Journal of Functional Analysis **174**, 128–154 (2000)
doi:10.1006/jfan.2000.3581, available online at http://www.idealibrary.com on IDEAL®

Gradients on Fractals

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Communicated by L. Gross

Received November 8, 1999; accepted January 12, 2000

In this paper we define and study a gradient on p.c.f. (post critically finite, or finitely ramified) fractals. We use Dirichlet (energy) form analysis developed for such fractals by Kigami. We consider both nondegenerate and degenerate harmonic structures (where a nonzero harmonic function can be identically zero on an open set). We show that the energy is equal to the integral of a certain seminorm of the gradient if the harmonic structure is weakly nondegenerate. This result was proved by Kusuoka in a different form. We show that for a C^1-function on the Sierpiński gasket the gradient considered here and Kusuoka's gradient essentially coincide with a gradient considered by Kigami. The gradient at a junction point was studied by Strichartz in relation to the Taylor approximation on fractals. He also proved the existence of the gradient almost everywhere with respect to the Hausdorff (Bernoulli) measure for a function in the domain of the Laplacian. In this paper we obtain certain continuity properties of the gradient for a function in the domain of the Laplacian. As an appendix, we prove an estimate of the local energy of harmonic functions which was stated by Strichartz as a hypothesis.

Key Words: fractal; gradient; harmonic structure; Dirichlet form; Laplacian.

Contents.

1. INTRODUCTION

The Laplacian on fractals was first constructed as the generator of a diffusion process by S. Goldstein, S. Kusuoka, and T. Lindstrøm in [Ku1, Go, Li].

[1] Research supported in part by the National Science Foundation of the U.S.A. and the Natural Sciences and Engineering Research Council of Canada.

0022-1236/00 $35.00

Later an analytic approach was developed by J. Kigami, who constructed the Laplacian using the theory of Dirichlet forms [Ki1, Ki2]. These two approaches were unified in [Ku2, Ku3, Fu, Ba]. There are a number of papers on the properties of the diffusion process ([Ba, BP, Fu, Ku2, Ku3, MS] and references therein) and on the spectral properties of the Laplacian ([BK, FS, Ki8, KL, MT, T] and references therein). Recently there were several works in the general direction of creating a calculus on fractals [BST, DSV, Ki4–Ki7, St1–St5] *et al.*

Although the Laplacian on fractals is now relatively well understood, the first order derivatives are less studied. In this paper we give an approach to define a gradient on fractals. We also compare our work with the definitions and results in [Ku2, Ki3, St5], where different questions related to gradients on fractals where considered.

In the future it would be interesting to relate the gradient and energy measures to the volume measures considered in [La1, La2] and to obtain similar results in the case of the Sierpiński carpet (see [BB] and references therein).

This paper is organized as follows. In Section 2 we give notation most of which was introduced in [Ki2, Ku2, St5]. Then in Section 3 we give definitions and examples of nondegenerate harmonic structures. Also we define a gradient in a nondegenerate situation and prove a form of the chain rule for this gradient. In Section 4 we describe the relation between our definitions and results, and those of J. Kigami, S. Kusuoka, and R. Strichartz. In Section 5 we give some sufficient conditions for the existence and continuity of the gradient for a function in the domain of the Laplacian on a nondegenerate harmonic structure. In Section 6 we apply the results of the previous section to the case of the Sierpiński gasket, and also prove some results on discontinuities of the gradient. In Section 7 we define a weak gradient for a degenerate harmonic structure. Then we prove that for a weakly nondegenerate harmonic structure the Dirichlet (energy) form can be recovered as an integral of a certain semi-norm of the weak gradient, which is a generalization of a result by Kusuoka in [Ku2]. In [Ki3] Kusuoka proved a similar result for nested fractals, which can be degenerate, using a different notion of a weak gradient. Finally, in Appendix we prove an estimate of the local energy of harmonic functions which was stated by Strichartz in [St5] as a hypothesis.

2. NOTATION

In this paper we suppose that a post critically finite self-similar structure $(K, S, \{F_s\}_{s\in S})$ and a harmonic structure $(D, \mathbf{r})$ are fixed. The reader can find all the related definitions and basic results in [Ki2]. Here we recall the facts and give notation which will be used in this paper.

2.1. *P.c.f. Self-similar Structure.* The post critically finite (p.c.f.) self-similar set K is a compact metric space, $S=\{1, 2, ..., N\}$, $F_s: K\to K$ are continuous injections such that $K=\bigcup_{j\in S} F_j(K)$.

We define W_n as the space of finite sequences (words) $w=w_1\cdots w_n$, $w_n\in S$, of the length n and $W_*=\bigcup_{n\geqslant 1} W_n$. Then we denote

$$F_w=F_{\omega_1}\circ\cdots\circ F_{\omega_n}$$

and

$$K_w=F_w(K).$$

The p.c.f. property implies, in particular, that the self-similar set K has a finite *boundary* $V_0\subset K$, and the boundary of K_w is $V_w=F_w(V_0)$. The important feature of a p.c.f. structure is that the intersection of the sets K_w and $K_{w'}$ is contained in the boundary of these sets if $w, w'\in W_n$, $w\neq w'$.

Let $\Omega=\{1, ..., N\}^{\mathbb{N}}$ be the space of infinite sequences $\omega=\omega_1\cdots\omega_n\cdots$, $\omega_n\in S$. It is a topological (metric) space with a distance, say, $\delta(\omega,\psi)=\sum_{n=1}^{\infty} 2^{-n}|\omega_n-\psi_n|$. In fact, many different metrics will yield the same topology.

For $\omega\in\Omega$ we denote $[\omega]_n=\omega_1\cdots\omega_n\in W_n$. If $\omega\in\Omega$ and $w\in W_n$ then $w\omega=w_1\cdots w_n\omega_1\cdots\omega_k\cdots$. Similarly a product $ww'\in W_{n+m}$ is defined for $w\in W_n$, $w'\in W_m$.

There is a continuous map $\pi:\Omega\to K$ such that $F_j\circ\pi(\omega)=\pi(j\omega)$ for $j=1, ..., N$. For any $\omega\in\Omega$ there is a unique $x\in K$ such that $\{x\}=\bigcap_{m\geqslant 1} K_{[\omega]_m}$. Then $\pi(\omega)=x$. Note that for all $x\in K$, except a countable subset, there corresponds a unique sequence ω such that $\pi(\omega)=x$. The p.c.f. assumptions imply that $\pi^{-1}\{x\}$ is a finite set for any $x\in K$.

We denote $V_n=\bigcup_{w\in W_n} V_w$ and $V_*=\bigcup_{n\geqslant 1} V_n$. A point $x\in V_*$ is called a junction point of order n if there are at least two different $w, w'\in W_n$ such that $x\in K_w\cap K_{w'}$. Thus x is a junction point if and only if $\pi^{-1}\{x\}$ consists of more than one element.

2.2. *Self-similar Harmonic Structure.* We suppose that a harmonic structure $(D, \mathbf{r})$, as defined in [Ki2], is fixed on $(K, S, \{F_s\}_{s\in S})$. Here D is the matrix of a certain nonnegative quadratic form in $\ell^2(V_0)$ and $\mathbf{r}=(r_1, ..., r_N)$ is a collection of positive numbers. This harmonic structure defines a local regular Dirichlet form $\mathscr{E}$ which satisfies a self-similarity relation

$$\mathscr{E}(f, f)=\sum_{i=1}^{N} r_i^{-1}\mathscr{E}(f\circ F_i, f\circ F_i) \tag{2.1}$$

which implies that

$$\mathscr{E}(f, f) = \sum_{w \in W_n} r_w^{-1} \mathscr{E}(f \circ F_w, f \circ F_w) \tag{2.2}$$

for any $n \geqslant 0$. The domain $Dom\mathscr{E}$ of $\mathscr{E}$ consists of continuous functions f such that $\mathscr{E}(f, f) < \infty$. This Dirichlet form $\mathscr{E}(f, f)$ is often referred to as the *energy* of f.

2.3. *The Space of Harmonic Functions.* A continuous function h is called *harmonic* if it minimizes $\mathscr{E}(h, h)$ given the boundary values $h|_{V_0}$. The space of harmonic functions $\mathscr{H}$ is $|V_0|$-dimensional since any harmonic function is uniquely determined by its boundary values. We define the norm of $\mathscr{H}$ by $\|h\|^2_{\mathscr{H}} = \mathscr{E}(h, h) + (\sum_{x \in V_0} h(x))^2$. Let $\tilde{\mathscr{H}}$ be the orthogonal complement to constant harmonic functions and $\tilde{P}$ be the orthogonal projection from $\mathscr{H}$ onto $\tilde{\mathscr{H}}$.

Let for every $i = 1, ..., N$ the linear map $M_i: \mathscr{H} \to \mathscr{H}$ be defined by $M_i h = h \circ F_i$. We also define $\tilde{M}_i: \tilde{\mathscr{H}} \to \tilde{\mathscr{H}}$ by $\tilde{M}_i = \tilde{P} M_i \tilde{P}$. We define the norm on $\tilde{\mathscr{H}}$ by $\|h\|^2 = \mathscr{E}(h, h)$. Note that $\|\cdot\|$ is a semi-norm on $\mathscr{H}$. We have $\|h\| = 0$ if and only if h is constant.

By Eq. (2.1) for any harmonic function h and any $n \geqslant 0$ we have that

$$\|h\|^2 = \sum_{w \in W_n} r_w^{-1} \|\tilde{M}_w h\|^2, \tag{2.3}$$

where $\tilde{M}_w h = \tilde{M}_{w_n} \cdots \tilde{M}_{w_1} h = h \circ F_w$ (note the order in which this product is evaluated). This equation implies another basic relation

$$\sum_{w \in W_n} r_w^{-1} \tilde{M}_w^* \tilde{M}_w = I, \tag{2.4}$$

where the adjoint $\tilde{M}_w^*$ is with respect to $\|\cdot\|$-norm on $\tilde{\mathscr{H}}$ and I is the identity operator.

2.4. *Kusuoka and Energy Measures.* For each function $f \in Dom\mathscr{E}$ we associate an energy measure ν_f on K by assigning its value to each set K_w as

$$\nu_f(K_w) = r_w^{-1} \mathscr{E}(f \circ F_w, f \circ F_w). \tag{2.5}$$

By (2.1) and by the Carathéodory extension theorem relation (2.5) defines the Borel measure ν_f uniquely. Indeed, ν_f is finite since $\nu_f(K) = \mathscr{E}(f, f)$. It is shown in [BST] that ν_f is nonatomic under very mild assumptions.

For each harmonic function h we have

$$\nu_h(K_w) = r_w^{-1} \|M_w h\|^2. \tag{2.6}$$

Let $h_1, ..., h_m$ be an $\|\cdot\|$-orthonormal basis of $\tilde{\mathscr{H}}$ (here $m = \dim \tilde{\mathscr{H}} = |V_0| - 1$). Then we define the Kusuoka measure ν as

$$\nu = \sum_{i=1}^{m} \nu_{h_i}. \tag{2.7}$$

In fact, ν does not depend on the choice of the orthonormal basis because

$$\nu(K_w) = \sum_{i=1}^{m} r_w^{-1} \|M_w h_i\|^2 = r_w^{-1} \, Tr \, \tilde{M}_w^* \tilde{M}_w. \tag{2.8}$$

We will abuse notation by defining a measure ν on Ω as the pullback of the measure ν on K under the projection map π, that is $\nu(\pi^{-1}(\cdot)) = \nu(\cdot)$.

We also consider a Bernoulli measure μ on K such that $\mu(K_w) = \mu_w = \mu_{w_1} \cdots \mu_{w_m}$ where $\mu_i = \mu(K_i)$. Again we denote also by μ a measure on Ω which is the pullback of the measure μ on K under the projection map π, that is, $\mu(\pi^{-1}(\cdot)) = \mu(\cdot)$ or $\mu(w\Omega) = \mu_w$.

Let for $w \in W_n$

$$Z_n(w) = \begin{cases} \dfrac{\tilde{M}_w^* \tilde{M}_w}{Tr \, \tilde{M}_w^* \tilde{M}_w} & \text{if} \quad \text{Rank } \tilde{M}_w > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Proposition 2.1. *$Z_n([\omega]_n)$ is a ν-martingale.*

Proof. It is so because of the relations

$$\begin{aligned} \sum_{j=1}^{N} Z_{n+1}(wj) \, \nu(K_{wj}) &= \sum_{j=1}^{N} r_{wj}^{-1} \tilde{M}_{wj}^* \tilde{M}_{wj} \\ &= r_w^{-1} \tilde{M}_w^* \left(\sum_{j=1}^{N} r_j^{-1} \tilde{M}_j^* \tilde{M}_j \right) \tilde{M}_w \\ &= r_w^{-1} \tilde{M}_w^* \tilde{M}_w = Z_n(w) \, \nu(K_w), \end{aligned}$$

where the next to the last equality follows from (2.4). ∎

Corollary 2.2. *$Z(\omega) = \lim_{n \to \infty} Z_n(\omega)$ exists for ν-almost all ω as any bounded martingale converges a.s.*

Remark 2.3. The results and definitions included above in this subsection were first given in [Ku2] for nondegenerate harmonic structures on fractals and in [Ku3] for any harmonic structure (see Sections 3 and 4). We would like to note that traditionally we denote a self-similar Bernoulli measure by μ; in [Ku2, Ku3, Ki3] this measure is denoted by ν and the energy measures are denoted using letter μ.

The following result is a generalization of some of the results of Kusuoka in [Ku2, Ku3] and was proved in [BST] in a slightly different form. Note that there are no extra assumptions on the p.c.f. self-similar structure and the harmonic structure.

THEOREM. (1) *The measure ν has no atoms.*

(2) *For any $f \in \mathrm{Dom}\,\mathcal{E}$ the measure ν_f is absolutely continuous with respect to $\tilde{\nu} = \sum_{n=1}^{\infty} (1/(2N)^n) \sum_{w \in W_n} \nu \circ F_w^{-1}$.*

(3) *If ν_h and μ are singular for any harmonic function h then ν_f and μ are singular for any $f \in \mathrm{Dom}\,\mathcal{E}$.*

(4) *Suppose that $\mu_j = r_j/(r_1 + \cdots + r_N)$ for $j = 1, ..., N$. Then either the measures ν_h and μ are singular for any harmonic function h or $\nu_h = \|h\|^2 \mu$ for some nonconstant harmonic function h. The latter happens if and only if $\|M_w h\|^2 = r_w \mu_w \|h\|^2$ for any $w \in W_*$.*

Remark 2.4. If the harmonic structure is nondegenerate (as defined in the next section) then ν_f is absolutely continuous with respect to ν for any $f \in \mathrm{Dom}\,\mathcal{E}$.

Remark 2.5. In part (4) of this theorem the condition for nonsingularity is true for the standard harmonic function on an interval. We conjecture that an interval is the only situation when ν_h is not singular with respect to μ.

We also conjecture that if ν_h is not singular with respect to any Bernoulli measure then the harmonic structure contains an interval as a “substructure.” The Vicsek set is an example of such a situation (see Example 7.5).

3. GRADIENT FOR NONDEGENERATE HARMONIC STRUCTURES

DEFINITION 3.1. A harmonic structure is said to be *nondegenerate* if the restriction of any nonconstant harmonic function to any K_w, $w \in W_*$, is not constant.

PROPOSITION 3.2. *A harmonic structure is nondegenerate if and only if $\tilde{M}_j$ is invertible for every $j = 1, ..., N$.*

Definition 3.3. Suppose the harmonic structure is nondegenerate. For any $\omega \in \Omega$ the harmonic *tangent* to (the graph of) f is the element of $\mathscr{H}$ defined by

$$\operatorname{Tan}_{\omega} f = \lim_{n \to \infty} \operatorname{Tan}_{n, [\omega]_n} f$$

if the limit exists, where for $w \in W_n$

$$\operatorname{Tan}_{n, w} f = M_w^{-1} H(f \circ F_w). \tag{3.1}$$

The *gradient* is the element of $\tilde{\mathscr{H}}$ defined by

$$\operatorname{Grad}_{\omega} f = \lim_{n \to \infty} \operatorname{Grad}_{n, [\omega]_n} f$$

if the limits exist, where for $w \in W_n$

$$\operatorname{Grad}_{n, w} f = \tilde{M}_w^{-1} \tilde{H}(f \circ F_w). \tag{3.2}$$

Here Hg is a unique harmonic function which coincides with g on the boundary of K and $\tilde{H} = \tilde{P}H$.

Remark 3.4. One may think about the tangent as a harmonic approximation to f at $x = \pi(\omega)$. Indeed, $\operatorname{Tan}_{n, w} f$ is a unique harmonic function which coincides with f on the boundary of K_w. However, if x is a junction point, then the best harmonic approximation may not exists even for such "regular" fractals as Sierpiński gasket (see Proposition 6.3). In [St5] this difficulty is dealt with by introducing so called local tangents (see discussion in Subsection 4.2).

It is easy to see that if f is continuous and $\omega \in \Omega$ then $\operatorname{Tan}_{\omega} f$ exists if and only if $\operatorname{Grad}_{\omega} f$ exists. In this work we will consider only the gradient because the tangent can be expressed easily as $\operatorname{Tan}_{\omega} f = f(\pi(\omega)) + \operatorname{Grad}_{\omega} f$.

The next lemma gives a form of the chain rule for the gradient defined above. Let $F \in C^2(\mathbb{R}^d)$ and $\partial_1 F, ..., \partial_d F$ be the first order partial derivatives of F. Suppose $g = F(f_1, ..., f_d)$ where $f_1, ..., f_d \in C(K)$.

Lemma 3.5. *If gradients* $\operatorname{Grad}_{\omega} f_1, ..., \operatorname{Grad}_{\omega} f_d$ *exists and*

$$\lim_{n \to \infty} \|\tilde{M}_{[\omega]_n}^{-1}\| \, \|\tilde{M}_{[\omega]_n}\|^2 = 0 \tag{3.3}$$

then gradient $\operatorname{Grad}_\omega g$ *exists and*

$$\operatorname{Grad}_\omega g = \sum_{k=1}^{d} \partial_k F(a_1, ..., a_d) \operatorname{Grad}_\omega f_k, \tag{3.4}$$

where $a_k = f_k(\pi(\omega))$.

Proof. It follows from Definition 3.3 that

$$\|\tilde{H}(f_k \circ F_{[\omega]_n})\| = O(\|\tilde{M}_{[\omega]_n}\|)_{n \to \infty}$$

since $\operatorname{Grad}_\omega f_k$ exists. Therefore

$$\begin{aligned}&\tilde{H}F(f_1 \circ F_{[\omega]_n}, ..., f_d \circ F_{[\omega]_n})\\ &\quad = \sum_{k=1}^{d} \partial_k F(a_1, ..., a_d)\, \tilde{H}(f_k \circ F_{[\omega]_n}) + O(\|\tilde{M}_{[\omega]_n}\|^2)_{n \to \infty}\end{aligned}$$

which proves the lemma because of Definition 3.3. ∎

Remark 3.6. In particular, one has the product formula for the gradient: if $\operatorname{Grad}_\omega u$ and $\operatorname{Grad}_\omega v$ exist and (3.3) is satisfied for a fixed ω then

$$\operatorname{Grad}_\omega(uv) = u(\pi(\omega)) \operatorname{Grad}_\omega v + v(\pi(\omega)) \operatorname{Grad}_\omega u. \tag{3.5}$$

Note that on the Sierpiński gasket condition (3.3) is satisfied for μ-almost all ω (see the proof of Lemma 4.1(1)).

One can see that the chain rule (3.4) also holds in the same situations when the gradient $\operatorname{Grad}_\omega f$ exists under conditions of Lemma 4.1(2) and Theorems 1, 2, 3.

EXAMPLE 3.7. *Interval* (nondegenerate harmonic structure). Interval $K = [-1, 1]$ is a p.c.f. selfsimilar structure with $V_0 = \{-1, 1\}$, $F_1(x) = \frac{1}{2}(x-1)$ and $F_2(x) = \frac{1}{2}(x+1)$. The set V_m contains all the fractions $k/2^{m-1}$, $k = -2^{m-1}, ..., 2^{m-1}$ and the set V_* is the set of all dyadic rationals.

The harmonic structure is the usual harmonic structure, that is the energy form is the integral of the square of the derivative up to a constant multiple. The tangent is again the usual tangent line and $\operatorname{Grad}_\omega f$ is the tangent shifted to pass through the origin. So the usual derivative is the slope of $\operatorname{Grad}_\omega f$. The operators $\tilde{M}_i$ are just the multiplication by $\frac{1}{2}$.

The measures μ and ν are both multiples of the Lebesgue measure.

EXAMPLE 3.8. *Sierpiński gasket* (nondegenerate harmonic structure). Let p_1, p_2, p_3 be the corners of a equilateral triangle and $F_i(x) = \frac{1}{2}(x + p_i)$, $i = 1, 2, 3$ (see Fig. 1). The Sierpiński gasket (SG) is a unique compact subset K of $\mathbb{R}^2$ such that $K = F_1(K) \cup F_2(K) \cup F_3(K)$. Then $V_0 = \{p_1, p_2, p_3\}$.

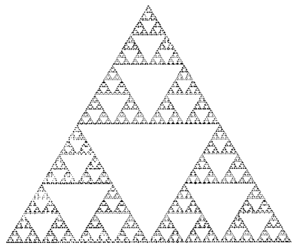

FIG. 1. Sierpiński gasket.

Note that there are three reflections R_1, R_2, R_3 such that R_i fixes p_i and interchanges the other two corners. These reflections (symmetries) will be used extensively in Section 6.

On the Sierpiński gasket there is an $\|\cdot\|$-orthonormal basis $\{h_1, h_2\}$ of $\tilde{\mathscr{H}}$ such that h_1 is R_1-symmetric and h_2 is R_1-skew symmetric in the sense that $h_1 \circ R_1 = h_1$ and $h_2 \circ R_1 = -h_2$. In this basis $\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}_3$ have the matrix representation

$$\tilde{M}_1 = \begin{pmatrix} 3/5 & 0 \\ 0 & 1/5 \end{pmatrix}, \qquad \tilde{M}_2 = \begin{pmatrix} 3/10 & \sqrt{3}/10 \\ \sqrt{3}/10 & 1/2 \end{pmatrix},$$

$$\tilde{M}_3 = \begin{pmatrix} 3/10 & -\sqrt{3}/10 \\ -\sqrt{3}/10 & 1/2 \end{pmatrix}.$$

4. GRADIENTS OF KIGAMI, KUSUOKA, AND STRICHARTZ

4.1. *Kusuoka Measure and Gradient for Nondegenerate Harmonic Structures.* In [Ku2, Ki3] S. Kusuoka proved a number of results describing the properties of the Dirichlet form $\mathscr{E}$. Here we sketch some of his ideas.

In [Ku2] only nondegenerate harmonic structures were considered. One can see that for an m-harmonic function f the gradient $\mathrm{Grad}_\omega f$ defined in the previous section is the same as $X(\omega, f)$ defined in [Ku2, Lemma 3.5].

A continuous function is called m-harmonic if $f \circ F_w$ is harmonic for any $w \in W_m$. One can see that if f is m-harmonic then $\mathrm{Grad}_{m, [\omega]_m} f = \mathrm{Grad}_{m+n, [\omega]_{m+n}} f$ for any $n \geqslant 0$.

It follows from the definitions in Section 2 and Section 3 that for an m-harmonic function f

$$\mathscr{E}(f, f) = \sum_{w \in W_m} \langle \mathrm{Grad}_{m,w} f, Z_m(w)\, \mathrm{Grad}_{m,w} f \rangle\, \nu(K_w)$$

which implies by Corollary 2.2 that

$$\mathscr{E}(f, f) = \int_K \langle \mathrm{Grad}_\omega f, Z(\omega) \, \mathrm{Grad}_\omega f \rangle \, dv. \tag{4.1}$$

Then one can extend (4.1) to $Dom\mathscr{E}$ because m-harmonic functions are dense in $Dom\mathscr{E}$ in $\mathscr{E}(\cdot, \cdot)$-norm. However, for $f \in Dom\mathscr{E}$ the limit in the definition of $\mathrm{Grad}_\omega f$ may not exists except as in a weak sense of the (semi-) norm $\int_K \langle \cdot, Z(\omega) \cdot \rangle \, dv$. In relation (4.1), $\mathrm{Grad}_\omega f$ can be substituted by $Y(\omega, f)$ which is equal to the orthogonal projection of $\mathrm{Grad}_\omega f$ onto the image of $Z(\omega)$.

In [Ku3] these results are extended to the case of all the nested fractals which may have a degenerate harmonic structure. However one needs to consider a modification $\tilde{Z}(\omega)$ of $Z(\omega)$ and a modification $u(f)(\omega)$ of $Y(\omega, f)$ in order to obtain a relation

$$\mathscr{E}(f, f) = \int_K \langle u(f)(\omega), \tilde{Z}(\omega) \, u(f)(\omega) \rangle \, dv$$

for any $f \in Dom\mathscr{E}$.

One of the main results in [Ku2, Ku3] is that under certain assumptions, which are satisfied for the Sierpiński gasket and many other fractals, $\mathrm{Rank}\, Z(\omega) = 1$ for v-almost all ω.

4.2. *Strichartz Gradient and Local Tangents.* In [St5] R. Strichartz studied approximation of functions by local tangents at junction and generic points. It is assumed that every boundary point is a fixed point of some F_s, and there are other assumptions (see [St5] for details).

Suppose a boundary point x is fixed by F_s. Then $x = \pi(\omega)$ where $\omega = \dot{s}$. If $\mathrm{Grad}_\omega f$ exists then for any harmonic function h we have a limit

$$d_h = \lim_{n \to \infty} \langle h, \mathrm{Grad}_\omega f \rangle = \langle (\tilde{M}_s^*)^{-n} h, \tilde{H} f \circ F_{[\omega]_n} \rangle$$

which can be called a directional derivative. If h is the kth eigenvector of $\tilde{M}_s^*$ with an eigenvalue λ_k ($|\lambda_k|$ are in decreasing order) then

$$d_k = \lim_{n \to \infty} (\lambda_k)^{-n} \langle h, \tilde{H} f \circ F_{[\omega]_n} \rangle$$

can be called kth derivative of f. In [St5] the collection of derivatives is called the gradient $df(x)$ at x. One can see that although the definitions of $\mathrm{Grad}_\omega f$ and df yield equivalent objects, $\mathrm{Grad}_\omega f$ is an element of $\tilde{\mathscr{H}}$ while $df(x)$ is, in a sense, an element of the adjoint space $\tilde{\mathscr{H}}^*$.

If x is a junction point then there are several $w_1, ..., w_l \in W_n$ such that $\{x\} = \bigcap_{j=1}^{l} K_{w_j}$. A local tangent is harmonic on the standard neighborhood $U = \bigcup_{j=1}^{l} K_{w_j}$ of x. The gradient $df(x)$ is defined as the collection of all the derivatives associated with each K_{w_j}.

The gradient $\mathrm{Grad}_\omega f$ defined in this paper can be called "global" because the corresponding harmonic function is defined on the entire fractal K. This "global" definition has an advantage that the gradient at any point is an element of the same vector space $\tilde{\mathscr{H}}$. In many situations the gradient $\mathrm{Grad}_\omega f$ depends continuously on the variable $\omega \in \Omega$. The gradient df defined in [St5] is better suitable for studying local approximation although it may vary very irregular if we move the point $x \in K$ it is computed at.

Without going into details, we would like to mention that [St5] contains a number of results on the existence of and the rate of approximation by harmonic tangents, and by tangents of higher order. It also contains a detailed study of the one dimensional case and of the structures with dihedral-3 symmetry. In particular, Theorem 2 was proved in [St5] for μ-almost all x and Theorem 3 was proved at every junction point.

4.3. *Kigami Gradient and Harmonic Metric on the Sierpiński Gasket.* In this subsection we deal only with the standard symmetric harmonic structure on the Sierpiński gasket. In [Ki3] Kigami considered functions of the form $f = F(h_1, h_2)$ where F is a C^1 function on $\mathbb{R}^2$ and $\{h_1, h_2\}$ is an $\|\cdot\|$-orthonormal basis of $\tilde{\mathscr{H}}$. Then he proved, among other results, that $f \in Dom\mathscr{E}$ and $\mathscr{E}(f, f) = \int_{SG} \langle \nabla f, Z\, \nabla f \rangle \, dv$ where for every $x \in K$ a gradient $\nabla f(x)$ is an element of $\tilde{\mathscr{H}}$ defined by

$$\nabla f(x) = \partial_1 F(a_1, a_2)\, h_1 + \partial_2 F(a_1, a_2)\, h_2, \tag{4.2}$$

where $a_k = h_k(x)$. It means that $\nabla f(x)$ is a linear combination of h_1 and h_2 with coefficients that are the partial derivatives of F evaluated at the point $(h_1(x), h_2(x)) \in \mathbb{R}^2$. This notion of a gradient was used in [MS]. Also note that (4.2) is a particular case of (3.4) with $f_k = h_k$.

The natural question is whether $\nabla f(x) = \mathrm{Grad}_{\pi^{-1}(x)} f$. We conjecture that this is not necessarily true for all x as it is suggested by another result of Kigami in the same paper: there is a dense set of x such that the limit in the definition of Z does not exist (see Corollary 2.2). However, we can give a partial answer to this question.

LEMMA 4.1. (1) *If $F \in C^2(\mathbb{R}^2)$ then $\nabla f(x) = \mathrm{Grad}_{\pi^{-1}(x)} f$ for μ-almost all x*

(2) *If $F \in C^4(\mathbb{R}^2)$ then $\nabla f(x) = \mathrm{Grad}_{\pi^{-1}(x)} f$ for any junction point x.*

Proof of (1). By the Lemma 3.5 it is enough to show that (3.3) holds for μ-almost all $\omega \in \Omega$. We claim that in fact

$$\lim_{n\to\infty} \exp\left(\frac{1}{n}\log(\|\tilde{M}^{-1}_{[\omega]_n}\| \, \|\tilde{M}_{[\omega]_n}\|^2)\right) < \frac{\sqrt{5}}{3} < 1$$

for μ-almost all ω, and so the result follows. To prove the claim note that $\|\tilde{M}_w\|/\|\tilde{M}_w^{-1}\| = |\det \tilde{M}_w| = (3/25)^n$ and for μ-almost all ω

$$\alpha = \lim_{n\to\infty} \exp\left(\frac{1}{n}\log\|\tilde{M}_{[\omega]_n}\|\right) < \frac{1}{\sqrt{5}}.$$

The last inequality was proved in [BST] and then improved in [St5, V]. ∎

Remark 4.2. Condition (1) of this lemma can be replaced by the assumption that $F \in C^1(\mathbb{R}^2)$ and the partial derivatives of F are Hölder continuous. The condition on the Hölder exponent depends on the value of the Lyapunov exponent α. Although the estimate for α we use here is relatively easy to prove, more precise estimations are very difficult to obtain even numerically (see [St5, V]).

Another way to improve the result in part 1 is to use pointwise estimates using the changes counting function $C(\omega, n)$, similarly to the proof of Theorem 2. The conclusion may hold for some non symmetric Bernoulli measures and, hopefully, for the Kusuoka measure ν.

Proof of (2). It is enough to prove the statement if x is a boundary point, say $x = p_1$. The result for any junction point can be obtained by a linear change of variable, that is by choosing a different basis in $\tilde{\mathscr{H}}$, not necessarily orthogonal. If F is linear then f is harmonic and $\nabla f(x) = \mathrm{Grad}_{\pi^{-1}(x)} f = \tilde{P} f$ for any x. Therefore we can assume that $F(h_1(x), h_1(x)) = 0$ and $\nabla F(h_1(x), h_1(x)) = 0$. To further simplify the situation we assume that h_1 is an R_1-symmetric and h_2 is an R_1-skew symmetric harmonic function and $h_1(x) = h_2(x) = 0$. Although the latter assumption is impossible if $h_1 \in \tilde{\mathscr{H}}$, the addition of a constant does not change the argument.

Let $\omega = \dot{1}$ and $w = [\omega]_n$. Since $F \in C^4(\mathbb{R}^2)$ we have

$$F(h_1 \circ F_w, h_2 \circ F_w) = A(\tfrac{9}{25})^n h_1^2 + 2B(\tfrac{3}{25})^n h_1 h_2 + C(\tfrac{1}{25})^n h_2^2 + D(\tfrac{27}{125})^n h_1^3 + O(\tfrac{3}{5})^{4n}_{n\to\infty} \tag{4.3}$$

because $M_w h_1 = (\frac{3}{5})^n h_1$ and $M_w h_2 = (\frac{1}{5})^n h_1$. Here A, B, C, and D are the appropriate second and third order partial derivatives of F at $(0, 0)$. Note that the other third order terms in (4.3) are $O((9/125)^n)_{n\to\infty}$.

We have that $\|\tilde{M}_w^{-1}\| = 5^n$ and $\tilde{M}_w^{-1}\tilde{H}g = (\frac{5}{3})^n\,\tilde{H}g$ for any R_1 symmetric continuous function g (we apply this relation to $g = h_1^2$ and $g = h_1^3$). Then we obtain $\mathrm{Grad}_\omega f = 0$ by the Definition 3.3.

Again, condition (2) of this lemma can be replaced by the assumption that $F \in C^3(\mathbb{R}^2)$ and the partial derivatives of F of the third order are Hölder continuous with a certain Hölder exponent. ∎

Remark 4.3. The proof of this lemma shows that at a junction point x we have $(f - \mathrm{Tan}_{\pi^{-1}(x)} f)\,|_{K_w} = O((9/25)^m)_{m \to \infty}$ if $F \in C^4(\mathbb{R}^2)$, $x \in K_w$, $w \in W_m$. This rate of approximation can be faster than that for a function in the domain of the Laplacian Δ (see Section 6), however, a function which is strongly differentiable in the sense of [St5] is approximated by local tangents at a rate of $O(5^{-m})$ (see Section 6 in [St5]).

At a generic (in terms of the measure μ) point x we have $(f - \mathrm{Tan}_{\pi^{-1}(x)} f)|_{K_w} = O(\alpha^{2m}) = O(5^{-m})_{m \to \infty}$ if $F \in C^2(\mathbb{R}^2)$, $x \in K_w$, $w \in W_m$. This rate of approximation by tangents is faster than that in [St5, Theorems 7.3]. However, this not an improvement of any result in [St5] because the function f considered in Lemma 4.1 is not in the domain of the Laplacian unlike functions considered in [St5] (see discussion below).

In the end of this section we compare informally some objects of similar nature which are not equal in the case of the Sierpiński gasket. Define $Y(\omega, f)$ as the orthogonal projection of $\mathrm{Grad}_\omega f$ onto the image of $Z(\omega)$ (see Subsection 2.4). Then $Y(\omega, f)$ typically is not equal to either $\mathrm{Grad}_\omega f$ or ∇f. The explanation is that the later two objects are (often) continuous in ω (as in Lemma 4.1 or Theorem 3) but $Y(\omega, f)$ can not be continuous since $Z(\omega)$ is discontinuous at every $\omega \in \Omega$. By a similar reason, even if a nonlinear function F is $C^\infty(\mathbb{R}^2)$ then $f = F(h_1, h_2)$ is not in the domain of the Laplacian because ∇f is continuous (it contradicts Propositions 6.3 and 6.4). However, one can expect f to be in the domain of the Laplacian defined with respect to the Kusuoka measure ν but not the Bernoulli measure μ, as considered in this paper.

5. LAPLACIAN AND THE CONTINUITY OF THE GRADIENT

In this section we assume the harmonic structure to be nondegenerate and regular, that is $r_j < 1$ for $j = 1, ..., N$.

Let μ be a finite nonatomic measure on K such that $\mu(O) > 0$ for any nonempty open set O. Then there is a dense set of continuous functions $Dom\Delta_\mu$ and an unbounded linear operator Δ_μ (Laplacian) such that

$$\mathscr{E}(u, v) = -\int_K u\Delta_\mu v\, d\mu + \sum_{p \in V_0} u(p)\, dv(p), \tag{5.1}$$

where $dv(p)$ is a certain normal (Neumann) derivative of v (see [Ki2, Proposition 7.3, Ki9]). If we fix boundary conditions, say Dirichlet or Neumann, and an appropriate domain then the Laplacian Δ_μ is a nonpositive self-adjoint operator. Alternatively, $\Delta_\mu f$ can be defined as a pointwise limit of difference operators $\Delta_{\mu,n} f$ (see [Ki2, Definition 6.1] or [Ki9]). In this paper we will use yet another equivalent definition.

We will say that $\Delta_\mu f = g$ if f and g are continuous functions and

$$f = G_\mu g + Hf, \tag{5.2}$$

where Hf is the unique harmonic function which coincides with f on the boundary of K and

$$G_\mu f(x) = \int_K f(y)\, g(x, y)\, d\mu(y). \tag{5.3}$$

Here $g(x, y)$ is a so-called Green's function, which is nonnegative and symmetric (see [Ki2, Ki9] and also (5.9)). Green's function is jointly continuous in x and y if $x \neq y$, and $g(x, y) = 0$ if x or y is a boundary point.

Since we assume in this section that the harmonic structure is regular, Green's function $g(x, y)$ is jointly continuous in x and y (see [Ki2, Proposition 5.4, Ki9]). Also we assume that μ is a fixed probability Bernoulli measure with weights $\mu_1, ..., \mu_N$. Then we will write Δ and G instead of Δ_μ and G_μ.

THEOREM 1. *Suppose $f \in Dom \Delta$. Then* $\text{Grad}_\omega f$ *exists for every $\omega \in \Omega$ such that*

$$\sum_{n \geqslant 1} r_{[\omega]_n}\, \mu_{[\omega]_n}\, \|M_{[\omega]_n}^{-1}\| < \infty. \tag{5.4}$$

Proof. Let $f \in Dom \Delta$. Then we have

$$\Delta(f \circ F_w) = r_w \mu_m (\Delta f) \circ F_w \tag{5.5}$$

that is the same as

$$f \circ F_w = r_w \mu_w G(\Delta f \circ F_w) + H(f \circ F_w). \tag{5.6}$$

Let ω be fixed. Then (5.6) implies

$$\begin{aligned} f \circ F_{[\omega]_{n+1}} &= r_{[\omega]_n} \mu_{[\omega]_n} G(\Delta f \circ F_{[\omega]_n}) \circ F_{\omega_{n+1}} + H(f \circ F_{[\omega]_n}) \circ F_{\omega_{n+1}} \\ f \circ F_{[\omega]_{n+1}} &= r_{[\omega]_{n+1}} \mu_{[\omega]_{n+1}} G(\Delta f \circ F_{[\omega]_{n+1}}) + H(f \circ F_{[\omega]_{n+1}}) \end{aligned} \tag{5.7}$$

and therefore

$$\begin{aligned}\operatorname{Grad}_{n+1,[\omega]_{n+1}} f - \operatorname{Grad}_{n,[\omega]_n} f &= \tilde{M}^{-1}_{[\omega]_{n+1}}(H(f\circ F_{[\omega]_{n+1}}) - H(f\circ F_{[\omega]_n})\circ F_{\omega_{n+1}}) \\ &= r_{[\omega]_n}\mu_{[\omega]_n}\tilde{M}^{-1}_{[\omega]_{n+1}} H(G(\Delta f\circ F_{[\omega]_n})\circ F_{\omega_{n+1}}) \end{aligned} \tag{5.8}$$

because of the fact that $HG\equiv 0$ by the definition of the Green's operator G (since $g(x, y)=0$ if $x\in V_0$).

The Green's function $g(x, y)$ has a representation

$$g(x, y) = \sum_{u\in W_*\cup\varnothing} r_u \sum_{p, q\in V_1\backslash V_0} X_{p,q}\psi_p(F_u^{-1}x)\,\psi_q(F_u^{-1}y), \tag{5.9}$$

where $X_{p,q}$ are certain positive coefficients and ψ_p is a unique 1-harmonic function which is one at p and zero at every other point of V_1 (see [Ki2, Definition 5.1]). Then

$$\begin{aligned} &H((Gu)\circ F_j) \\ &\quad = H\left(\int_K u(y)\sum_{u\in W_*} r_u \sum_{p, q\in V_1\backslash V_0} X_{p,q}\psi_p(F_u^{-1}x)\,\psi_q(F_u^{-1}y)\,d\mu(y)\circ F_j\right) \\ &\quad = \sum_{v\in W_*} r_v \sum_{p, q\in V_1\backslash V_0} X_{p,q}H(\psi_p\circ F_v^{-1}\circ F_j)\int_K u(y)\,\psi_q(F_v^{-1}y)\,d\mu(y) \\ &\quad = \sum_{p, q\in V_1\backslash V_0}\left(X_{p,q}\int_K u(y)\,\psi_q(y)\,d\mu(y)\right)\psi_p\circ F_j \end{aligned} \tag{5.10}$$

because $H(\psi_p\circ F_v^{-1}\circ F_j)$ is zero unless u is an empty word. Note that $\psi_p\circ F_j$ is a harmonic function because ψ_p is 1-harmonic.

By (5.8) and (5.10) with $u=\Delta(f\circ F_{[\omega]_n})$ and $j=\omega_{n+1}$ we have

$$\begin{aligned} &\operatorname{Grad}_{n+1,[\omega]_{n+1}} f - \operatorname{Grad}_{n,[\omega]_n} f \\ &\quad = r_{[\omega]_n}\mu_{[\omega]_n}\tilde{M}^{-1}_{[\omega]_n}\sum_{p, q\in V_1\backslash V_0}\left(X_{p,q}\int_K \Delta f(F_{[\omega]_n}(y))\,\psi_q(y)\,d\mu(y)\right) \\ &\qquad \times \tilde{M}^{-1}_{\omega_{n+1}}(\psi_p\circ F_{\omega_{n+1}}). \end{aligned} \tag{5.11}$$

There is constant C such that for any j

$$\sum_{p, q\in V_1\backslash V_0} \|X_{p,q}\tilde{M}_j^{-1}(\psi_p\circ F_j)\| \int_K |\psi_q(y)|\,d\mu(y) \leqslant C. \tag{5.12}$$

Then

$$\|\mathrm{Grad}_{n+1,\,[\omega]_{n+1}} f - \mathrm{Grad}_{n,\,[\omega]_n} f\| \leqslant C\,\|\Delta f(x)\|_\infty\, r_{[\omega]_n}\mu_{[\omega]_n}\,\|\tilde{M}^{-1}_{[\omega]_n}\|. \tag{5.13}$$

Thus $\mathrm{Grad}_{n,\,[\omega]_n} f$ is a Cauchy sequence. ▮

COROLLARY 5.1. *Suppose $f \in \mathrm{Dom}\Delta$. Then* $\mathrm{Grad}_\omega f$ *exists for all $\omega \in \Omega$ if*

$$r_j\mu_j\,\|\tilde{M}_j^{-1}\| < 1 \tag{5.14}$$

for $j = 1, ..., N$. Moreover, in this case $\mathrm{Grad}_\omega f$ *is continuous in $\omega \in \Omega$.*

Proof. Under condition (5.14) the sequence $\{\mathrm{Grad}_{n,\,[\omega]_n} f\}$ is a uniformly convergent sequence of continuous $\mathscr{H}$-valued functions on Ω because

$$\|\mathrm{Grad}_{n,\,[\omega]_n} f - \mathrm{Grad}_{m,\,[\omega]_m} f\| \leqslant C\,\|\Delta f(x)\|_\infty \sum_{k=m}^{n-1} r_{[\omega]_k}\mu_{[\omega]_k}\,\|\tilde{M}^{-1}_{[\omega]_k}\| \tag{5.15}$$

by (5.13). ▮

The conditions of this proposition are true for the standard harmonic structure on an interval, but we do not know any other nondegenerate fractal which satisfies (5.14).

Remark 5.2. It is easy to see that the results of these theorem and proposition hold if $f = Gg$ where g is bounded measurable, not necessarily continuous.

A function $F(\omega)$ is continuous on Ω if and only if a function $\tilde{F}(x) = \tilde{F}(\pi^{-1}(x))$ is continuous at any nonjunction point x and $\lim_{y \to x,\, y \in K_w} \tilde{F}(y)$ exists for any junction point x on the boundary of K_w, $w \in W_*$.

Let x be a junction point, say x be a common boundary point of several K_w, $w \in W_n$. Then for each such w we can define a "directional" gradient $\mathrm{Grad}_{n,x,w} f$. Namely, $\mathrm{Grad}_{n,x,w} f = \mathrm{Grad}_\omega f$ where ω is a unique element of Ω such that $\pi(\omega) = x$ and $[\omega]_n = w$.

6. GRADIENT ON THE SIERPIŃSKI GASKET

Let $C(\omega, n) = \#\{\omega_j \neq \omega_{j+1}, 1 \leqslant j \leqslant n-1\}$, that is, let $C(\omega, n)$ be the number of changes in the sequences $\{\omega_1, ..., \omega_n\}$.

THEOREM 2. *If Δf is continuous on the Sierpiński gasket then* $\mathrm{Grad}_\omega f$ *is defined at every $\omega \in \Omega$ such that*

$$\liminf_{n\to\infty} \frac{C(\omega, n)}{\log n} \geqslant \gamma \tag{6.1}$$

where γ is a certain constant.

Elements $\omega \in \Omega$ which satisfy (6.1) are generic in the sense that they represent a set of full measure for any Bernoulli measure on the Sierpiński gasket.

Proof of Theorem 2. If $i \neq j$ then $\|\tilde{M}_i^{-1}\tilde{M}_j^{-1}\| = 25\beta^2$ where $\beta = \sqrt{(7+\sqrt{13})/18} < 1$. Then $\|\tilde{M}_{[\omega]_n}^{-1}\| \leqslant 5^n \beta^{C(\omega, n)}$. Since $r_{[\omega]_n}\mu_{[\omega]_n} = 5^{-n}$, the series in (5.4) converges if

$$\sum_{n=1}^{\infty} \beta^{C(\omega, n)} < \infty. \tag{6.2}$$

Thus the assertion is true for any $\gamma > -1/\log\beta$. ∎

PROPOSITION 6.1. *There exists a function f such that Δf is continuous but* $\mathrm{Grad}_\omega f$ *is not defined on a dense set of $\omega \in \Omega$.*

Proof. First, we construct a function f such that Δf is continuous but $\mathrm{Grad}_\omega f$ does not exists for $\omega = \dot{1}$.

Let f_0 be a continuous nonzero function that satisfies the following three conditions; (a) $f_0 \circ F_{12}$ is nonnegative; (b) f_0 is skew-symmetric in the sense that $f_0 \circ R_j = -f_0$ for any $j = 1, 2, 3$; (c) f_0 is zero on $F_{11}(\mathrm{SG})$. Here R_j is the reflection of Sierpiński gasket which fixes the corner p_j.

From (a), (b), and (c) we have that $f_0 \circ F_{12}$, $f_0 \circ F_{23}$ and $f_0 \circ F_{31}$ are nonnegative, $f_0 \circ F_{21}$, $f_0 \circ F_{32}$ and $f_0 \circ F_{13}$ are nonpositive and $f_0 \circ F_{11} = f_0 \circ F_{22} = f_0 \circ F_{33} = 0$. In fact, there are 3-harmonic functions which satisfy (a), (b), and (c) but no 2-harmonic function.

It is easy to see by the definition (5.9) of G that Gf_0 is also skew-symmetric. Moreover, $h_0 = (Gf_0) \circ F_{11}$ is a nonzero R_1-skew symmetric harmonic function, that is $h_0 \circ R_1 = -h_0$. Then $\mathrm{Grad}_{n, [\omega]_n}(Gf_0) = 5^2 h_0$ for $n \geqslant 2$. Let

$$f = \sum_{n=0}^{\infty} \frac{1}{n+1} 5^{-2n} (Gf_0) \circ F_{[\omega]_{2n}}^{-1},$$

where $\omega = \dot{1}$ and $(Gf_0) \circ F_{[\omega]_{2n}}^{-1} = 0$ outside of $F_{[\omega]_{2n}}(\mathrm{SG})$. Then $\Delta f = \sum_{n=0}^{\infty} \frac{1}{n+1} f_0 \circ F_{[\omega]_{2n}}^{-1}$ is a continuous function because functions $f_0 \circ F_{[\omega]_{2n}}^{-1}$ have disjoint support for different n. We see that $\mathrm{Grad}_{2m, [\omega]_{2m}} f = \sum_{n=0}^{m-1} \frac{1}{n+1} 5^{-2n+2} h_0 \circ F_{[\omega]_{2n}}^{-1} = 5^2 h_0 \sum_{n=0}^{m-1} \frac{1}{n+1}$ and so $\mathrm{Grad}_\omega f$ does not exist.

Then it is easy to construct a function such that the gradient does not exist for any ω for which $\pi(\omega)$ is a junction point. ∎

THEOREM 3. *Suppose Δf is Hölder continuous on the Sierpiński gasket, that is $|\Delta f(x) - \Delta f(y)| \leqslant c\rho^n$ if $x, y \in F_w SG$, $w \in W_n$. Then* $\mathrm{Grad}_\omega f$ *is defined for every $\omega \in \Omega$ and*

$$\|\mathrm{Grad}_\omega f\| \leqslant const \left(\frac{c}{(1-\rho)} + \|\Delta f(x)\|_\infty \right). \tag{6.3}$$

Moreover, $\mathrm{Grad}_\omega f$ *is continuous at $\omega \in \Omega$ unless $\pi(\omega)$ is a boundary or junction point. If $\pi(\omega)$ is a boundary or junction point, then* $\mathrm{Grad}_\omega f$ *is continuous at $\omega \in \Omega$ if and only if $\Delta f(\pi(\omega)) = 0$.*

Remark 6.2. It is proved in [St5] that any function in the domain of the Laplacian is Hölder continuous with $\rho = \frac{3}{5}$ (see also Appendix). Therefore the conclusions of this theorem hold if both f and Δf are in the domain of the Laplacian, say if f is an eigenfunction of Δ.

Proof of Theorem 3. By (5.11) we have

$$\begin{aligned} &\mathrm{Grad}_{n+1, [\omega]_{n+1}} f - \mathrm{Grad}_{n, [\omega]_n} f \\ &\quad = r_{[\omega]_n} \mu_{[\omega]_n} \tilde{M}^{-1}_{[\omega]_n} \sum_{p, q \in V_1 \setminus V_0} \left(X_{p,q} \int_K \Delta f(F_{[\omega]_n}(y))\, \psi_q(y)\, d\mu(y) \right) \\ &\qquad \times \tilde{M}^{-1}_{\omega_{n+1}} (\psi_p \circ F_{\omega_{n+1}}). \end{aligned} \tag{6.4}$$

Let $h = \sum_{p, q \in V_1 \setminus V_0} (X_{p,q} \int_K \Delta f(F_{[\omega]_n}(y))\ \psi_q(y)\, d\mu(y))\ \tilde{M}^{-1}_{\omega_{n+1}} (\psi_p \circ F_{\omega_{n+1}})$. Denote $h_s = \frac{1}{2}(h + h \circ R_{\omega_{n+1}})$ and $h_a = \frac{1}{2}(h - h \circ R_{\omega_{n+1}})$, that is, h_s and h_a are $R_{\omega_{n+1}}$-symmetric and $R_{\omega_{n+1}}$-skew symmetric parts of h. Then $\|h_a\| \leqslant C\rho^n$ and $\|h_s\| \leqslant C\, \|\Delta f\|_\infty$. Therefore

$$r_{[\omega]_n} \mu_{[\omega]_n} \|\tilde{M}^{-1}_{[\omega]_n} h_a\| \leqslant C\rho^n \tag{6.5}$$

and

$$r_{[\omega]_n} \mu_{[\omega]_n} \|\tilde{M}^{-1}_{[\omega]_n} h_s\| \leqslant (\tfrac{3}{5})^m\, \beta^{C(\omega, m)} C\, \|\Delta f\|_\infty, \tag{6.6}$$

where $C(\omega, n)$ is defined before Theorem 2 and m is the smallest number such that $[\omega]_m = [\omega]_{m+1} = \cdots = [\omega]_{n+1}$.

Let $\omega \in \Omega$ be fixed. Denote by $\{n_k\}_{k=1}^{\infty}$ a unique increasing sequence such that $\omega_{n_k} = \omega_{n_k+1} = \cdots = \omega_{n_{k+1}-1} \neq \omega_{n_{k+1}}$. Then it is easy to see that $C(\omega, n) = k-1$ if $n_k \leqslant n < n_{k+1}$. We have

$$\sum_{m=n_k}^{n_{k+1}-1} \beta^{C(\omega, m)} \left(\tfrac{3}{5}\right)^{m-n_k} \leqslant \tfrac{5}{2} \beta^{k-1}$$

and so

$$\sum_{m=1}^{\infty} r_{[\omega]_n} \mu_{[\omega]_n} \|\tilde{M}_{[\omega]_n}^{-1} h_s\| \leqslant \tfrac{5}{2} \sum_{k=1}^{\infty} \beta^{k-1} < \infty.$$

The continuity of the gradient is implied by the fact that

$$\begin{aligned}\sum_{m=m_0}^{\infty} r_{[\omega]_n} \mu_{[\omega]_n} \|\tilde{M}_{[\omega]_n}^{-1} h_s\| &\leqslant \frac{5}{2} \beta^{C(\omega, m_0)} \sum_{k=1}^{\infty} \beta^{k-1} \\ &= \frac{5}{2} \frac{1}{\beta(1-\beta)} \beta^{C(\omega, m_0)},\end{aligned}$$

where the right hand side depends only on $[\omega]_{m_0}$ and $\beta^{C(\omega, m_0)} \to 0$ as $m_0 \to \infty$ unless $\pi(\omega)$ is a boundary or junction point.

If $\pi(\omega)$ is a boundary or junction point and $\Delta f(x) = 0$ then instead of (6.5) and (6.6) one can use

$$r_{[\omega]_n} \mu_{[\omega]_n} \|\tilde{M}_{[\omega]_n}^{-1} h\| \leqslant C\rho^n. \tag{6.7}$$

In the case $\pi(\omega)$ is a boundary or junction point and $\Delta f(x) \neq 0$ the discontinuity of the gradient is proved in Proposition 6.4. ∎

PROPOSITION 6.3. *Suppose Δf is Hölder continuous on the Sierpiński gasket. Then* $\mathrm{Grad}_{\pi^{-1}(x)} f$ *is not continuous at $x \in SG$ if x is a junction point and $\Delta f(x) \neq 0$.*

Proof. Let x be a junction point. Then there are two elements ω and ω' of Ω such that $\pi(\omega) = \pi(\omega') = x$. Theorem 3 implies that $\mathrm{Grad}_\omega f$ and $\mathrm{Grad}_{\omega'} f$ exist. It is easy to see that if $\mathrm{Grad}_\omega f = \mathrm{Grad}_{\omega'} f$ then $\Delta f(x) = 0$. Thus $\mathrm{Grad}_{\pi^{-1}(x)} f$ is discontinuous at x unless $\Delta f(x) = 0$. ∎

This proposition also follows from the next one. We give a separate proof of Proposition 6.3 because it is much simpler than that of Proposition 6.4, and in a sense it provides a different reason for the discontinuities of $\mathrm{Grad}_{\pi^{-1}(x)} f$.

PROPOSITION 6.4. *Suppose Δf is Hölder continuous on the Sierpiński gasket. Then* $\mathrm{Grad}_\omega f$ *is not continuous at $\omega \in \Omega$ if $\pi(\omega)$ is a boundary or junction point and $\Delta f(\pi(\omega)) \neq 0$.*

Proof. The result for any junction point will follow if we prove it for $\omega = \dot{1}$. Moreover, since Δf is Hölder continuous, it is enough to prove it only for the case $f = G1$. Thus we assume $\omega = \dot{1}$, $\Delta f = 1$ and f is zero at the boundary. Let $\alpha_n = [\omega]_n \dot{2} \in \Omega$. Then $\alpha_n \to \omega$ as $n \to \infty$. We claim $\lim_{n \to \infty} \mathrm{Grad}_{\alpha_n} f \neq \mathrm{Grad}_\omega f$. This is true because the R_1-skew symmetric part of $\mathrm{Grad}_{\alpha_n} f - \mathrm{Grad}_{n, [\omega]_n} f$ is not zero and does not depend on n. ∎

7. GRADIENT FOR WEAKLY NONDEGENERATE HARMONIC STRUCTURES

DEFINITION 7.1. For a degenerate harmonic structure the *weak gradient* is the element of $\tilde{\mathscr{H}}$ defined by

$$\mathrm{Grad}_\omega f = \lim_{n \to \infty} \mathrm{Grad}_{n, [\omega]_n} f$$

if the limit exists. Here for $w \in W_n$

$$\mathrm{Grad}_{n, w} f = P_w^{Ker^\perp} \tilde{M}_w^{-1} P_w^{Im} \tilde{H}(f \circ F_w) \tag{7.2}$$

and P_w^{Im}, $P_w^{Ker^\perp}$ are the orthogonal projectors onto the image of $\tilde{M}_w$ and the orthogonal complement of *Ker* $\tilde{M}_w$, respectively.

Let us describe this definition of $\mathrm{Grad}_{n, w} f$ informally. The first difficulty in the case of a degenerate harmonic structure is that there may not exist a harmonic function which coincides with f on the boundary of K_w. So we introduce the orthogonal projector P_w^{Im} which gives us a harmonic function that minimizes the energy of $(h - f)|_{K_w}$. The next difficulty is that there may exist more than one harmonic function with the same values on K_w. We take among them the harmonic function of the smallest energy by introducing the orthogonal projector $P_w^{Ker^\perp}$. In other words, $\mathrm{Grad}_{n, w} f$ is the harmonic function h of the smallest energy such that h minimizes the energy of $(h - f)|_{K_w}$. Note that $P_w^{Ker^\perp} \tilde{M}_w^{-1} P_w^{Im}$ is a well defined linear operator even if $\tilde{M}_w$ is not invertible.

DEFINITION 7.2. A harmonic structure is said to be *weakly nondegenerate* if for any $w' \in W_*$ and any nonconstant harmonic function h there exists $w \in W_*$ such that h is not constant on $K_{ww'}$.

PROPOSITION 7.3. *A harmonic structure is weakly nondegenerate if and only if for any nonzero invariant subspace* $\widetilde{\mathscr{H}}'$ *of* $\widetilde{\mathscr{H}}$ *and any* $w \in W_*$ *there exists* $h' \in \widetilde{\mathscr{H}}'$ *such that* $\widetilde{M}_w h' \neq 0$, *that is* $\operatorname{Rank} \widetilde{M}_w|_{\widetilde{\mathscr{H}}'} > 0$.

A subspace $\widetilde{\mathscr{H}}'$ *of* $\widetilde{\mathscr{H}}$ *is called invariant if it is invariant for any* $\widetilde{M}_j$, $j = 1, ..., N$.

EXAMPLE 7.4. *Hexagasket* (*fractal star of David*) (weakly nondegenerate harmonic structure). Let $p_1, ..., p_6$ be the corners of a regular hexagon. We define $F_i(x) = \frac{1}{3}(x + 2p_i)$, $i = 1, ..., 6$. The hexagasket is a unique compact subset K of $\mathbb{R}^2$ such that $K = \bigcup_{i=1}^{6} F_i(K)$. Then $V_0 = \{p_1, ..., p_6\}$. There is an alternative construction which uses only three of the corners of the large hexagon as the boundary (some of the maps F_i involve rotations after contractions). Then the approximating graphs are made from the stars of David, which gives the second name (see Fig. 2).

EXAMPLE 7.5. *Vicsek set* (degenerate harmonic structure). Let p_1, p_2, p_3, p_4 be the corners and p_5 be the center of a square. We define $F_i(x) = \frac{1}{3}(x + 2p_i)$, $i = 1, ..., 5$. The Vicsek set is a unique compact subset K of $\mathbb{R}^2$ such that $K = \bigcup_{i=1}^{5} F_i(K)$. Then $V_0 = \{p_1, ..., p_4\}$ (see Fig. 3).

It is easy to see that the measure ν is concentrated on the main diagonals. It is a multiple of the Lebesgue measure on these two line segments. Fractals which are topological trees, like this one, were considered in [Ki6].

We define a (semi-) norm

$$\|u\|^2_{\nu, \widetilde{\mathscr{H}}, Z} = \int_K \langle u(\omega), Z(\omega)\, u(\omega) \rangle \, d\nu(\omega)$$

on the space $L^2_{\nu, \widetilde{\mathscr{H}}}$ of $\widetilde{\mathscr{H}}$-valued functions.

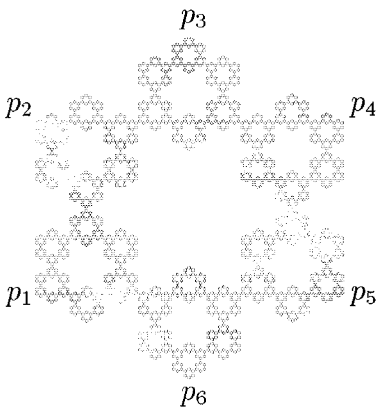

FIG. 2. Hexagasket (fractal star of David).

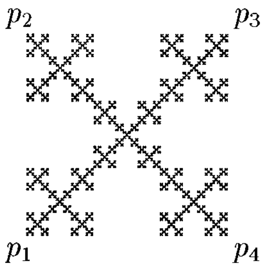

FIG. 3. Vicsek set.

THEOREM 4. *Suppose the harmonic structure* $(K, S, \{F_s\}_{s \in S})$ *is weakly nondegenerate and* $f \in \mathrm{Dom}\mathscr{E}$. *Then* $\mathrm{Grad}_\omega f$ *exists for* ν*-almost all* ω *in the sense that the limit in* (7.1) *exists in* $\|\cdot\|_{\nu, \mathscr{H}, Z}$ (*semi-*) *norm. Moreover*

$$\mathscr{E}(f, f) = \|\mathrm{Grad}_\omega f\|^2_{\nu, \mathscr{H}, Z} = \int_K \langle \mathrm{Grad}_\omega f, Z(\omega)\, \mathrm{Grad}_\omega f \rangle \, d\nu(\omega). \tag{7.3}$$

This theorem is a generalization of a result in [Ku2]; it is similar to a result in [Ku3] (see discussion in Subsection 4.1).

Before we prove this theorem we need the following lemma:

LEMMA 7.6. *Let* $W'_* \subseteq W_*$ *be a nonempty collection of words such that if* $w \in W_*$ *and* $w' \in W'_*$ *then* $ww' \in W'_*$. *If the harmonic structure is weakly nondegenerate then for any harmonic function* h

$$\mathscr{E}(h, h) = \lim_{n \to \infty} \sum_{w \in W_n \cap W'_*} r_w^{-1} \|M_w h\|^2. \tag{7.4}$$

Proof. Let $h \in \tilde{\mathscr{H}}$ be nonzero. By the Definition 7.2 for any $w' \in W_*$ there exists $w \in W_*$ such that $\tilde{M}_{ww'} h \neq 0$. Using the compactness argument one can show that there are m' and $\varepsilon' > 0$ such that for any $h \in \tilde{\mathscr{H}}$ we have

$$\sum_{w \in W_{m'} \cap W'_*} r_w^{-1} \|\tilde{M}_w h\|^2 \geqslant \varepsilon' \|h\|^2.$$

Then

$$\|h\| = \sum_{w \in W_{m'}} r_w^{-1} \|\tilde{M}_w h\|^2 \geqslant \varepsilon' \|h\|^2 + \sum_{w \in W_{m'} \setminus W'_*} r_w^{-1} \|\tilde{M}_w h\|^2.$$

Therefore for any $n \geqslant 0$

$$(1-\varepsilon') \sum_{w \in W_n \backslash W'_*} r_w^{-1} \|\tilde{M}_w h\|^2 \geqslant \sum_{w \in W_{n+m'} \backslash W'_*} r_w^{-1} \|\tilde{M}_w h\|^2.$$

This implies (7.4) because of (2.3). ∎

COROLLARY 7.7. *If the conditions of Lemma* 7.6 *are satisfied and* $\Omega' = \{\omega \in \Omega\colon [\omega]_n \in W'_*$ *for some* $n\}$, *then* $\nu(\Omega') = \nu(\Omega)$.

Proof of the Theorem 4. Recall that a function f is called m-harmonic if it is continuous and $f \circ F_w$ is harmonic for any $w \in W_m$. It is known that the space of m-harmonic functions is dense in $C(K)$ and also is dense in $Dom\mathscr{E}$ in $\mathscr{E}(\cdot, \cdot)$-norm. It is easy to show for any m-harmonic function f that

$$\mathscr{E}(f, f) \geqslant \int_K \langle \mathrm{Grad}_\omega f, Z(\omega)\, \mathrm{Grad}_\omega\, \omega f \rangle \, d\nu(\omega).$$

Then the statement follows from Lemma 7.6 with $W'_* = \{w\colon \mathrm{Rank}\, \tilde{M}_w = R_{min}\}$ where $R_{min} = \min_{w \in W_*} \mathrm{Rank}\, \tilde{M}_w$. ∎

APPENDIX

An Estimate of the Local Energy of Harmonic Functions

In this appendix we give a proof of inequality (A.1). In [St5] R. Strichartz stated a hypothesis that $\|\tilde{M}_j\| \leqslant r_j$ for any $j = 1, ..., N$ (Hypothesis 8.1 in [St5]). The theorem we prove here implies a slightly weaker statement: for any $j = 1, ..., N$ there is a matrix norm $\|\cdot\|_j$ such that $\|\tilde{M}_j\|_j \leqslant r_j$. It also implies that $\rho(\tilde{M}_j) \leqslant r_j$ where $\rho(\tilde{M}_j)$ is the spectral radius of $\tilde{M}_j$ (the information on the matrix norms can be found in [HJ]).

As it was shown in [St5], if F_j fixes a boundary point then r_j is the largest eigenvalue of $\tilde{M}_j$ and its multiplicity is one. It follows that for such j we have $\|\tilde{M}_j\|_j = r_j$ for some matrix norm $\|\cdot\|_j$. Moreover, if we deal with a harmonic structure which is dihedral-3 symmetric then $\|\tilde{M}_j\| = r_j$.

THEOREM 5. *For any harmonic structure there exists a constant* B *such that*

$$\|\tilde{M}_w\| \leqslant B r_w \tag{A.1}$$

for any $w \in W$.

For any harmonic function h this means $\mathscr{E}(h \circ F_w, h \circ F_w) \leqslant B^2 r_w^2 \mathscr{E}(h, h)$ because of the definition of M_j, $\tilde{M}_j$ and $\|\cdot\|$ (see Subsection 2.3). In other words, by (2.6) it means that the energy of h concentrated in the set K_w is at most $B^2 r_w \mathscr{E}(h, h)$. Interestingly, it follows in particular that $\sum_{j=1}^{N} r_j \geqslant 1$.

It is proved in [St5] that inequality (A.1) implies that any function in the domain of the Laplacian satisfies an estimate $|f(x) - f(y)| \leqslant c r_w$ for any $x, y \in K_w$, where the constant c may be taken to be a multiple of $\|f\|_\infty + \|\Delta f\|_\infty$.

Before the proof of Theorem 5 we need to introduce some notation and prove Lemma A.1.

There is a Dirichlet form $\mathscr{E}_m$ on $\ell^2(V_m)$ such that for any harmonic function h and any $m \geqslant 0$ we have $\mathscr{E}_m(h, h) = \mathscr{E}(h, h)$ (see [Ki2, Ki9]). This form can be defined by

$$\mathscr{E}_m(f, f) = \sum_{w \in W_m} \sum_{x, y \in V_w} D^w_{x, y}(f(x) - f(y))^2, \tag{A.2}$$

where $D^w_{x, y} = r_w^{-1} D_{F_w^{-1}(x), F_w^{-1}(y)}$ if $x, y \in V_w = F_w(V_0)$. Here $\{D_{p, q}\}_{p, q \in V_0}$ is a nonpositive matrix and $D_{p, q} \geqslant 0$ if $p \neq q$.

In the next lemma $m \geqslant 0$ is fixed.

LEMMA A.1. *Let h be a harmonic function such that its value at each boundary point is either 0 or $M > 0$. Then for any $w \in W_m$ and any $x, y \in V_w$ we have*

$$D^w_{x, y} |h(x) - h(y)| \leqslant \frac{\mathscr{E}(h, h)}{M}. \tag{A.3}$$

Proof. Let E_a be the set of ordered triples (x, y, w) such that $w \in W_m$, $x, y \in V_w$, $h(x) < a$, $h(y) \geqslant a$. Denote

$$F(a) = \sum_{(x, y, w) \in E_a} D^w_{x, y}(h(x) - h(y)). \tag{A.4}$$

It is easy to see that $F(a)$ is zero if $a \notin (0, M]$. We claim that $F(a) = \frac{\mathscr{E}(h, h)}{M}$ if $a \in (0, M]$. Then (A.3) follows because all the terms in (A.4) are nonnegative.

To prove the claim note that $F(a)$ is constant for $0 < a < M$ because for any nonboundary point $x \in V_m$ we have

$$\sum_{w, y:\, w \in W_m,\, x, y \in V_w} D^w_{x, y}(h(x) - h(y)) = 0 \tag{A.5}$$

since h is harmonic. So the claim holds because Lemma 6.7 in [Ki2] (or formulas (A.2) and (A.5)) implies $\mathscr{E}(h, h) = MF(M)$. ∎

Remark A.2. Inequality (A.3) has a very clear meaning in terms of electrical networks. Suppose we have a network where points x and y are connected by a resistor with the conductance $D^w_{x,y}$. To each boundary point we apply electric potential either zero or M. Then $\frac{\mathscr{E}(h,h)}{M}$ is the total electrical current through the network because of the formula "E = IU," the energy is the current times the change of the potential. Then the inequality (A.3) says that the current through any particular resistor is not greater than the total current.

Indeed, the inequality (A.3) holds not only in the self-similar situation we consider but for any network (that is for any Dirichlet form on a finite graph). Moreover, one can show for any function that $\mathscr{E}(f,f) = -\int_{-\infty}^{\infty} a\, dF(a)$.

Proof of the Theorem 5. There is a constant C_1 such that for any harmonic function h we have $\|h\| \leqslant C_1 \max_{x,y \in V_0} D_{x,y} |h(x) - h(y)|$ because all the norms on a finite dimensional vector space $\mathscr{H}$ are equivalent.

Lemma A.1 implies that if h is a harmonic function which is 1 at one boundary point and 0 at the others, then $\|\tilde{M}_w h\| \leqslant C_1 r_w \|h\|^2$. Since such harmonic functions span $\mathscr{H}$, there is a constant C_2 such that $\|\tilde{M}_w\| \leqslant C_1 C_2 r_w$ for any $w \in W_*$. ∎

ACKNOWLEDGMENTS

The author is extremely grateful to J. Kigami and R. Strichartz for many important remarks and suggestions. The author thanks M. Barlow, R. Bass, and V. Metz for very helpful and interesting discussions related to this work.

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