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Self-Similarity, Operators and Dynamics

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Abstract. We construct a large class of infinite self-similar (fractal, hierarchical or substitution) graphs and show, under a certain strong symmetry assumption, that the spectrum of the Laplacian can be described in terms of iterations of an associated rational function (so-called 'spectral decimation'). We prove that the spectrum consists of the Julia set of the rational function and a (possibly empty) set of isolated eigenvalues which accumulate to the Julia set. In order to obtain our results, we start with investigation of abstract spectral self-similarity of operators.

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1. Introduction

In this paper we construct a large class of infinite self-similar graphs for which the spectrum of the Laplacian is self-similar in the sense that it can be completely described in terms of iterations of a rational function. This phenomenon, usually called 'spectral decimation', was first observed for Sierpiński lattice in the physics literature [16, 17], and later a mathematical theory was developed [2, 4, 19, 20, 23]. Other fractals were considered in [7, 12, 14, 21], and the complex multi-dimensional case in [18]. Graphs of certain fractal groups studied in [1, 5, 6] have a fractal spectrum, although they seem to be different from the graphs considered here. For more information on analysis on fractals, see [9] and references therein.

We show that the spectrum of self-similar graphs satisfying a certain strong symmetry assumption consists of the Julia set of a rational function and a (possibly empty) set of isolated eigenvalues which accumulate to the Julia set. These eigenvalues, typically, have infinite multiplicity.

The graphs we consider can be called fractal, hierarchical or substitution graphs. These graphs are often related to nested fractals [13, 22]. A class of such graphs

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with only two boundary points, two-point self-similar graphs (TPSG), was considered in [15], where completeness of localized eigenfunctions was proved under mild conditions.

Our construction of a self-similar graph is based on a finite symmetric M-point model graph. The model graph determines a self-similar sequence of finite graphs which then defines an infinite self-similar graph. This infinite graph always has polynomial growth and is assumed to be connected. There is no spectral gap that is zero (or one in the probabilistic statement), is a point of the spectrum, and is not an isolated point. The graph random walk is recurrent. The spectrum of the Laplacian is described in terms of rational functions $R(z)$, $\varphi_1(z)$ and $\varphi_0(z)$ which depend only on the model graph. These functions are difficult to read directly from the structure of the model graph, but they can be computed effectively by (3.2). By (3.3), $\varphi_1(z)$ and $\varphi_0(z)$ are two Green's functions, and $R(z)$ is the ratio of them.

The necessary and sufficient conditions for spectral similarity on fractals were first considered in [21]. However, these conditions are algebraic in nature and difficult to verify. Our work gives sufficient conditions that are slightly different than those of [21], where only nested fractals were considered. We also give certain simple necessary conditions for spectral similarity. The main difference between our paper and [21] is that we analyze the spectrum of infinite graphs, while [21] studied the fractals, which are compact sets and give rise to discrete spectrum. We note that our main results were obtained independently of [21]. Another independent study of the spectrum of self-similar graphs has appeared recently in [10, 11].

2. Abstract Spectral Self-Similarity

Let $\mathcal{H}$ and $\mathcal{H}_0$ be Hilbert spaces, and $U\colon \mathcal{H}_0 \to \mathcal{H}$ be an isometry. Suppose that H and H_0 are operators on $\mathcal{H}$ and $\mathcal{H}_0$, respectively, and that φ_0 and φ_1 are complex-valued functions defined on a set $\Lambda \subseteq \mathbb{C}$. Here and in what follows, an operator always means a bounded linear operator unless stated otherwise.

DEFINITION 2.1. We call an operator H *spectrally similar* to an operator H_0 with functions φ_0 and φ_1 if

$$U^*(H - z)^{-1}U = (\varphi_0(z)H_0 - \varphi_1(z))^{-1} \tag{2.1}$$

$z \in \Lambda_0$, where Λ_0 is the set of those $z \in \Lambda$ for which the two sides of (2.1) are well defined. We always assume that Λ_0 is open and not empty.

In particular, if $\varphi_0(z) \neq 0$ and $R(z) = \varphi_1(z)/\varphi_0(z)$, then

$$U^*(H - z)^{-1}U = \frac{1}{\varphi_0(z)}(H - R(z))^{-1}. \tag{2.2}$$

Remark 2.2. The functions φ_0 and φ_1 are defined uniquely by (2.1) if and only if H_0 is not a multiple of the identity operator on $\mathcal{H}_0$. In what follows, we always assume that this is the case.

We also often assume that $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$. In this case, U is the inclusion operator, which we will omit, and $U^* = P_0$, the orthogonal projector onto $\mathcal{H}_0$.

EXAMPLE 2.3. Suppose $\mathcal{H} = \mathbb{R}^3$ and $\mathcal{H}_0 = \mathbb{R}^2$ in the decomposition $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}^1$. Then the operators H and H_0 given by

$$H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad H_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

are spectrally similar. In view of (2.1) or (3.2) below, it is easy to compute that $\varphi_1(z) = z/(z-1)$, $\varphi_0(z) = z$ and $R(z) = z - 1$.

EXAMPLE 2.4. Suppose again that $\mathcal{H} = \mathbb{R}^3$ and $\mathcal{H}_0 = \mathbb{R}^2$ in the decomposition $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}^1$. Then the operators H and H_0 given by

$$H = \begin{pmatrix} -1 & 0 & 1/2 \\ 0 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{pmatrix} \quad \text{and} \quad H_0 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

are spectrally similar, with $\varphi_1(z) = 1 + z - 1/(z+1)$, $\varphi_0(z) = 1/2(z+1)$ and $R(z) = 2z^2 + 4z$ by (2.1), or by (3.2) below.

DEFINITION 2.5. We call an operator H *spectrally self-similar* with functions φ_0 and φ_1 if there exists a co-isometry D on $\mathcal{H}$, that is, $DD^* = I$, such that

$$D(H - z)^{-1}D^* = (\varphi_0(z)H - \varphi_1(z))^{-1} \tag{2.3}$$

for any $z \in \Lambda$ for which the two sides are well defined.

We note that Equation (2.3) was used by J. Béllissard [2] for a particular case of the Laplacian on Sierpiński lattice.

PROPOSITION 2.6. *The operator H is spectrally self-similar if and only if H is spectrally similar to an operator H_0 on a closed subspace $\mathcal{H}_0$ such that $H = UH_0U^{-1}$ for an isometry $U\colon \mathcal{H}_0 \to \mathcal{H}$.*

3. Properties of Spectrally Similar Operators

Most of the results in this section appeared in [23] and [21] in a slightly different setting.

In this section we assume that $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$. Then U is the inclusion operator, which we will omit, and $U^* = P_0$, the orthogonal projector onto $\mathcal{H}_0$. (If $\mathcal{H}_0$ is not a subspace of $\mathcal{H}$ one can use U to identify $\mathcal{H}_0$ with $\operatorname{Im} U$ which is a closed subspace of $\mathcal{H}$.)

NOTATION 3.1. Let $\mathcal{H}_1$ be the orthogonal complement to $\mathcal{H}_0$ and $P_1 = I - P_0$ be the orthogonal projector onto $\mathcal{H}_1$. The operators S: $\mathcal{H}_0 \to \mathcal{H}_0$, X: $\mathcal{H}_0 \to \mathcal{H}_1$, $\bar{X}$: $\mathcal{H}_1 \to \mathcal{H}_0$ and Q: $\mathcal{H}_1 \to \mathcal{H}_1$ are defined by $S = P_0 H$, $X = P_1 H$, $\bar{X} = P_0 H$ and $Q = P_1 H$. This means that H has the following block structure with respect to the representation $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$:

$$H = \begin{pmatrix} S & \bar{X} \\ X & Q \end{pmatrix}. \tag{3.1}$$

For $i = 1, 2$, I_i denotes the identity operator on $\mathcal{H}_i$. The resolvent set of an operator A is denoted by $\rho(A)$.

Remark 3.2. Note that the operators S and X, as well as H_0, are defined on $\mathcal{H}_0$, and the operators Q and $\bar{X}$ are defined on $\mathcal{H}_1$. Throughout this paper, we slightly abuse the notation by omitting the operators of inclusion of $\mathcal{H}_0$ and $\mathcal{H}_1$ into $\mathcal{H}$ in all formulas. Also, we often write a constant instead of an identity operator multiplied by this constant, and we do not use any the notation for a restriction of an operator to a particular subspace. It allows us to simplify many expressions and, we hope, will not confuse the reader.

The next lemma will be an important tool for dealing with spectrally similar operators.

LEMMA 3.3. *For $z \in \rho(H) \cap \rho(Q)$ relation* (2.1) *holds if and only if*

$$S - zI_0 - \bar{X}(Q - zI_1)^{-1}X = \varphi_0(z)H_0 - \varphi_1(z)I_0. \tag{3.2}$$

Proof. We have that

$$\begin{aligned} &P_0(H - z)^{-1}(S - z - \bar{X}(Q - z)^{-1}X)P_0 \\ &\quad = P_0(H - z)^{-1}P_0(H - z)P_0 - \\ &\quad\quad - P_0(H - z)^{-1}P_0(H - z)(Q - z)^{-1}P_1(H - z)P_0 \\ &\quad = P_0(H - z)^{-1}P_0(H - z)P_0 + P_0(H - z)^{-1}P_1(H - z)P_0 = P_0. \end{aligned}$$

Therefore (2.1) is equivalent to (3.2). □

COROLLARY 3.4. *Suppose H is spectrally similar to H_0 on $\mathcal{H}_0$. Then there exists a unique analytic continuation of φ_0 and φ_1 from the set $\Lambda_0 \cap \rho(Q)$ to its connected component in $\rho(Q)$ such that* (3.2) *holds.*

In particular, if the spectrum of Q consists only of isolated eigenvalues, then φ_0 and φ_1 have a meromorphic continuation to all of $\mathbb{C}$, with all the poles contained in $\sigma(Q)$.

Proof. By Remark 2.2, there are two vectors $e, e' \in \mathcal{H}_0$ such that $\langle e, e' \rangle = 0$ but $\langle H_0 e, e' \rangle \neq 0$, then, by (3.2),

$$\varphi_0(z) = \frac{\langle S - \bar{X}(Q-z)^{-1}X)e, e'\rangle}{\langle H_0 e, e'\rangle},$$
$$\varphi_1(z) = \frac{\langle (S - z - \bar{X}(Q-z)^{-1}X - \varphi_0(z)H_0)e, e\rangle}{\langle e, e\rangle} \tag{3.3}$$

for any $z \in \rho(Q)$. These formulas give the required unique analytic continuation. □

By this corollary, without loss of generality, we can assume that $\varphi_0(z)$ and $\varphi_1(z)$ are defined on $\rho(Q)$ by (3.3).

DEFINITION 3.5. If H is spectrally similar to H_0, then the set

$$\mathcal{E}(H, H_0) = \{z \in \mathbb{C} : z \notin \rho(Q) \text{ or } \varphi_0(z) = 0\}$$

is called the exceptional set of the operators H and H_0.

THEOREM 3.6. *Let H be spectrally similar to H_0 on $\mathcal{H}_0$ and $z \notin \mathcal{E}(H, H_0)$. Then*

(1) *$R(z) \in \rho(H_0)$ if and only if $z \in \rho(H)$.*
(2) *$R(z)$ is an eigenvalue of H_0 if and only if z is an eigenvalue of H. Moreover, there is a one-to-one map*

$$f_0 \mapsto f = f_0 - (Q-z)^{-1}Xf_0$$

from the eigenspace of H_0 corresponding to $R(z)$ onto the eigenspace of H corresponding to z.

Proof of (1). Suppose $z \in \rho(H)$. Then the operator, $\varphi_0(z)H_0 - \varphi_1(z)$ has a bounded inverse, as it follows from the proof of Lemma 3.3 and corollary following it. Hence $R(z) \in \rho(H_0)$ by the definition of $R(z)$, see Definition 2.5, since $\varphi_0(z) \neq 0$.

Suppose now that $z \notin \rho(H)$. Then there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of elements of $\mathcal{H}$ such that $\|g_n\| = 1$ and $(H - z)g_n \to 0$ as $n \to \infty$. That is

$$(Q - z)P_1 g_n + XP_0 g_n \to 0 \tag{3.4}$$

and

$$(S - z)P_0 g_n + \bar{X}P_1 g_n \to 0 \tag{3.5}$$

as $n \to \infty$. Since $z \notin \rho(Q)$, relation (3.4) implies that $P_1 g_n + (Q-z)^{-1}XP_0 g_n \to 0$ and so $P_0 g_n \not\to 0$. At the same time, by (3.4) and (3.5), $(S - z - \bar{X}(Q - z)^{-1}X)P_0 g_n \to 0$ as $n \to \infty$. Therefore, $R(z) \notin \rho(H_0)$ by Lemma 3.3. □

Proof of (2). Suppose that $R(z)$ is an eigenvalue of H_0 and f_0 is a corresponding eigenvector. Then, by the definition of $R(z)$, $\varphi_0(z)H_0 f_0 - \varphi_1(z) f_0 = 0$ and so, by (3.2),

$$zf_0 = (S - \bar{X}(Q-z)^{-1}X)f_0.$$

Define $f_1 = -(Q-z)^{-1}Xf_0$ and $f = f_0+f_1$, then $P_0HP_1f = -\bar{X}(Q-z)^{-1}Xf_0$. Hence

$$(P_0HP_0 + P_0HP_1)f = S - \bar{X}(Q-z)^{-1}Xf_0 = zf_0. \tag{3.6}$$

Also we have $(Q-z)f_1 = -Xf_0$ and so

$$(P_1HP_1 + P_1HP_0)f = Qf_1 + Xf_0 = zf_1. \tag{3.7}$$

Combining relations (3.6) and (3.7), we get

$$Hf = (P_1HP_1 + P_1HP_0 + P_0HP_0 + P_0HP_1)f = zf_1 + zf_0 = zf.$$

Therefore the required map is well defined. It is, obviously, one-to-one.

Suppose now that z is an eigenvalue of H and f is a corresponding eigenvector. Define $f_0 = P_0f$, $f_1 = P_1f$. Then, clearly,

$$zf_0 = Sf_0 + \bar{X}f_1, \qquad zf_1 = Qf_1 + Xf_0.$$

Therefore $f_1 = -(Q-z)^{-1}Xf_0$. On the one hand, this implies $zf_0 = Sf_0 - \bar{X}(Q-z)^{-1}Xf_0$, and so $H_0f_0 = R(z)f_0$ by (3.2). On the other hand, $f = f_0 + f_1 = f_0 - (Q-z)^{-1}Xf_0$. Therefore, the map under consideration is onto. □

PROPOSITION 3.7. *If H is spectrally similar to H_0 on $\mathcal{H}_0$ then $S = aH_0 + bI_0$ for some $a, b \in \mathbb{C}$, and for any $k \geqslant 0$ there are coefficients $a_k, b_k \in \mathbb{C}$ such that*

$$\bar{X}Q^kX = a_kH_0 + b_kI_0. \tag{3.8}$$

If for any $k \geqslant 0$ there are coefficients a_k, b_k, $c_k \in \mathbb{C}$ such that

$$\bar{X}Q^kX = a_kH_0 + b_kI_0 + c_kS, \tag{3.9}$$

then H is spectrally similar to H_0 on $\mathcal{H}_0$.

Proof. Suppose there exists a continuous linear functional λ on the space $\mathcal{B}(\mathcal{H}_0)$ of bounded operators on $\mathcal{H}_0$ such that $\lambda(H_0) = \lambda(I_0) = 0$ but $\lambda(S) \neq 0$. Then (3.2) implies that $\lambda(S - \bar{X}(Q-z)^{-1}X) = 0$. This is impossible since $\|\bar{X}(Q-z)^{-1}X\| \to 0$ as $|z| \to \infty$. Therefore, such λ does not exist. By the Hahn–Banach theorem, S belongs to the linear span of H_0 and I_0.

If $|z| > \|Q\|$ then $\bar{X}(Q-z)^{-1}X = \sum_{k=0}^{\infty} z^{-k-1}\bar{X}Q^kX$. Hence, the result follows if one considers the expansion of the right-hand side of (3.2) into Laurent series. □

The next proposition describes a few examples of trivial situations when we have spectral similarity.

PROPOSITION 3.8. (1) *Let $H = \tilde{H}_0 \oplus \tilde{H}_1$ be an orthogonal sum of operators $\tilde{H}_i\colon \mathcal{H}_i \to \mathcal{H}_i$, $i = 0, 1$. Then H is spectrally similar to H_0 on $\mathcal{H}_0$ if and only if*

$\tilde{H}_0 = aH_0 + b$ *for some* $a, b \in \mathbb{C}$. *In this case Equation* (2.1) *holds with* $\varphi_0(z) = a$ *and* $\varphi_1(z) = z - b$.

(2) *Suppose* H *is spectrally similar to* H_0 *on* $\mathcal{H}_0$. *Then* $\varphi_0(z) = a$ *and* $\varphi_1(z) = z - b$ *if and only if* $\bar{X}Q^kX = 0$ *for any* $k \geqslant 0$.

(3) *If* $S = aH_0 + b$ *and* $\bar{X}Q^kX = 0$ *for any* $k \geqslant 0$ *then* H *is spectrally similar to* H_0.

Proof. (1) We see that in this case Equation (2.1) has the form $(\tilde{H}_0 - z)^{-1} = (\varphi_0(z)H_0 - \varphi_1(z))^{-1}$ or $\tilde{H}_0 - z = \varphi_0(z)H_0 - \varphi_1(z)$. The statement follows since $\tilde{H}_0$ does not depend on z.

(2) Note that for $|z| > r_0$, r_0 is large enough, (3.2) is equivalent to

$$S - z + \sum_{k=0}^{\infty} z^{-k-1}\bar{X}Q^kX = \varphi_0(z)H_0 - \varphi_1(z). \tag{3.10}$$

If (2.1) holds with $\varphi_0(z) = a$ and $\varphi_1(z) = z - b$, then

$$S + \sum_{k=0}^{\infty} z^{-k-1}\bar{X}Q^kX = aH_0 + b$$

by Lemma 3.3 for $|z| > r_0$. Therefore $\bar{X}Q^kX = 0$ for any $k \geqslant 0$.

Conversely, if $\bar{X}Q^kX = 0$ for any $k \geqslant 0$ then, by (3.10),

$$S - z = \varphi_0(z)H_0 - \varphi_1(z), \quad |z| > r_0.$$

Hence, $\varphi_0(z) = a$ and $\varphi_1(z) = z - b$ (see Remark 2.2).

(3) Follows from Lemma 3.3 and (3.10). □

EXAMPLE 3.9. Let operators H and $\tilde{H}$ have the following block structure with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$

$$H = \begin{pmatrix} aH_0 + bI_0 & B \\ 0 & C \end{pmatrix}, \qquad \tilde{H} = \begin{pmatrix} aH_0 + bI_0 & 0 \\ B & C \end{pmatrix}.$$

Then both H and $\tilde{H}$ are spectrally similar to H_0 with functions $\varphi_0(z) = a$ and $\varphi_1(z) = z - b$.

The next lemma allows us to construct a pair of spectrally similar operators from a given family of spectrally similar operators. It will play an important role in the next section. It allows us, in particular, to prove the existence of spectral similarity when a graph is not symmetric as in Example 4.8.

For each $\alpha \in \mathcal{A}$, let $\mathcal{H}^\alpha$ be a closed subspace of $\mathcal{H}$, $\mathcal{H}^\alpha = \mathcal{H}_0^\alpha \oplus \mathcal{H}_1^\alpha$, P_i^α be the orthogonal projector onto $\mathcal{H}_i^\alpha$, $i = 0, 1$. Also assume that $P_i^\alpha P_1^\beta = 0$ if $\alpha \neq \beta$, $i = 0, 1$. Then $P_1 = \sum_{\alpha\in\mathcal{A}} P_1^\alpha$ is the orthogonal projector onto $\mathcal{H}_1 = \bigoplus_{\alpha\in\mathcal{A}} \mathcal{H}_1^\alpha$. We define $\mathcal{H}_0$ as the orthogonal complement to $\mathcal{H}_1$.

For $\alpha \in \mathcal{A}$ let H^α and H_0^α be operators on $\mathcal{H}^\alpha$ and $\mathcal{H}_0^\alpha$ respectively. Suppose that for given functions $\varphi_1(z)$ and $\varphi_0(z)$ defined on $\bigcap_{\alpha\in\mathcal{A}} \rho(Q^\alpha)$, we have

$$S^\alpha - z - \tilde{X}^\alpha (Q^\alpha - z)^{-1} X^\alpha = \varphi_0(z) H_0^\alpha - \varphi_1(z) P_0^\alpha$$

for each α where S^α, X^α, $\bar{X}^\alpha$, Q^α are defined in the same way as S, X, $\bar{X}$, Q in Lemma 3.3. This implies that each H^α is spectrally similar to H_0^α with functions $\varphi_0(z)$ and $\varphi_1(z)$ which do not depend on α.

LEMMA 3.10. *Let for families of operators $\{L^\alpha\}_{\alpha\in\mathcal{A}}$, $\{R^\alpha\}_{\alpha\in\mathcal{A}}$ we have that $P_0 = \sum_{\alpha\in\mathcal{A}} L^\alpha P_0^\alpha R^\alpha$ and for each α, $P_1 L^\alpha = R^\alpha P_1 = P_1^\alpha$, $P_0 L^\alpha = L^\alpha P_0^\alpha$, $R^\alpha P_0 = P_0^\alpha R^\alpha$. Then the operators $H = \sum_{\alpha\in\mathcal{A}} L^\alpha H^\alpha R^\alpha$ and $H_0 = \sum_{\alpha\in\mathcal{A}} L^\alpha H_0^\alpha R^\alpha$ are spectrally similar with functions $\varphi_0(z)$, $\varphi_1(z)$.*

Proof. We have

$$SP_0 = P_0 \sum_{\alpha\in\mathcal{A}} L^\alpha H^\alpha R^\alpha P_0 = \sum_{\alpha\in\mathcal{A}} L^\alpha P_0^\alpha H^\alpha P_0^\alpha R^\alpha = \sum_{\alpha\in\mathcal{A}} L^\alpha S^\alpha R^\alpha.$$

Also

$$\begin{aligned} P_0 H P_1^\alpha &= \sum_{\beta\in\mathcal{A}} P_0 L^\beta H^\beta R^\beta P_1^\alpha = \sum_{\beta\in\mathcal{A}} P_0 L^\beta H^\beta R^\beta P_1 P_1^\alpha \\ &= P_0 L^\alpha H^\alpha P_1^\alpha = L^\alpha P_0^\alpha H^\alpha P_1^\alpha = L^\alpha \bar{X}^\alpha P_1^\alpha, \end{aligned}$$

and similarly $P_1^\alpha H P_0 = P_1^\alpha H^\alpha P_0^\alpha R^\alpha = X^\alpha P_0^\alpha R^\alpha$. In addition, on $\mathcal{H}_1$

$$\begin{aligned} (Q-z)^{-1} &= \left(\sum_{\alpha\in\mathcal{A}} P_1 L^\alpha H^\alpha R^\alpha P_1 - z\right)^{-1} = \left(\sum_{\alpha\in\mathcal{A}} P_1^\alpha H^\alpha P_1^\alpha - z\right)^{-1} \\ &= \sum_{\alpha\in\mathcal{A}} (P_1^\alpha H^\alpha P_1^\alpha - z)^{-1} P_1^\alpha = \sum_{\alpha\in\mathcal{A}} (Q^\alpha - z)^{-1} P_1^\alpha. \end{aligned}$$

Thus

$$\begin{aligned} &S - \bar{X}(Q-z)^{-1} X \\ &\quad = \sum_{\alpha\in\mathcal{A}} (L^\alpha S^\alpha R^\alpha + L^\alpha \bar{X}^\alpha (Q^\alpha - z)^{-1} X^\alpha R^\alpha) \\ &\quad = \sum_{\alpha\in\mathcal{A}} (z - \varphi_1(z)) L^\alpha P_0^\alpha R^\alpha + \varphi_0(z) L^\alpha H_0^\alpha R^\alpha \\ &\quad = (z - \varphi_1(z)) P_0 + \varphi_0(z) H_0. \end{aligned}$$

□

4. Symmetric Graphs and Spectral Similarity of Laplacians

For a graph G we denote by $V(G)$ and $E(G)$ the set of its vertices and edges, respectively. By $\ell(V)$ we denote the linear space of functions on V. We always assume that a graph is locally finite, that is each vertex is contained in a finite number of edges. A complete graph is a graph which has one edge between any two vertices, and does not have any loops or multiple edges. A graph isomorphism is a bijective map from one graph to another which preserves the graph structure (vertices are mapped into vertices, edges are mapped into edges, and the vertices contained in an edge are mapped into the vertices contained in the image of this edge). A graph automorphism is an isomorphism onto itself.

DEFINITION 4.1. Let G be a graph and $V_0 \subseteq V(G)$. We say that G is *symmetric with respect to* V_0 if any bijection $\sigma: V_0 \to V_0$ can be extended to a graph automorphism $\psi_\sigma: G \to G$. We denote the set of these automorphisms by $\Sigma(G, V_0)$.

For the next lemma we assume that the inner product on $\mathcal{H} = \ell(V(G))$ is invariant under the action of the symmetries. If $V_0 \subseteq V(G)$ then we define $\mathcal{H}_0$ to be a subset of $\mathcal{H}$ of the functions vanishing on $V(G) \setminus V_0$.

LEMMA 4.2. *Let G be a graph symmetric with respect to $V_0 \subseteq V(G)$ and H_0 be an operator on $\mathcal{H}_0$ invariant under any permutation $\sigma: V_0 \to V_0$. If an operator H on $\mathcal{H}$ is invariant under $\Sigma(G, V_0)$ then H is spectrally similar to H_0.*

The proof of this lemma follows easily from Lemma 4.3.

LEMMA 4.3. *Suppose for a family of operators $\{T_\sigma\}_{\sigma\in\Sigma}$ on $\mathcal{H}$ the following two assumptions hold:*

(1) *An operator H and an orthogonal projector P_0 commute with each T_σ.*
(2) *There exists an operator H_0 on $\mathcal{H}_0 = \mathrm{Im} P_0$ such that an operator $\tilde{H}_0$ on $\mathcal{H}_0$ commutes with each T_σ if and only if $\tilde{H}_0 = aH_0 + b$ for some $a, b \in \mathbb{C}$.*

Then H and H_0 are spectrally similar.

Proof. Let the operators $S, X, \bar{X}, Q$ be defined as in the Lemma 3.3. Assumption (1) implies that, for any $z \notin \rho(Q)$, operator $S - z - \bar{X}(Q - z)^{-1}X$ commutes with each T_σ. Then assumption (2) implies that for a fixed z there are two complex numbers, say $\varphi_0(z)$ and $\varphi_1(z)$, such that $S - z - \bar{X}(Q - z)^{-1}X = \varphi_0(z)H_0 - \varphi_1(z)$. Hence, H and H_0 are spectrally similar by Lemma 3.3. □

Remark 4.4. Indeed, the lemma above is used in a situation when $\{T_\sigma\}_{\sigma\in S}$ is a representation of a group of symmetries. The conditions (1) and (2) can be written in the language of representation theory, but we will not use it here.

In the next definition and throughout this paper, 'Laplacian' is always a discrete difference operator.

DEFINITION 4.5. The *graph* or *probabilistic Laplacian* of a function $f \in \ell(V(G))$ is defined by

$$\Delta_G f(x) = -f(x) + \frac{1}{\deg(x)} \sum_{(x,y)\in E(G)} f(y), \tag{4.1}$$

where $\deg(x)$ is the degree of the vertex x. The *Markov operator* (generator of the simple random walk) is defined by

$$\Delta_M f(x) = \frac{1}{\deg(x)} \sum_{(x,y)\in E(G)} f(y) \tag{4.2}$$

and the *adjacency matrix* (*combinatorial*) *Laplacian* is defined by

$$\Delta_A f(x) = -\deg(x) f(x) + \sum_{(x,y)\in E(G)} f(y). \tag{4.3}$$

It is easy to see that the graph Laplacian and the Markov operator are bounded and symmetric with respect to the graph inner product

$$\langle f, g\rangle_G = \sum_{x\in V(G)} f(x)g(x)\deg(x). \tag{4.4}$$

The Hilbert space with this inner product will be denoted by $\mathcal{H}(G)$ and the corresponding norm $\|\cdot\|_G$. The adjacency matrix Laplacian is symmetric in $\ell^2(V(G))$ and may be unbounded on an infinite graph. Note that the graph Laplacian is the Markov operator minus the identity operator and so any information on the first one is easy to translate into information on the second one. If the graph is regular, as in the case of a group, then the adjacency matrix Laplacian is a multiple of the graph Laplacian, and the same is true about its spectrum.

While Lemma 4.2 gives only sufficient conditions for spectral similarity, the next simple lemma gives a weaker necessary condition (but strong enough to be applicable to an example in Section 6).

For the next lemma, we assume that the inner product on $\mathcal{H} = \ell(V(G))$ is one of the two defined above. If $V_0 \subseteq V(G)$ then again we define $\mathcal{H}_0$ to be a subset of $\mathcal{H}$ of the functions vanishing on $V(G)\setminus V_0$.

LEMMA 4.6. *Let G be a locally finite graph, and G_0 be a finite complete graph with $V(G_0) = V_0 \subseteq V(G)$. Let H and H_0 be Laplacians on G and G_0 respectively* (*defined in Definition* 4.5). *If H is spectrally similar to H_0, then* $\operatorname{dist}_G(a,b) = \operatorname{dist}_G(a,c)$ *for any three distinct points $a, b, c \in V_0$, where* dist_G *is the usual graph distance in G.*

Proof. First note that all the nondiagonal entries of the matrix of H_0 are strictly positive. Suppose that $\operatorname{dist}_G(a,b) > \operatorname{dist}_G(a,c)$. If $\operatorname{dist}_G(a,c) = 1$ then some nondiagonal entries of the matrix of S are positive while some are zero, which contradicts (3.2). If $\operatorname{dist}_G(a,c) \geqslant 2$ then consider the term in the left-hand side series in (3.10) corresponding to $k = \operatorname{dist}_G(a,c) - 2$. Then again, some nondiagonal entries of the matrix of $\bar{X}Q^kX$ are positive while some are zero which contradicts (3.10) for $|z|$ large enough. □

Let G be a graph symmetric with respect to $V_0 \subseteq V(G)$ and G_0 be a complete graph with $V(G_0) = V_0$. By Lemma 4.2, Δ_G and Δ_{G_0} are spectrally similar with some functions $\varphi_1(z)$ and $\varphi_0(z)$. If $\mathcal{A}$ is a set then $G \times \mathcal{A}$ is a naturally defined graph with the set of vertices $V(G) \times \mathcal{A}$ and the set of edges $E(G) \times \mathcal{A}$. Let $\sim$ be an equivalence relation on the set $V_0 \times \mathcal{A}$ such that each element is equivalent to a finite set of elements. Let $\widetilde{G} = \{G \times \mathcal{A}\}/\sim$. The graph $G \times \mathcal{A}$ can be considered as a disjoint union of $|\mathcal{A}|$ copies of G and $\widetilde{G}$ is a union of $|\mathcal{A}|$ copies of G joined at the equivalent vertices. Similarly we define $\widetilde{G}_0 = \{G_0 \times \mathcal{A}\}/\sim$. The Hilbert space under consideration is $\widetilde{\mathcal{H}} = \mathcal{H}(\widetilde{G})$.

LEMMA 4.7. *The graph Laplacian on $\widetilde{G}$ is spectrally similar to the graph Laplacian on $\widetilde{G}_0$ with the same functions $\varphi_1(z)$ and $\varphi_0(z)$.*

Proof. Let $\mathcal{H} = \mathcal{H}(G \times \mathcal{A})$. Note that we can identify $\widetilde{\mathcal{H}}$ with the subspace $\{f \in \mathcal{H} : f(x) = f(y) \text{ if } x \sim y\}$ of $\mathcal{H}$, and the norms coincide by (4.4). Denote $V^\alpha = V(G \times \alpha)$ and $V_0^\alpha = V(G_0 \times \alpha)$. Let for each $\alpha \in \mathcal{A}$ we define $\mathcal{H}^\alpha = \mathcal{H} \cap \ell(V^\alpha)$ and $\mathcal{H}_0^\alpha = \mathcal{H} \cap \ell(V_0^\alpha)$. Let H^α and H_0^α, $\alpha \in \mathcal{A}$, be the corresponding Laplacians on $\mathcal{H}^\alpha$ and $\mathcal{H}_0^\alpha$, respectively, that is $H^\alpha = \Delta_{G\times\alpha}$ and $H_0^\alpha = \Delta_{G_0\times\alpha}$. Then H^α and H_0^α are spectrally similar for each $\alpha \in \mathcal{A}$ with $\varphi_1(z)$ and $\varphi_0(z)$ by Lemma 4.2 (clearly $\varphi_1(z)$, $\varphi_0(z)$ and $\rho(Q^\alpha)$ do not depend on α).

In order to apply Lemma 3.10 we need to define families of operators $\{L^\alpha\}_{\alpha\in\mathcal{A}}$, $\{R^\alpha\}_{\alpha\in\mathcal{A}}$ with the required properties. We define $R^\alpha = P^\alpha$ and $L^\alpha = P_{\widetilde{\mathcal{H}}} P^\alpha$. Then $H = \sum_{\alpha\in\mathcal{A}} L^\alpha H^\alpha R^\alpha = \Delta_{\widetilde{G}}$, $H_0 = \sum_{\alpha\in\mathcal{A}} L^\alpha H_0^\alpha R^\alpha = \Delta_{\widetilde{G}_0}$ and the conditions of Lemma 3.10 are easy to verify. □

EXAMPLE 4.8. Let $\widetilde{G}_0$ be any graph and $\widetilde{G}$ be a graph obtained from $\widetilde{G}_0$ by substituting each edge with two consecutive edges. Then by this proposition, the graph Laplacian on $\widetilde{G}$ is spectrally similar to the graph Laplacian on $\widetilde{G}_0$ with $\varphi_1(z) = 1 + z - 1/(z+1)$, $\varphi_0(z) = 1/2(z+1)$ and $R(z) = 2z^2 + 4z$ (see Example 2.4).

More generally, let G_0 and $\widetilde{G}_0$ be any graphs, and suppose that G_0 is symmetric with respect to a two-point set $V_0 = \{v_1, v_2\}$. Let $\widetilde{G}$ be a graph obtained from $\widetilde{G}_0$ by substituting each edge (x_1, x_2) with a copy of G_0 in such a way that, after the substitution, v_1 coincides with x_1 and v_2 coincides with x_2. Then by the proposition above the graph Laplacian on $\widetilde{G}$ is spectrally similar to the graph Laplacian on $\widetilde{G}_0$. Two-point self-similar graphs defined in [15] fall into this example.

We emphasize that in this example the graphs $\widetilde{G}$ may have no symmetries in any sense. However the spectral similarity holds due to symmetries in the 'substituting' graph G_0.

LEMMA 4.9. *Let G and G_0 be finite graphs with $V(G_0) = V_0 \subsetneq V(G)$, and suppose the graph Laplacians Δ_G and Δ_{G_0} are spectrally similar. If G is connected then $R(0) = 0$. If G_0 is also connected then, in addition, $R'(0) > 1$.*

Proof. First note that 0 is not in the spectrum of Q because G is connected and finite (even if the graphs are infinite, in the examples we consider Q can be represented as a product of identical finite-dimensional matrices and so 0 is not in the spectrum of Q).

We have $\Delta_G 1 = 0$ where 1 stands for a function on $V(G)$ that is identically one. Then by (3.1) we have $XP_0 1 + QP_1 1 = 0$ and so $P_1 1 = -Q^{-1}XP_0 1$. Also by (3.1) we have $\bar{X}P_1 1 + SP_0 1 = 0$, which implies

$$SP_0 1 - \bar{X}Q^{-1}XP_0 1 = 0.$$

By (3.2) we have $\varphi_1(0) = 0$ since $\Delta_{G_0} P_0 1 = 0$. It is known that Q^{-1} has nonpositive matrix entries (see Lemma 2.7.1 in [8]) and $\bar{X}Q^{-1}X$ has at least one strictly negative nondiagonal entry since G is connected. Hence, $\varphi_0(0) \neq 0$ that implies $R(0) = 0$.

By the same argument every diagonal matrix entry of $\bar{X}Q^{-1}X$ is strictly negative that implies $0 < \varphi_0(0) < 1$. By differentiating (3.2), we have

$$-I_0 - \bar{X}Q^{-2}X = \varphi_0'(z)H_0 - \varphi_1'(z)I_0.$$

Then $\varphi_0'(0) \leqslant 0$ since at least some nondiagonal entries of H_0 are positive if G_0 is connected, and all the entries if $\bar{X}Q^{-2}X$ are nonnegative. Hence, $\varphi_1'(0) \geqslant 1$ by the comparison of the diagonal matrix entries. Thus $R'(0) = \varphi_1'(0)/\varphi_0(0) > 1$. □

5. Symmetric Self-Similar Graphs and Self-Similar Spectrum

DEFINITION 5.1. An *M-point model graph G* is a finite connected graph symmetric (Definition 4.1) with respect to an M point set $\partial G = V_0 \subset V(G)$ if

(1) there are complete graphs G^s of M vertices such that $G = \bigcup_{s\in S} G^s$ where S is a finite set and $|S| \geqslant M \geqslant 2$;
(2) we have $G^s \cap G^{s'} = V(G^s) \cap V(G^{s'})$ for all distinct $s, s' \in S$, and this intersection is either empty or has only one point;
(3) we have $|G^s \cap \partial G| \leqslant 1$ for any $s \in S$;
(4) any bijection $\sigma\colon \partial G \to \partial G$ has an extention (see Definition 4.1) to a graph automorphism $\psi_\sigma\colon G \to G$, such that $\psi_\sigma G^s = G^{\bar{\sigma}s}$ for a bijection $\bar{\sigma}\colon S \to S$.

DEFINITION 5.2. If an M-point model graph G is given then we define the corresponding *self-similar symmetric sequence of finite graphs* $\{G_n\}_{n=0}^{\infty}$ inductively as follows:

(1) G_0 is a complete graph of M vertices with $\partial G_0 = V(G_0)$;
(2) if $\partial G_n \subset V(G_n)$ is an M point set, then G_{n+1} is obtained by substituting each G^s in G by a copy G_n^s of G_n, so that $\partial G^s = V(G^s)$ is substituted by ∂G_n^s;
(3) ∂G_{n+1} is defined as ∂G after this substitution.

For this self-similar sequence of finite graphs $\{G_n\}_{n=0}^{\infty}$ there are bijections $B_n\colon \partial G \to \partial G_n \subset V(G_n)$ and graph monomorphisms $F_s^n\colon G_n \to G_{n+1}$, $s \in S$, such that for all $n \geqslant 0$

(1) each F_s^n is a graph isomorphism from G_n to G_n^s and $G_{n+1} = \bigcup_{s\in S} G_n^s$;
(2) $G_n^s \cap G_n^{s'} = \partial G_n^s \cap \partial G_n^{s'}$ for all $s, s' \in S$, $s \neq s'$, where $\partial G_n^s = F_s^n(\partial G_n)$;
(3) for $n \geqslant 1$, we have $B_{n+1}(x) = F_s^n(B^n(x))$ if $x \in \partial G \cap G^s$;
(4) for all $s, s' \in S$, $s \neq s'$, we have $F_s^n(x) = F_{s'}^n(x')$ if and only if there are $\bar{x}, \bar{x}' \in \partial G$ such that $x = B_n(\bar{x})$, $x' = B_n(\bar{x}')$ and $F_s^0 B_0^{-1}(\bar{x}) = F_{s'}^0 B_0^{-1}(\bar{x}')$.

Note that G_1 can be naturally identified with G in such a way that ∂G_1 is identified with ∂G and $B_1(x) = x$ for all $x \in \partial G$.

LEMMA 5.3. *Each G_n is symmetric with respect to ∂G_n.*

Proof. First note note that bijections σ of $V_0 = \partial G$ are in one-to-one correspondence with bijections σ_n of ∂G_n via $\sigma_n = B_n \sigma B_n^{-1}$. Let $\sigma, \bar{\sigma}$ be as in Definition 5.1. We have that G_0 is always symmetric with respect to ∂G_0. For $n \geqslant 0$ we define $\psi_{\sigma_{n+1}}\colon G_{n+1} \to G_{n+1}$ by $\psi_{\sigma_{n+1}}(x) = F_{\bar{\sigma}s}^n (F_s^n)^{-1}(x)$ if $x \in F_s^n(G_n)$. Then ψ_{σ_n} are the required well-defined graph automorphisms. □

One can see that for a given M-point model graph there is a unique self-similar symmetric sequence of finite graphs up to a natural isomorphism. This sequence can be constructed inductively by Lemma 5.4. The proof of the lemma is elementary.

LEMMA 5.4. *For each $n \geqslant 0$ the graph G_{n+1} is isomorphic to a graph $G_n \times S/\sim$ (see Lemma 4.7) where the relation $\sim$ on $V(G_n) \times S$ is defined as follows: if $(x, s), (x', s') \in V(G_n) \times S$ then $(x, s) \sim (x', s')$ if and only if there are $\bar{x}, \bar{x}' \in V_0 = \partial G$ such that $x = B_n(\bar{x})$, $x' = B_n(\bar{x}')$ and $F_s^0 B_0^{-1}(\bar{x}) = F_{s'}^0 B_0^{-1}(\bar{x}')$. Moreover, each maps F_n^s is the map $x \mapsto (x, s)$ modulo $\sim$.*

DEFINITION 5.5. Suppose an M-point model graph and a sequence $\mathcal{K} = \{k_n\}_{n=0}^{\infty}$, $k_n \in S$, are fixed. If $G_n \subset G_{n+1}$ for each $n \geqslant 0$, and each $F_{k_n}^n$ is the identity (inclusion) map then the corresponding *self-similar infinite graph* is $G_\infty = \bigcup_{n=0}^{\infty} G_n$. We define $\partial G_\infty = \bigcap_{n=0}^{\infty} \partial G_n$.

Clearly, for any given M-point model graph and a sequence $\mathcal{K}$ there exists a unique self-similar infinite graph (up to isomorphism). At the same time isomorphic self-similar infinite graphs may correspond to different model graphs, and for different sequences $\mathcal{K}$ even if the M-point model graph is the same. The graph G_∞ is always of polynomial growth.

DEFINITION 5.6. *Expansion maps* $K_n\colon V(G_n) \to V(G_{n+1})$ are defined inductively by $K_n(x) = F_s^n K_{n-1}(F_s^{n-1})^{-1}(x)$ if $x \in V(G_{n-1}^s) = F_s^{n-1}(V(G_{n-1}))$. Although such s may not be unique, the expansion map does not depend on a particular choice of s by Definition 5.2.

For a self-similar infinite graph an *expansion map* $K_\infty\colon V(G_\infty) \to V(G_\infty)$ is defined by $K_\infty|_{V(G_n)} = K_n$.

LEMMA 5.7. *The expansion map* $K_n: V(G_n) \to V(G_{n+1})$ *induces an isometry* $U_n: \mathcal{H}(G_n) \to \mathcal{H}(G_{n+1})$ *defined by* $U_n f(x) = 0$ *if* $x \notin \operatorname{Im} K_n$ *and* $U_n f(x) = f(K_n^{-1}(x))$ *otherwise.*

Similarly, K_∞ *induces an isometry* $U_\infty: \mathcal{H}(G_\infty) \to y\mathcal{H}(G_\infty)$.

Proof. By the definition, K_n is an injection and so is U_n. It is isometric since by a simple induction we have $\deg K_n(x) = \deg x$ for any $x \in V(G_n)$. □

By Lemmas 4.7 and 5.4, the graph Laplacian on G_n is spectrally similar to that on G_0. The next theorem gives a more useful spectral similarity result.

THEOREM 5.8. *Let* $\Delta_n = \Delta_{G_n}$ *and* $\Delta_\infty = \Delta_{G_\infty}$ *be the graph Laplacians on* G_n *and* G_∞ *respectively for a self-similar symmetric sequence of finite graphs. Then*

(1) *For any* $n \geqslant 0$, *the operator* Δ_{n+1} *is spectrally similar to* Δ_n *with the isometry* U_n *and rational functions* $\varphi_0(z)$ *and* $\varphi_1(z)$ *which do not depend on* n. *The exceptional set* (*see Definition* 3.5) $\mathcal{E} = \mathcal{E}(\Delta_{n+1}, \Delta_n) = \mathcal{E}(\Delta_1, \Delta_0)$ *also does not depend on* n.
(2) *Let* $\mathcal{D}_n = \bigcup_{m=0}^{n} R^{-m}(\mathcal{E} \cup \sigma(\Delta_0))$, *where* R^{-m} *is the preimage of order* m *under* $R(z) = \varphi_1(z)/\varphi_0(z)$. *Then* $\sigma(\Delta_n) \subseteq \mathcal{D}_n$, *where* $\sigma(\cdot)$ *is the spectrum of an operator.*
(3) *The operator* Δ_∞ *is spectrally self-similar with the isometry* U_∞, *rational functions* $\varphi_0(z)$ *and* $\varphi_1(z)$ *and the exceptional set* $\mathcal{E}$.

$$\mathcal{J}(R) \subseteq \sigma(\Delta_\infty) \subseteq \mathcal{J}(R) \cup \mathcal{D}_\infty,$$

where $\mathcal{D}_\infty = \bigcup_{n=0}^{\infty} \mathcal{D}_n$ *and* $\mathcal{J}(R)$ *is the Julia set of the rational function* R.

Remark 5.9. In particular, the Julia set of R is real. Moreover, one can show easily that $\sigma(\Delta_\infty) \subseteq [-2, 0]$, and so $\mathcal{J}(R) \subseteq [-2, 0]$. Note also that $\mathcal{D}_\infty \setminus \mathcal{J}(R)$ contains only isolated points, if any.

By this theorem many eigenvalues and eigenfunctions of Δ_{n+1} are 'offsprings' of those of Δ_n via maps defined in Theorem 3.6. (so-called 'spectral decimation'). However a significant number of eigenfunctions might not fall into this category (see [4, 14, 19, 23]).

Proof. In order to prove (1) we apply Lemma 4.7 where we define $G = G_1$ and $\mathcal{A} = S^n$. The role of V_0 will be played by ∂G_1. Let the relation $\sim$ on $\partial G_1 \times \mathcal{A}$ be as follows: if $x, x' \in \partial G_1$, $\alpha = s_n \cdots s_1$, $\alpha' = s'_n \cdots s'_1$ then $(x, \alpha) \sim (x', \alpha')$ if and only if $F^n_{s_n} \cdots F^1_{s_1}(x) = F^n_{s'_n} \cdots F^1_{s'_1}(x')$. By induction we have that the graph $\widetilde{G} = \{G_1 \times \mathcal{A}\}/\sim$ is isomorphic to G_{n+1}. Similarly we define $\widetilde{G}_0 = \{G_0 \times \mathcal{A}\}/\sim$, where G_0 is (temporarily, for this part of the proof only) identified with the complete graph over the set of vertices ∂G_1. Note that ∂G_1 and V_0 are in one-to-one correspondence via B_1, and so G_0 is naturally isomorphic to the complete graph over ∂G_1 via the map induced by B_1. Then $\widetilde{G}_0$ is isomorphic to G_n. Moreover, the expansion map K_n is a bijection from $V(G_n)$ to $V(\widetilde{G}_0)$ such that the degree of $\deg_{G_{n-1}} x = \deg_{\widetilde{G}_0} K_n(x)$.

The proof of (2) follows from (1) and Theorem 3.6 by induction.

Clearly, $0 \in \sigma(\Delta_\infty)$ but 0 is not an eigenvalue. Hence, 0 is a point of spectrum which is not isolated. Therefore, by Theorem 3.6, for any $\epsilon > 0$ and $n \geqslant 1$ there exists $z \in \sigma(\Delta_\infty)$ such that $|z| < \epsilon$ and $R_{-n}(z) \subseteq \sigma(\Delta_\infty)$. Since $\sigma(\Delta_\infty)$ is a closed set, this implies

$$\text{Closure}\left(\bigcup_{n\geqslant 1} R_{-n}(0)\right) \subseteq \sigma(\Delta_\infty).$$

We have by Lemma 4.9 that the point 0 is a repulsive fixed point of the rational function R, and so $0 \in \mathcal{J}(R)$ by Theorem 2.2 in [3]. Then

$$\text{Closure}\left(\bigcup_{n\geqslant 1} R_{-n}(0)\right) = \mathcal{J}(R)$$

by Corollary 2.2 in [3].

By Theorem 3.6, $\sigma(\Delta_m) \subseteq \mathcal{D}_\infty$ for any $m \geqslant 1$. Therefore, $\sigma(\Delta_\infty) \subseteq$ Closure $(\mathcal{D}_\infty)$ since Δ_m converges strongly to Δ_∞. We have

$$\text{Closure}(\mathcal{D}_\infty) = \mathcal{J}(R) \cup \mathcal{D}_\infty$$

by [3]. □

6. Examples

A large class of infinite graphs with spectral self-similarity is given by two-point self-similar graphs in [15]. Any nested fractal [13] with two or three essential fixed points gives rise to a self-similar symmetric sequence of finite graphs and thus to spectral similarity. The same is true for a nested fractal which has its essential fixed points in general position; that is, for the essential fixed points $P_0, \ldots, P_k$ the vectors $\{[\overrightarrow{P_0, P_i}]\}_{i=1}^k$ are linearly independent.

Below we give a few concrete examples. All the examples here have a common feature that the infinite self-similar graphs can be realized as infinite self-similar lattices in $\mathbb{R}^2$. Moreover, the corresponding fractals can also be realized as linear

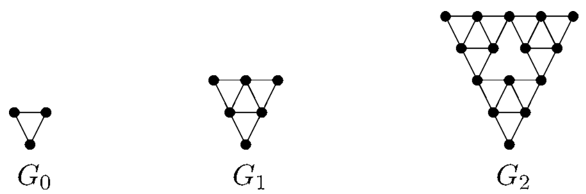

Figure 1. Sierpiński gasket self-similar sequence of finite graphs.

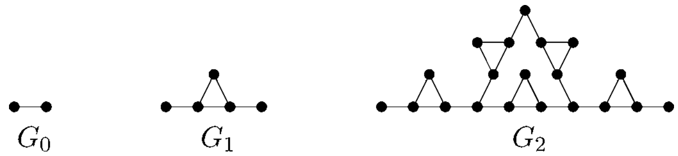

Figure 2. Modified Koch self-similar sequence of finite graphs.

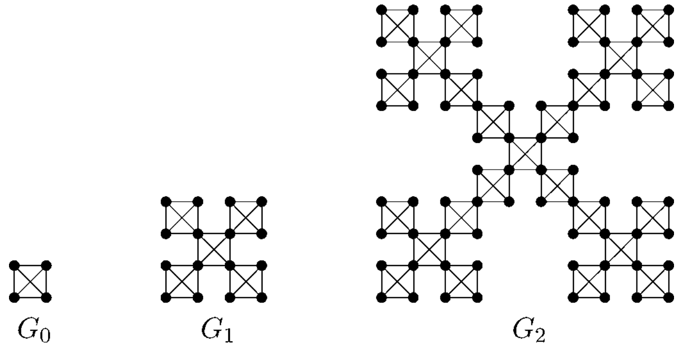

Figure 3. Vicsek set self-similar sequence of finite graphs.

self-similar fractals in $\mathbb{R}^2$. This means that these fractals are the limit sets of an iteration function system of contracting similitudes of $\mathbb{R}^2$. Then the maps F_s^n are expanding similitudes that are inverse of the just mentioned contractions. Such lattices are described in detail in [22]. Therefore we will avoid giving precise definitions since they are either obvious, or are given in the references provided. Note that the model graph G is the same as G_1.

EXAMPLE 6.1. Sierpiński gasket self-similar sequence of finite graphs (Figure 1) [2, 4, 16, 17, 19, 20, 23]. Historically this is the first example of the spectral self-similarity we are interested in. Here $R(z) = z(4z + 5)$.

EXAMPLE 6.2. Modified Koch self-similar sequence of finite graphs (Figure 2) [14]. In this example

$$R(z) = \frac{2z(z-1)(3z-4)(3z-5)}{2z-3}.$$

EXAMPLE 6.3. Vicsek set self-similar sequence of finite graphs (Figure 3). Note that these graphs are symmetric in the sense of Definition 4.1 although the given $\mathbb{R}^2$-embedding does not have all the required symmetries (the essential fixed points are not in general position). By [7, 21] we have $R(z) = z(6z + 3)(6z + 5)$.

EXAMPLE 6.4. Lindstrøm snowflake self-similar sequence of finite graphs (Figure 4) [13]. This example is different from the previous ones in that the spectral similarity does not hold. One can see that the pairwise distances between the boundary points are not the same, and so the necessary condition of Lemma 4.6 is not satisfied.

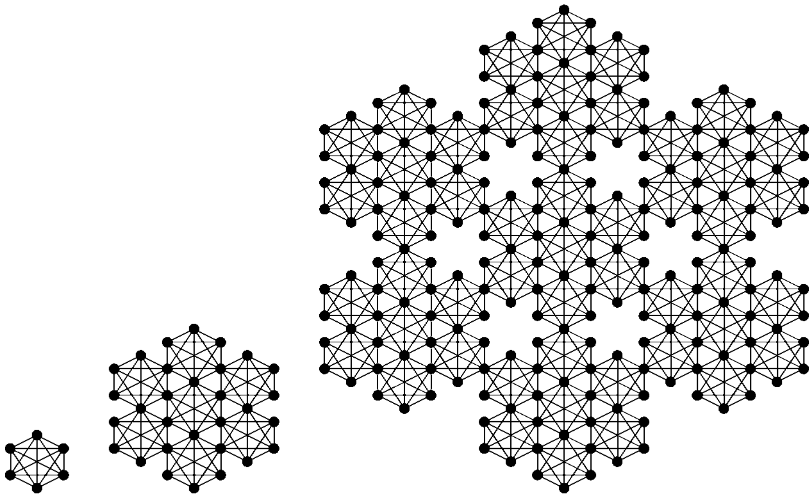

Figure 4. Lindstrøm snowflake self-similar sequence of finite graphs.

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