lambda_{p_k} (1 - c_{p_k}) \mu_{p_k}[c_{p_k}, 1] \\
&\le \frac{1}{2} + C(1 - c_{p_k}),
\end{aligned}$$

which is impossible by (4.15). This contradiction completes the proof. □

Theorem 4.6. *Assume the same hypotheses as in Theorem 4.5 and assume in addition that each eigenfunction u_p satisfies $u_p'(0) > 0$ and is normalized, i.e., $\|u_p\|_2 = 1$. Then*

$$\lim_{p \to 0} |u_p'(1)| = \infty. \tag{4.17}$$

Proof. Let ξ_p be the unique zero of u_p' in $(0, 1)$. The normalization condition $\|u_p\|_2 = 1$ implies that

$$\|u_p\|_\infty = u_p(\xi_p) \ge 1. \tag{4.18}$$

We suffices to prove that (4.17) holds for each sequence $p_k \to 0$. We prove this by contradiction. Suppose there exist a sequence $\{p_k\}$ and a positive constant C_1 such that $p_k \to 0$ but

$$|u'_{p_k}(1)| \le C_1 \quad \text{for all } k. \tag{4.19}$$

If there exists some subsequence $\{p_{k_j}\}$ such that $\xi_{p_{k_j}} \to 1$ or $\|u_{p_{k_j}}\|_\infty \to \infty$, then obviously we would have $|u'_{p_{k_j}}(1)| \to \infty$ (cf. (4.18)), contradicting (4.19). Hence we will assume there exist positive constants C_2 and C_3 such that

$$\xi_{p_k} \le C_2 < 1 \quad \text{and} \quad \|u_{p_k}\|_\infty \le C_3 \quad \text{for all } k.$$

Now, for any fixed $c \in (C_2, 1)$, we have

$$\begin{aligned} 1 = \|u_{p_k}\|_2^2 &= \int_0^c u_{p_k}(y)^2\, d\mu_{p_k}(y) + \int_c^1 u_{p_k}(y)^2\, d\mu_{p_k}(y) \\ &= C_3^2 \int_0^c \left(\frac{u_{p_k}(y)}{C_3}\right)^2 d\mu_{p_k}(y) + C_3^2 \int_c^1 \left(\frac{u_{p_k}(y)}{C_3}\right)^2 d\mu_{p_k}(y) \\ &\le C_3^2 \mu_{p_k}[0,c] + C_3^2 \int_c^1 \frac{u_{p_k}}{C_3}\, d\mu_{p_k}(y) \;\le\; C_3^2 \mu_{p_k}[0,c] + C_3 u_{p_k}(c). \end{aligned}$$

Consequently, for all k sufficiently large, we have

$$u_{p_k}(c) \ge \frac{1}{2C_3}.$$

This obviously implies that $|u'_{p_k}(1)| \to \infty$, contradicting (4.18) again. This completes the proof. $\square$

Example 4.7. Assume the same hypotheses of Theorem 4.6 and let $z_2 = z_2(p) \in (0,1)$ be such that $u_p(z_2)$ is the maximum of the first Dirichlet eigenfunction. In this example we describe two situations such that:

$$\text{(A)} \quad \lim_{p\to 0} z_2(p) = \frac{1}{2} \qquad \text{and} \qquad \text{(B)} \quad \lim_{p\to 0} z_2(p) = 1.$$

Therefore the analog of Theorem 4.2 does not hold for the first Dirichlet eigenfunction. In fact, one can construct an example such that $\lim_{p\to 0} z_2(p) = 0$.

(A) To simplify computations, instead of the interval $[0,1]$ we consider the interval $[0, 1+p^2]$, and the zero boundary conditions are at 0 and at $1+p^2$. Clearly, this will not change the limiting behavior of $z_2(p)$ as $p \to 0$. Let measure μ_p have a density with respect to Lebesgue measure that is equal to p on $[0,1)$ and equal to $1/p^2$ on $[1, 1+p^2]$. This μ_p is not a probability measure, but again it will not change the limiting behavior of $z_2(p)$ as $p \to 0$. Equation (1.4) or Theorem 2.1(b) implies that $u_p(x) = C_1 \sin\left(x\sqrt{\lambda p}\right)$ on the interval $[0,1]$, and $u_p(x) = C_2 \sin\left(\sqrt{\lambda}(1+p^2-x)/p\right)$ on the interval $[1, 1+p^2]$. The function $u_p(x)$, as well as its derivative, must be continuous at $x = 1$, which means

$$C_1 \sin\left(\sqrt{\lambda p}\right) = C_2 \sin\left(p\sqrt{\lambda}\right) \quad \text{and} \quad C_1\, p\sqrt{p} \cos\left(\sqrt{\lambda p}\right) = -C_2 \cos\left(p\sqrt{\lambda}\right).$$

Therefore λ is the lowest positive solution of the equation

$$\tan\left(\sqrt{\lambda p}\right) = -p\sqrt{p}\tan\left(p\sqrt{\lambda}\right).$$

If $\sqrt{p\lambda} = y$, then y is the lowest positive solution of the equation

$$\tan(y) = -p\sqrt{p}\tan\left(y\sqrt{p}\right).$$

It is easy to see that $\lim_{p\to 0}\sqrt{p\lambda} = \lim_{p\to 0} y = \pi$. This implies $\lim_{p\to 0} z_2(p) = \frac{1}{2}$.

In this example we have $\lim_{p\to 0} u_p(x) = \infty$ uniformly on any compact subset of $(0,1)$. However, $\lim_{p\to 0}\left(u_p(x)/\|u_p\|_\infty\right) = \sin(\pi x)$ uniformly on $[0,1]$.

(B) Here, in order to simplify computations, we consider the interval $[0, 1+p]$ instead of the interval $[0,1]$, and measure μ_p that has density p on $[0,1)$ and density $1/p$ on $[1, 1+p]$. The zero boundary conditions are at 0 and at $1+p$. Then we have $u_p(x) = C_1\sin\left(x\sqrt{\lambda p}\right)$ on the interval $[0,1]$, and $u_p(x) = C_2\sin\left(\sqrt{\lambda}(1+p^2-x)/p\right)$ on the interval $[1, 1+p]$. Moreover,

$$C_1\sin\left(\sqrt{\lambda p}\right) = C_2\sin\left(\sqrt{p\lambda}\right) \quad \text{and} \quad C_1\,p\cos\left(\sqrt{\lambda p}\right) = -C_2\cos\left(\sqrt{p\lambda}\right).$$

Therefore $\sqrt{p\lambda} = \frac{\pi}{2}$ and $\lim_{p\to 0} z_2(p) = 1$.

In fact, a simple computation shows that $\lim_{p\to 0} u_p(x) = \sqrt{2}\sin(\frac{\pi}{2}x)$ uniformly on any compact subset of $[0,1)$. In particular, $\lim_{p\to 0}\|u_p\|_\infty = \lim_{p\to 0} u_p(z_2) = \sqrt{2}$.

(B') Here we modify example (B) so that $\lim_{p\to 0} z_2(p) = 1$ and $\lim_{p\to 0} u_p(x) = \sqrt{2}x$ uniformly on any compact subset of $[0,1)$. We consider the interval $[0, 1+\sqrt{p}]$ instead of the interval $[0,1]$, and measure μ_p with density p on $[0,1)$ and density $\frac{1}{\sqrt{p}}$ on $[1, 1+\sqrt{p}]$. The zero boundary conditions are at 0 and at $1+\sqrt{p}$. Then λ is the lowest positive solution of the equation $\tan(\sqrt{\lambda p}) = -\sqrt{p\sqrt{p}}\tan\left(\sqrt{\lambda\sqrt{p}}\right)$. It is easy to see that $\lim_{p\to 0}\sqrt{\lambda\sqrt{p}} = \frac{\pi}{2}$. This implies $\lim_{p\to 0} z_2(p) = 1$, and also $\lim_{p\to 0} u_p(x) = \sqrt{2}x$ for $0 \le x < 1$.

Example 4.7 shows that under the same hypotheses as in Theorem 4.6, we can have $\limsup_{p\to 0}\|u_p\|_\infty = \infty$ as well as $\limsup_{p\to 0}\|u_p\|_\infty < \infty$. The same is true for the first Neumann eigenfunction.

Conjectures 4.8. *We conjecture that* $\lim_{p\to 0}\|u_p\|_\infty = \infty$ *if* μ_p *and* u_p *are as in Figures 1 and 2. However, the numerical approximations show that the growth rate is very low. We also conjecture that in the same situation there is a limit* $\lim_{p\to 0} u_p(x)/\|u_p\|_\infty$ *uniform on any compact subset of* $[0,1)$. *Moreover, this limit is a convex nonnegative piecewise linear function (different in Dirichlet and Neumann cases). In addition, we conjecture that* $\lim_{p\to 0} z_2(p) = \frac{1}{2}$ *in the Dirichlet case, and* $\lim_{p\to 0} u_p(1) = 0$ *in the Neumann case.*

5. Eigenvalues and their asymptotics. We begin by recovering two well-known properties of the eigenvalues for the Laplacians corresponding to general measures on $[0,1]$.

Proposition 5.1. *Let μ be a bounded positive continuous measure with* $\operatorname{supp}(\mu) = [0,1]$. *Let λ_n^D and λ_n^N denote the nth Dirichlet and Neumann eigenvalues, respectively. Then*

(a) $0 \leq \lambda_{n-1}^N \leq \lambda_n^D$;
(b) *The Dirichlet and Neumann eigenvalues are discrete.*

Proof. Part (a) follows from the variational formula (see e.g., [D, K2]) because $W_0^{1,2} \subseteq W_1^{1,2}$. Part (b) follows from Theorem 2.5 (e). The discreteness of the Dirichlet eigenvalues follows from the inequality

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \leq \int_0^1 \int_0^1 g(x,y)\, d\mu(x)\, d\mu(y) < \infty,$$

a consequence of Bessel's inequality and the Green's function representation of the eigenfunctions (Theorem 3.1). The discreteness of the Neumann eigenvalues follows by combining this with part (a). □

For the rest of this section we restrict our attention to a p.c.f. self-similar structure. Consider the iterated function system

$$S_1(x) = rx, \quad S_2(x) = (1-r)x + r, \quad 0 < r < 1, \tag{5.1}$$

and let μ be the self-similar measure defined by

$$\mu = p\mu \circ S_1^{-1} + (1-p)\mu \circ S_2^{-1}, \quad 0 < p < 1. \tag{5.2}$$

Recall that the *spectral dimension* (see [K1]), d_s, for this μ is given by

$$(pr)^{d_s/2} + \big((1-p)(1-r)\big)^{d_s/2} = 1. \tag{5.3}$$

Let $\rho : \mathbb{R} \to \mathbb{N}$ be the *eigenvalue counting function* defined as

$$\rho(x) = \#\{\lambda : \lambda \text{ is an eigenvalue and } \lambda \leq x\}.$$

Then according to a theorem of Kigami and Lapidus (see [Kl, K1, SV]), $\rho(x)$ is related to d_s as follows:

(i) (Non-arithmetic case) If $\log(pr)/\log((1-p)(1-r)) \in \mathbb{R}\backslash\mathbb{Q}$, then $\lim_{x\to\infty} \rho(x)/x^{d_s/2}$ exists and is positive and finite.
(ii) (Arithmetic case) On the other hand, suppose $\log(pr)/\log((1-p)(1-r)) \in \mathbb{Q}$ and suppose $T > 0$ is the generator of the group $(\log(pr)/2)\mathbb{Z} + (\log((1-p)(1-r))/2)\mathbb{Z}$ (i.e. $T\mathbb{Z}$ equals the group). Then

$$\rho(x) = \left(G\Big(\frac{\log x}{2}\Big) + o(1)\right) x^{d_s/2},$$

where G is a non-zero bounded periodic function of period T.

Using the above result we can prove

Proposition 5.2. *Fix $0 < p < 1$ and let d_s be the spectral dimension of the measure μ in (5.2). Let ρ be the eigenvalue counting function for the Laplacian defined in (1.1) with the Dirichlet or Neumann boundary condition. Then*

(a) $\lim_{n\to\infty} \dfrac{\rho(pr\lambda_n)}{\rho(\lambda_n)} = (pr)^{d_s/2}$.

(b) $\lim_{n\to\infty} \dfrac{\rho\big((1-p)(1-r)\lambda_n\big)}{\rho(\lambda_n)} = \big((1-p)(1-r)\big)^{d_s/2}$.

Proof. We only prove (a); the proof of (b) is similar. We first consider the non-arithmetic case. By (i) above and the fact that $\lim_{n\to\infty} \lambda_n = \infty$ (Theorem 2.5 (e)), we have

$$\frac{\rho(pr\lambda_n)}{\rho(\lambda_n)} = \frac{\rho(pr\lambda_n)(pr\lambda_n)^{-d_s/2}(pr)^{d_s/2}}{\rho(\lambda_n)\lambda_n^{-d_s/2}} \quad \to \quad (pr)^{d_s/2} \quad \text{as} \quad n\to\infty.$$

For the arithmetic case, by using (ii) above, we have

$$\frac{\rho(pr\lambda_n)}{\rho(\lambda_n)} = \frac{\left(G\left(\frac{1}{2}\log(pr\lambda_n)\right)+o(1)\right)(pr\lambda_n)^{d_s/2}}{\left(G\left(\frac{1}{2}\log(\lambda_n)\right)+o(1)\right)\lambda_n^{d_s/2}} = \frac{\left(G\left(\frac{1}{2}\log(\lambda_n)\right)+o(1)\right)(pr\lambda_n)^{d_s/2}}{\left(G\left(\frac{1}{2}\log(\lambda_n)\right)+o(1)\right)\lambda_n^{d_s/2}}.$$

The last equality follows from the periodicity of G, because $\log(pr)/2 = kT$ for some $k \in \mathbb{Z}$. Again $\rho(pr\lambda_n)/\rho(\lambda_n) \to (pr)^{d_s/2}$ as $n \to \infty$. $\square$

Remark. $\lim_{n\to\infty} \rho(pr\lambda_n)/\rho(\lambda_n)$ (resp. $\lim_{n\to\infty} \rho((1-p)(1-r)\lambda_n)/\rho(\lambda_n)$) is the asymptotic ratio of the number of zeros of an eigenfunction in $[0,r]$ (resp. $[r,1]$) to the number of zeros in $[0,1]$. In the case $r = 1/2$, it is observed that if this ratio is

equal to α^2, where $\alpha = (\sqrt{5}-1)/2$ be the reciprocal of the golden ratio, then there exists a subsequence of rapidly decaying Neumann eigenfunctions. To find the corresponding value of p, we set the result in Proposition 5.2 equal to α^2 to get

$$\left(\frac{p}{2}\right)^{d_s/2} = \alpha^2. \tag{5.4}$$

(5.3) then implies that

$$\left(\frac{1-p}{2}\right)^{d_s/2} = 1-\alpha^2 = \alpha. \tag{5.5}$$

Taking logarithms on both sides of (5.4) and (5.5) and dividing, we get

$$\frac{\log(p/2)}{\log((1-p)/2)} = 2 \quad \Rightarrow \quad p = 2-\sqrt{3}.$$

The graphs in Section 6 are plotted with this value of p. See Figures 3 and 4.

Notation. Given two real functions f and g, we denote by

$$f(x) \sim g(x) \quad \text{as} \quad x \to \infty$$

if there exist two constants $C_1, C_2 > 0$ such that $C_1 g(x) \le f(x) \le C_2 g(x)$ for all sufficiently large x. Similarly, if $\{a_n\}, \{b_n\}$ are two sequences, then

$$a_n \sim b_n \quad \text{as} \quad n \to \infty$$

means that there exist two positive constants C_1, C_2 such that $C_1 b_n \le a_n \le C_2 b_n$ for all sufficiently large n.

Proposition 5.3. *Fix $0 < p < 1$ and let $\mu = \mu_p$ be a self-similar measure defined by (5.2), let d_s denote the spectral dimension of μ, and let λ_n denote the nth Dirichlet or Neumann eigenvalue. Then*

$$\lambda_n \sim n^{2/d_s}.$$

Proof. First, consider the non-arithmetic case. From the result in [KL] again,

$$\lim_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} = L \quad \text{for some} \quad 0 < L < \infty.$$

In particular, for all x sufficiently large, we have

$$L - \epsilon \le \frac{\rho(x)}{x^{d_s/2}} \le L + \epsilon,$$

where $\epsilon > 0$ is small enough so that $L - \epsilon > 0$. This implies that for all n sufficiently large,

$$L - \epsilon \le \frac{\rho(\lambda_n)}{\lambda_n^{d_s/2}} \le L + \epsilon. \tag{5.6}$$

We consider the Neumann case $\rho(\lambda_n) = n + 1$; the Dirichlet case $\rho(\lambda_n) = n$ is similar. For all n sufficiently large,

$$\begin{aligned} L - \epsilon \; &\le \; \frac{n+1}{\lambda_n^{d_s/2}} \; \le \; L + \epsilon \\ \frac{1}{(L+\epsilon)^{2/d_s}}\left(1 + \frac{1}{n}\right)^{2/d_s} n^{2/d_s} \; &\le \quad \lambda_n \quad \le \; \frac{1}{(L-\epsilon)^{2/d_s}}\left(1 + \frac{1}{n}\right)^{2/d_s} n^{2/d_s} \\ \frac{1}{(L+\epsilon)^{2/d_s}} n^{2/d_s} \; &\le \quad \lambda_n \quad \le \; \frac{1}{(L-\epsilon)^{2/d_s}} 2^{2/d_s} n^{2/d_s}. \end{aligned}$$

This proves the assertion for the non-arithmetic case. For the arithmetic case,

$$\rho(x) = \left(G\Big(\frac{\log x}{2}\Big) + o(1)\right) x^{d_s/2}.$$

Since G is a bounded non-zero periodic function and $o(1) \to 0$ as $x \to \infty$, there exist constants $C_1, C_2 > 0$ such that for all sufficiently large x, we have $C_1 \le \rho(x)/x^{d_s/2} \le C_2$. In particular, for all n sufficiently large, $C_1 \le \rho(\lambda_n)/\lambda_n^{d_s/2} \le C_2$. We are back to (5.6) and the assertion follows from the same argument. □

6. Numerical Approximations. Let $\{S_1, S_2\}$ be defined as in (1.10) and let μ be the corresponding self-similar measure as defined in (1.9). Then for any $f \in L^1([0,1], \mu)$,

$$\int f(x)\, d\mu = p \int f(S_1 x)\, d\mu + (1-p) \int f(S_2 x)\, d\mu. \tag{6.1}$$

All integrals in this section are over the interval $[0, 1]$. The following identities can be derived by using (6.1) (see [S1]):

$$
\text{(6.2)}\qquad
\begin{aligned}
&\text{(a)} \quad \int x\, d\mu = 1 - p \\
&\text{(b)} \quad \int x^2\, d\mu = \frac{1}{3}(1-p)(3-2p) \\
&\text{(c)} \quad \int (1-x)^2\, d\mu = \frac{1}{3}p(1+2p) \\
&\text{(d)} \quad \int x(1-x)\, d\mu = \frac{2}{3}p(1-p).
\end{aligned}
$$

For a multi-index $J = (j_1, \dots, j_m)$, $j_i = 1$ or 2, we let $|J|$ denote the length of J and define

$$S_J = S_{j_1} \circ \cdots \circ S_{j_m} \quad \text{and} \quad m_J = m_J(m) := \#\{i : 1 \le i \le m, j_i = 1\}.$$

(i.e., m_J is the number of S_1's in the composition $S_{j_1} \circ \cdots \circ S_{j_m}$.) By iterating (6.1), we get the following useful identity

$$\text{(6.3)}\qquad \int f\, d\mu = \sum_{|J|=m} p^{m_J}(1-p)^{m-m_J} \int f(S_J x)\, d\mu, \quad m \ge 1,$$

where the summation runs over all indices $J = (j_1, \dots, j_m)$ with $j_i = 1$ or 2. Identities (6.2) and (6.3) are useful in the finite element method, one of the methods we use to obtain approximate eigenvalues and eigenfunctions.

6.1. The finite element method. To numerically approximate the solutions of (1.1), we first solve it for $u, v \in \mathcal{S}_m$, the space of bounded continuous piecewise linear functions with knots at points $k/2^m$, $k = 0, 1, \dots, 2^m$. Suppose $u, v \in \mathcal{S}_m$ satisfy

$$u\Big(\frac{k}{2^m}\Big) = a_k, \quad v\Big(\frac{k}{2^m}\Big) = b_k, \quad k = 0, 1, \dots, 2^m.$$

That is, for all $x \in [(k-1)/2^m, k/2^m]$, $k = 0, 1, \dots, 2^m$,

$$
\begin{aligned}
u(x) &= 2^m(a_k - a_{k-1})\Big(x - \frac{k-1}{2^m}\Big) + a_{k-1} \\
v(x) &= 2^m(b_k - b_{k-1})\Big(x - \frac{k-1}{2^m}\Big) + b_{k-1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int u'v'\, dx = 2^m \sum_{k=1}^{2^m} (a_k - a_{k-1})(b_k - b_{k-1}) \\
\text{(6.4)}\quad &= 2^m \sum_{k=1}^{2^m} (a_k - a_{k-1})b_k - 2^m \sum_{k=1}^{2^m} (a_k - a_{k-1})b_{k-1} \\
&= 2^m(a_0 - a_1)b_0 + 2^m \sum_{k=1}^{2^m-1} (-a_{k-1} + 2a_k - a_{k+1})b_k + 2^m(-a_{2^m-1} + a_{2^m})b_{2^m}.
\end{aligned}
$$

On the other hand, by using identity (6.3) we can express the right-hand side of (1.1) as

$$
\begin{aligned}
\lambda\int uv\,d\mu &= \lambda\sum_{|J|=m} p^{m_J}(1-p)^{m-m_J}\int u(S_Jx)v(S_Jx)\,d\mu \\
&= \lambda\sum_{k=1}^{2^m} p^{m_k}(1-p)^{m-m_k}\int \big(a_{k-1}(1-x)+a_kx\big)\big(b_{k-1}(1-x)+b_kx\big)\,d\mu,
\end{aligned} \tag{6.5}
$$

where $m_k = m_k(m) := m_J$, and J is the unique index such that $S_J[0,1] = [(k-1)/2^m, k/2^m]$. The second equality in (6.5) follows from the definitions of u and v and the formula $S_J(x) = x/2^m + (k-1)/2^m$.

Substituting the identities in (6.2) into (6.5) and regrouping terms, we get

$$
\begin{aligned}
3\int uv\,d\mu &= \sum_{k=1}^{2^m} p^{m_k}(1-p)^{m-m_k}\Big(p(1+2p)a_{k-1}b_{k-1} \\
&\qquad + 2p(1-p)(a_{k-1}b_k + a_kb_{k-1}) + (1-p)(3-2p)a_kb_k\Big) \\
&= \big((1+2p)p^{m+1}a_0 + 2(1-p)p^{m+1}a_1\big)\,b_0 + \sum_{k=1}^{2^m-1}\Big\{2p^{1+m_k}(1-p)^{m+1-m_k}a_{k-1} \\
&\qquad + \big((1+2p)p^{1+m_{k+1}}(1-p)^{m-m_{k+1}} + (3-2p)p^{m_k}(1-p)^{m+1-m_k}\big)\,a_k + \\
&2p^{1+m_{k+1}}(1-p)^{m+1-m_{k+1}}a_{k+1}\Big\}b_k + \Big(2p(1-p)^{m+1}a_{2^m-1} + (3-2p)(1-p)^{m+1}a_{2^m}\Big)b_{2^m}.
\end{aligned}
$$

Comparing coefficients of the b_k in this equation and in (6.4) leads to the following system of linear equations

$$
\begin{aligned}
2^m(a_0-a_1) &= \frac{\lambda}{3}\Big((1+2p)p^{m+1}a_0 + 2(1-p)p^{m+1}a_1\Big) \\
2^m(-a_{k-1}+2a_k-a_{k+1}) &= \frac{\lambda}{3}\Big(2p^{1+m_k}(1-p)^{m+1-m_k}a_{k-1} \\
&\quad + \Big((1+2p)p^{1+m_{k+1}}(1-p)^{m-m_{k+1}} + (3-2p)p^{m_k}(1-p)^{m+1-m_k}\Big)a_k \\
&\quad + 2p^{1+m_{k+1}}(1-p)^{m+1-m_k}a_{k+1}\Big), \qquad k=1,\dots,2^m-1 \\
2^m(-a_{2^m-1}+a_{2^m}) &= \frac{\lambda}{3}\Big(2p(1-p)^{m+1}a_{2^m-1} + (3-2p)(1-p)^{m+1}a_{2^m}\Big).
\end{aligned} \tag{6.6}
$$

By writing $\boldsymbol{u} = [u_0, u_1, \dots, u_{2^m}]^T$, we can express these equations in matrix form as

$$
M_m\boldsymbol{u} = \lambda N_m\boldsymbol{u}. \tag{6.7}
$$

For example,

$$
M_1 = 2\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad M_2 = 2^2\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix},
$$

and N_1, N_2 are respectively

$$\frac{1}{3}\begin{bmatrix} p^2(1+2p) & 2p^2(1-p) & 0 \\ 2p^2(1-p) & 4p(1-p) & 2p(1-p)^2 \\ 0 & 2p(1-p)^2 & (3-2p)(1-p)^2 \end{bmatrix},$$

$$\frac{1}{3}\begin{bmatrix} p^3(1+2p) & 2p^3(1-p) & 0 & 0 & 0 \\ 2p^3(1-p) & 4p^2(1-p) & 2p^2(1-p)^2 & 0 & 0 \\ 0 & 2p^2(1-p)^2 & p(1-p)(3-4p+4p^2) & 2p^2(1-p)^2 & 0 \\ 0 & 0 & 2p^2(1-p)^2 & 4p(1-p)^2 & 2p(1-p)^3 \\ 0 & 0 & 0 & 2p(1-p)^3 & (3-2p)(1-p)^3 \end{bmatrix}.$$

We remark that both M_m and N_m are symmetric, tridiagonal, and of order 2^m+1. Moreover, they satisfy the recursive relations in the following proposition.

Proposition 6.1.

$$M_{m+1} = 2\begin{bmatrix} M_m & \mathbf{0} \\ \mathbf{0}_* & \mathbf{0} \end{bmatrix} + 2\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0}^* & M_m \end{bmatrix}$$
$$N_{m+1} = p\begin{bmatrix} N_m & \mathbf{0} \\ \mathbf{0}_* & \mathbf{0} \end{bmatrix} + (1-p)\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0}^* & N_m \end{bmatrix},$$

where $\mathbf{0}$, $\mathbf{0}_*$, *and* $\mathbf{0}^*$ *are zero matrices of orders* $2^m \times 2^m$, $2^m \times 2^{m+1}$, *and* $2^{m+1} \times 2^m$ *respectively.*

Proof. The first identity is quite obvious. The second one follows by using induction and applying the following simple relation to (6.6):

$$m_k(m+1) = \begin{cases} m_k(m)+1 & \text{for } k = 1,\dots,2^m \\ m_k(m) & \text{for } k = 2^m+1,\dots,2^{m+1}. \end{cases} \qquad \square$$

6.2. Difference approximation method. There is an alternative way to approximate the solutions of (1.1) by making use of the discrete Laplacian on the set of knots. Define

$$V_m = \Big\{\frac{k}{2^m} : k = 0,1,\dots,2^m\Big\}, \quad m = 1,2,\dots.$$

For any function $u : V_m \to \mathbb{R}$, the discrete Laplacian of u, denoted by $H_m u$, is defined as

$$H_m u(0) = -u(0) + u\Big(\frac{1}{2^m}\Big)$$

$$H_m u\Big(\frac{k}{2^m}\Big) = u\Big(\frac{k-1}{2^m}\Big) - 2u\Big(\frac{k}{2^m}\Big) + u\Big(\frac{k+1}{2^m}\Big), \quad 1 \le k \le 2^m - 1 \tag{6.8}$$

$$H_m u(1) = u\Big(1 - \frac{1}{2^m}\Big) - u(1).$$

Let $\Psi^m_{k/2^m}$, $k = 0, 1, \dots, 2^m$, be a sequence of triangular piecewise linear functions defined as

$$\Psi^m_0(x) = \begin{cases} -2^m x + 1, & 0 \le x \le 1/2^m \\ 0, & \text{otherwise,} \end{cases}$$

$$\Psi^m_{k/2^m}(x) = \begin{cases} 2^m x - (k-1), & (k-1)/2^m \le x \le k/2^m \\ -2^m x + k + 1, & k/2^m \le x \le (k+1)/2^m \\ 0, & \text{otherwise,} \end{cases} \qquad k = 1, \dots, 2^m - 1$$

$$\Psi^m_{2^m}(x) = \begin{cases} 2^m x - 2^m + 1, & 1 - 1/2^m \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then, solutions of (1.1) can also be approximated by using the formula (see [K2]):

$$(6.9) \qquad (H_m u)\Big(\frac{k}{2^m}\Big) = \lambda\Big(\int \Psi^m_{k/2^m}(x)\, d\mu\Big) u\Big(\frac{k}{2^m}\Big), \quad k = 0, 1, \dots, 2^m.$$

The left-hand side of (6.9) is determined by (6.8); the right-hand side is determined by the following proposition.

Proposition 6.2. *Let $m \ge 1$ and for $k = 1, \dots, 2^m - 1$, let J_1, J_2 be the unique indices such that*

$$S_{J_1}[0,1] = \Big[\frac{k-1}{2^m}, \frac{k}{2^m}\Big] \quad \textit{and} \quad S_{J_2}[0,1] = \Big[\frac{k}{2^m}, \frac{k+1}{2^m}\Big].$$

Then

(a) $\displaystyle\int \Psi^m_0\, d\mu = p^{m+1}$.

(b) $\displaystyle\int \Psi^m_{k/2^m}\, d\mu = p^{m_{J_1}}(1-p)^{m+1-m_{J_1}} + p^{m_{J_2}+1}(1-p)^{m-m_{J_2}}, \quad k = 1, \dots, 2^m - 1.$

(c) $\displaystyle\int \Psi^m_1\, d\mu = (1-p)^{m+1}$.

Proof. We will prove (b); the proofs of (a) and (c) are similar and easier. Using (6.3) and the fact that $\Psi^m_{k/2^m}$ is supported on $[(k-1)/2^m, (k+1)/2^m]$, we have

$$\begin{aligned} \int \Psi^m_{k/2^m}\, d\mu &= \sum_{|J|=m} p^{m_J}(1-p)^{m-m_J} \int \Psi^m_{k/2^m}(S_J x)\, d\mu \\ &= p^{m_{J_1}}(1-p)^{m-m_{J_1}} \int \Psi^m_{k/2^m}(S_{J_1} x)\, d\mu \\ &\qquad + p^{m_{J_2}}(1-p)^{m-m_{J_2}} \int \Psi^m_{k/2^m}(S_{J_2} x)\, d\mu \\ &= p^{m_{J_1}}(1-p)^{m-m_{J_1}} \int \Psi^m_{k/2^m}\Big(\frac{x}{2^m} + \frac{k-1}{2^m}\Big)\, d\mu \end{aligned}$$

$$
\begin{aligned}
&\quad + p^{m_{J_2}}(1-p)^{m-m_{J_2}} \int \Psi^m_{k/2^m}\Big(\frac{x}{2^m}+\frac{k}{2^m}\Big)\,d\mu \\
&= p^{m_{J_1}}(1-p)^{m-m_{J_1}} \int x\,d\mu \\
&\quad + p^{m_{J_2}}(1-p)^{m-m_{J_2}} \int (-x+1)\,d\mu \quad \text{(by definition of } \Psi^m_{k/2^m}) \\
&= p^{m_{J_1}}(1-p)^{m+1-m_{J_1}} + p^{m_{J_2}+1}(1-p)^{m-m_{J_2}} \qquad \text{(by (6.2))}. \qquad \square
\end{aligned}
$$

By combining (6.8) and Proposition 6.2 we can express (6.9) in a matrix form as

$$
\tilde{M}_m \tilde{\boldsymbol{u}} = \tilde{\lambda}\tilde{N}_m \tilde{\boldsymbol{u}}, \tag{6.10}
$$

where $\tilde{M}_m = M_m$ and, for example,

$$
\tilde{N}_1 = \begin{bmatrix} p^2 & 0 & 0 \\ 0 & 2p(1-p) & 0 \\ 0 & 0 & (1-p)^2 \end{bmatrix}
$$

$$
\tilde{N}_2 = \begin{bmatrix} p^3 & 0 & 0 & 0 & 0 \\ 0 & 2p^2(1-p) & 0 & 0 & 0 \\ 0 & 0 & p(1-p)^2 + p^2(1-p) & 0 & 0 \\ 0 & 0 & 0 & 2p(1-p)^2 & 0 \\ 0 & 0 & 0 & 0 & (1-p)^3 \end{bmatrix}.
$$

We remark that $\tilde{N}_m$ is a diagonal matrix of order $2^m + 1$.

As in the finite element method, there is a recursive relation governing $\tilde{N}_m$. We justify this in the following

Proposition 6.3. *For* $m = 1, 2, \ldots$, *we have*

$$
\int \Psi^{m+1}_{k/2^{m+1}}\,d\mu = \begin{cases} p\int \Psi^m_{k/2^m}\,d\mu, & \textit{if } k = 0,\ldots,2^m-1 \\ p\int \Psi^m_1\,d\mu + (1-p)\int \Psi^m_0\,d\mu, & \textit{if } k = 2^m \\ (1-p)\int \Psi^m_{(k-2^m)/2^m}\,d\mu, & \textit{if } k = 2^m+1,\ldots,2^{m+1}. \end{cases}
$$

Proof. If $k = 0$, then it follows easily from Proposition 6.1 (a) that

$$
\int \Psi^{m+1}_0\,d\mu = p^{m+2} = p\int \Psi^m_0\,d\mu.
$$

For $k = 1, \ldots, 2^m - 1$, Proposition 6.2 (b) gives

$$
\int \Psi^{m+1}_{k/2^{m+1}}\,d\mu = p^{m_{J_1}}(1-p)^{m+2-m_{J_1}} + p^{m_{J_2}+1}(1-p)^{m+1-m_{J_2}}, \tag{6.11}
$$

where

$$
S_{J_1}[0,1] = \Big[\frac{k-1}{2^m}, \frac{k}{2^m}\Big] \quad \text{and} \quad S_{J_2}[0,1] = \Big[\frac{k}{2^m}, \frac{k+1}{2^m}\Big].
$$

Since $S_{J_1}[0,1]$ and $S_{J_2}[0,1]$ are on the left-hand side of the unit interval $[0,1]$, J_1, J_2 can be expressed as $J_1 = (1, J_1')$ and $J_2 = (1, J_2')$, where

$$S_{J_1'}[0,1] = \Big[\frac{k-1}{2^m}, \frac{k}{2^m}\Big] \quad \text{and} \quad S_{J_2}[0,1] = \Big[\frac{k}{2^m}, \frac{k+1}{2^m}\Big].$$

Hence (6.11) becomes

$$\begin{aligned}\int \Psi^{m+1}_{k/2^{m+1}}\,d\mu &= p^{m_{J_1'}+1}(1-p)^{m+2-(m_{J_1'}+1)} + p^{m_{J_2'}+2}(1-p)^{m+1-(m_{J_2'}+1)} \\ &= p\Big(p^{m_{J_1'}}(1-p)^{m+1-m_{J_1'}} + p^{m_{J_2'}+1}(1-p)^{m-m_{J_2'}}\Big) \\ &= p\int \Psi^m_{k/2^m}\,d\mu.\end{aligned}$$

With a few modifications, equalities for the other cases can also be established similarly. □

By using Proposition 6.3, we obtain the following recursive relation for $\tilde{N}_m$:

$$\tilde{N}_{m+1} = p\begin{bmatrix} \tilde{N}_m & \mathbf{0} \\ \mathbf{0}_* & \mathbf{0}\end{bmatrix} + (1-p)\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0}^* & \tilde{N}_m\end{bmatrix}, \tag{6.12}$$

where $\mathbf{0}, \mathbf{0}_*, \mathbf{0}^*$ are defined as in Proposition 6.1.

6.3. Normalization. There are several different ways that we have used to normalize the eigenvectors numerically. We summarize these methods below.
(N1) Suppose $\boldsymbol{u}_n$ is an eigenvector (not yet normalized). Since

$$\int (\boldsymbol{u}_n')^2\,dx = \lambda_n \int (\boldsymbol{u}_n)^2\,d\mu,$$

we have

$$\int (c\boldsymbol{u}_n)^2\,d\mu = 1 \quad \Longleftrightarrow \quad c = \Big(\frac{\lambda}{\int (\boldsymbol{u}_n')^2\,dx}\Big)^{1/2}.$$

$c\boldsymbol{u}_n$ now is normalized by taking the above value of c. This method has the advantage that integration is with respect to Lebesgue measure.
(N2) Note that

$$1 = \int (c\boldsymbol{u}_n)^2\,d\mu \quad \Longleftrightarrow \quad c = \Big(\int \boldsymbol{u}_n^2\,d\mu\Big)^{-1/2}.$$

Another method we use is to create a list of μ measures on the dyadic intervals and find the approximate normalization factor directly by

$$\int \boldsymbol{u}_n^2\,d\mu \approx \sum_{k=0}^{2^m-1}\Big(\frac{\boldsymbol{u}_n(k)+\boldsymbol{u}_n(k+1)}{2}\Big)\mu\Big[\frac{k}{2^m}, \frac{k+1}{2^m}\Big].$$

(N3) Here we proceed as in (N2) but integrate $\int \boldsymbol{u}_n^2\,d\mu$ exactly using the fact that $\boldsymbol{u}_n^2$ is a piecewise quadratic function. This can be done since $\boldsymbol{u}_n$ is piecewise linear and we know the integral of all quadratic functions from the identities in (6.2).

6.4. Rapidly decaying eigenfunctions. Figure 3 shows the numerical solutions for the first 18 approximate normalized Neumann eigenfunctions for μ defined by equations (1.9) and (1.10) and $p = 2 - \sqrt{3}$ (see the remark following Proposition 5.2). It is observed that for this particular value of p, there exists a distinct subsequence of rapidly decaying eigenfunctions u_{n_k} where n_k satisfies the recursive Fibonacci type relation

$$n_1 = 1, \quad n_2 = 2, \ \dots \ , n_{k+1} = n_k + n_{k-1} \quad \text{for } k \geq 2.$$

Alternatively, one can say that these eigenfunctions are strongly localized. The same phenomenon occurs for Dirichlet eigenfunctions (we do not include the pictures for this case because of space limitations). Figure 4 shows the first ten Neumann eigenfunctions in this subsequence.

Conjectures 6.4. *We conjecture that, as the graphs suggest,* $\|u_{n_k}\|_\infty \to \infty$ *as* $k \to \infty$ *for the normalized eigenfunctions. We also conjecture that* $\max_{x\in[\varepsilon,1]} |u_{n_k}(x)| \to 0$ *as* $k \to \infty$ *for any* $\varepsilon > 0$*, which means a strong localization.*

6.5. Error estimates. We finish this section by discussing the error estimates of the finite element method. For any $u \in \mathrm{Dom}(\mathcal{E})$, and $x, y \in [0,1]$,

$$\big|u(y) - u(x)\big| = \left|\int_x^y u'(t)\,dt\right| \leq |y-x|^{1/2} \left(\int_x^y |u'|^2\,dt\right)^{1/2} \tag{6.13}$$

and so for all $x, y \in [0,1]$ we have

$$\big|u(y) - u(x)\big|^2 \leq |y-x|\mathcal{E}(u,u) \tag{6.14}$$

In particular, for each $u \in \mathrm{Dom}(\mathcal{E})$ that has a zero in $[0,1]$ (including all Dirichlet and Neumann eigenfunctions), we have

$$\|u\|_\infty \leq \mathcal{E}(u,u)^{1/2} \tag{6.15}$$

$$\|u\|_2 \leq \mathcal{E}(u,u)^{1/2}. \tag{6.16}$$

Moreover, by using the proof of [SU, Lemma 4.6], we have

$$\mathcal{E}(u,u)^{1/2} \leq \|\Delta_\mu u\|_2. \tag{6.17}$$

Using equations (6.13)–(6.17) and the proof of [SU, Theorem 4.8 and Corollary 4.9] with $j = 0$ (see also [GRS]), we have

Theorem 6.5. *Let* u *be an* $L^2(\mu)$*-normalized Dirichlet or Neumann eigenfunction of* (1.1) *with eigenvalue* λ*. Then there exists* $\tilde{u} \in \mathcal{S}_m$ *such that*

(a) $\mathcal{E}(u-\tilde{u}, u-\tilde{u})^{1/2} \leq \lambda\rho^{m/2}$,
(b) $\|u-\tilde{u}\|_\infty \leq \lambda\|u\|_\infty\rho^m$,
(c) $\|u-\tilde{u}\|_2 \leq \lambda\rho^{m/2}$,

where $\rho = \max\{p/2, (1-p)/2\}$*. Moreover,* $\tilde{u}$ *may be taken to be the spline that interpolates* u *on* $V_m := \{k/2^m : k = 0, 1, \dots, 2^m\}$*.*

Proof. Part (a) follows from the proof of [SU, Theorem 4.8] by using (6.17) and observing that for each multi-index J with $|J| = m$, $(u-\tilde{u}) \circ S_J$ vanishes on the boundary $\{0,1\}$. Part (b) follows from (a) and the same proof as [SU, Corollary 4.9]. Part (c) follows by combining (6.16) and (a). □

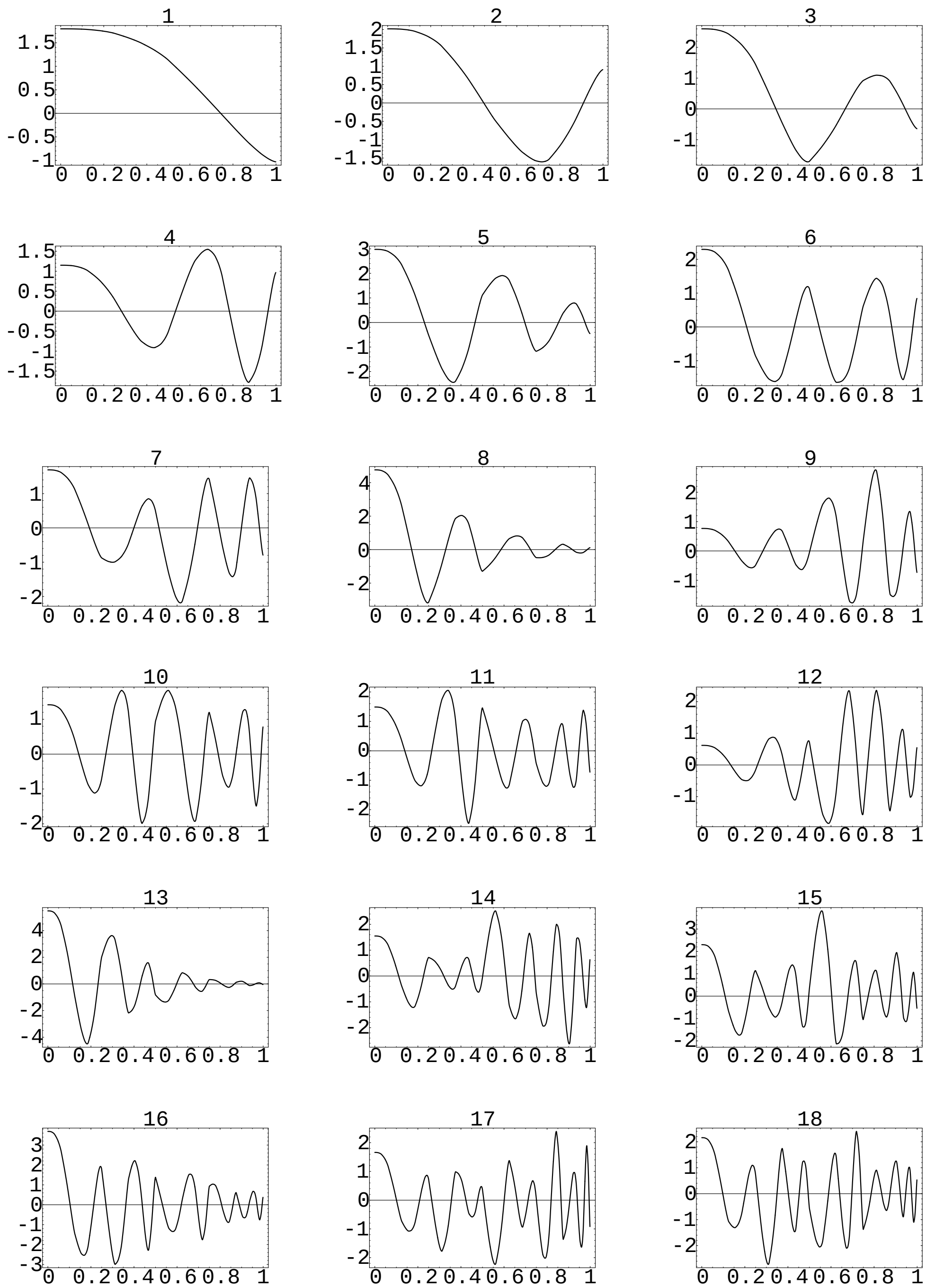


Figure 3. Approximate nth Neumann eigenfunctions of equation (1.1) for μ defined by equations (1.9) and (1.10) with $p = 2 - \sqrt{3}$, plotted with $m = 10$.

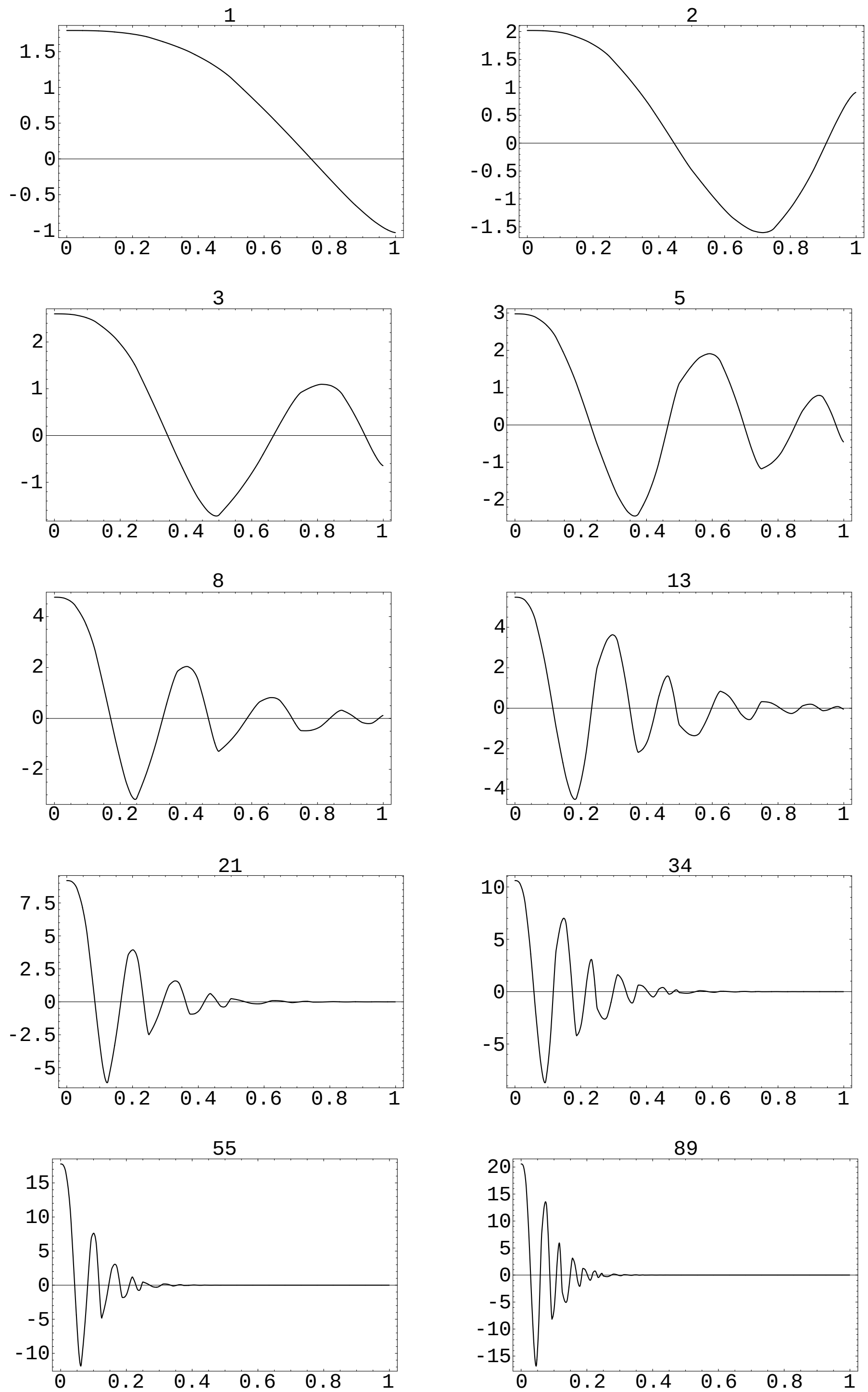


Figure 4. The sequence of rapidly decaying Neumann eigenfunctions.

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Résumé substantiel en français. Soit μ une mesure finie continue et positive à support $\operatorname{supp}(\mu) = [0,1]$. Typiquement, μ est une mesure auto-similaire. Nous étudions les valeurs propres λ et les fonctions propres u de l'équation (1.1), c'est-à-dire

$$\int_0^1 u'(x)v'(x)\,dx = \lambda \int_0^1 u(x)v(x)\,d\mu(x),$$

pour $v \in C_0^\infty(0,1)$, l'espace des fonctions infiniement différentiables à support dans $(0,1)$. Nous imposons soit la condition de Neumann $u'(0) = u'(1) = 0$ à la frontière, soit celle de Dirichlet $u(0) = u(1) = 0$. L'équation (1.1) définie comme distribution un laplacien $\Delta_\mu u$ pour lequel

$$\int_0^1 u'v'\,dx = \int_0^1 \left(-\Delta_\mu u\right) v\,d\mu$$

lorsque $v \in C_0^\infty(0,1)$. Les fonctions propres sont les analogues (fractals) des fonctions sinus et cosinus classiques de Fourier.

Pour une étude complète, nous démontrons aisément, en plus des résultats suivants essentiellement connus (voir, par exemple, [A]), l'existence d'une solution unique et lisse de l'équation précédente. La n-ième fonction propre de Neumann admet n zéros et sa dérivée $n+1$ zéros. La n-ième fonction propre de Dirichlet admet $n+1$ zéros et sa dérivée n zéros. Pour une fonction propre quelconque u, les zéros de u et de u' alternent entre eux et pareillement pour les zéros de la n-ième fonction propre de Dirichlet et de Neumann. Il existe une base orthonormale complète composée des fonctions propres de Dirichlet (Neumann). Les valeurs propres λ_n de Dirichlet (Neumann) sont simples. De plus, on a $\lim_{n\to\infty} \lambda_n = \infty$.

L'équation (1.1) est équivalente à l'équation intégrale de Volterra-Steiltjes (voir [A, Chapitre 11]):

$$u(x) = u(0) + u'(0)x - \lambda \int_0^x (x-y)u(y)\,d\mu(y), \qquad 0 \le x \le 1.$$

Il est connu qu'une solution quelconque u est différentiable et que la dérivée satisfait

$$u'(x) = u'(0) - \lambda \int_0^x u(y)\,d\mu(y), \qquad 0 \le x \le 1.$$

(Voir [A, Théorème 11.2.2]). L'inverse est aussi vrai. De plus, il existe une fonction de Green pour représenter u, qui est essentiellement celle que l'on trouve dans [S2, K2] et donnée par

$$u(x) = u(0) + \big(u(1) - u(0)\big)\,x + \lambda \int_0^1 g(x,y)u(y)\,d\mu(y), \qquad 0 \le x \le 1.$$

où $g(x,y)$ est telle que définie dans Section 2. On utilise ces formes équivalentes de l'équation pour obtenir des résultats de concavité pour les fonctions propres (Propositions 3.3 et 3.4) et d'autres résultats sur les propriétés des valeurs propres et des fonctions propres.

On étudie la première valeur propre et la première fonction propre lorsque la mesure varie. Soit $\{\mu_p : 0 < p < 1\}$ une famille de mesures de probabilités continues avec supp $\left(\mu_p\right) \subseteq [0,1]$ et telles que, pour $c \in (0,1)$ fixé quelconque, $\mu_p[0,c] \to 0$ lorsque $p \to 0$. Dans le cas Neumann, soit $z_1 = z_1(p)$ le zéro commun à toutes fonctions propres de Neumann associées à la première valeur propre. Alors $z_1 \to 1$ lorsque $p \to 0$ (Théorème 4.2). Soit $\lambda = \lambda_p$ la première valeure propre de Neumann. Alors $\lambda \to \infty$ lorsque $p \to 0$ (Théorème 4.3). On montre (Théorème 4.4) que

$$\left\|u_p'\right\|_\infty = \left|u_p'(z_1)\right| = \lambda \int_0^{z_1} u_p(y)\,d\mu(y) = \lambda \int_{z_1}^1 |u_p(y)|\;d\mu_p(y) \to \infty \text{ lorsque } p \to 0.$$

Si les fonctions propres sont normalisées, alors $\|u_p\|_2 = 1$.

Soit λ_p la première valeur propre de Dirichlet. Nous montrons que $\lim_{p\to 0} \lambda_p = \infty$ (Théorème 4.5). Pour la fonction propre de Dirichlet normalisée u_p, nous montrons que $\lim_{p\to 0} \left|u_p'(1)\right| = \infty$ (Théorème 4.6).

Nous donnons aussi un exemple qui montre que l'analogue du Théorème 4.2 est faux pour la première fonction propre de Dirichlet (Exemple 4.7).

Nous étudions aussi le comportement asymptotique des valeurs propres. Nous nous restreignons à une structure p.c.f. auto-similaire définie par le système itératif

$$S_1(x) = rx, \qquad S_2(x) = (1-r)x + r, \qquad 0 < r < 1,$$

et on prend pour μ la mesure auto-similaire définie par

$$\mu = p\mu \circ S_1^{-1} + (1-p)\mu \circ S_2^{-1}, \qquad 0 < p < 1.$$

Utilisant un résultat de Kigami et Lapidus [KL], on obtient des bornes asymptotiques pour les valeurs propres définies par cette classe de mesures auto-similaires.

Nous faisons la description des solutions numériques de l'équation (1.1). Pour estimer les valeurs propres et fonctions propres numériquement, nous restreignons μ davantage au cas d'une mesure auto-similaire définie par le système itératif

$$S_1(x) = \tfrac{1}{2}\,x, \quad S_2(x) = \tfrac{1}{2}\,x + \tfrac{1}{2}.$$

Moyennant l'identité

$$\int_0^1 f(x)\;d\mu = p\int_0^1 f\left(S_1 x\right)\;d\mu + (1-p)\int_0^1 f\left(S_2 x\right)\;d\mu,$$

on peut résoudre l'équation (1.1) pour $u, v \in S_m$, l'espace des fonctions continues linéaires par morceau avec noeuds à $k \ / \ 2^m, k = 0, 1, \ldots, 2^m$, où m est un entier positif quelconque. Ceci nous permet d'établir un système généralisé pour les valeurs propres

$$M_m \boldsymbol{u} = \lambda N_m \boldsymbol{u},$$

avec solutions $\boldsymbol{u}$ qui approximent les fonctions propres de Neumann. Les résultats numériques semblent indiquer que, pour certaines valeurs de p, il existe une sous-suite distincte de fonctions propres de Neumann associée aux nombres de Fibonacci et à décroissance rapide (localement). Nous obtenons des estimés d'erreur (Théorème 6.5) et faisons les Conjectures 4.8 et 6.4.

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