Energy on fractals and related questions: about the use of differential 1-forms on the Sierpinski Gasket and other fractals

Part 2

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Main works to discuss:

- D. Kelleher, M. Hinz, A. Teplyaev, Metrics and spectral triples for Dirichlet and resistance forms. J. Noncommut. Geom., to appear. arXiv:1309.5937
- M. Hinz, A. Teplyaev, Dirac and magnetic Schrödinger operators on fractals. J. Funct. Anal. 265 (2013), 2830–2854. arXiv:1207.3077
- M. Ionescu, L. G. Rogers, A. Teplyaev, Derivations and Dirichlet forms on fractals, arXiv:1106.1450, Journal of Functional Analysis, 263 (8), p.2141-2169, Oct 2012
- M. Hinz, A. Teplyaev, Local Dirichlet forms, Hodge theory, and the Navier-Stokes equations on topologically one-dimensional fractals. Trans. Amer. Math. Soc. 367 (2015), 1347–1380. arXiv:1206.6644
- M. Hinz, M. Röckner, A. Teplyaev, Vector analysis for local Dirichlet forms and quasilinear PDE and SPDE on fractals, Stochastic Process. Appl. 123 (2013), 4373–4406. arXiv:1202.0743

Metrics and Spectral Triples for Dirichlet and Resistance Spaces
To appear in Journal of Noncommutative Geometry

and

Measures and Dirichlet forms under the Gelfand transform
Journal of Mathematical Science, 2012, Volume 408

Idea: to deal with these problems by developing a differential geometry for Dirichlet spaces and hence fractals, based on:

Derivations and Dirichlet forms on fractals
by Ionescu, Rogers, T., JFA 2012

Vector analysis on Dirichlet Spaces
by Hinz, Roeckner, A.T., SPA 2013

related to
Spectral triples for the Sierpinski Gasket
Integrals and Potentials of Differential 1-forms on the Sierpinski Gasket
by
Fabio Cipriani, Daniele Guido, Tommaso Isola, Jean-Luc Sauvageot

Let (X, d) be a locally compact separable metric space with a nonnegative Radon measure μ on X, with $\mu(U) > 0$ for all non-empty $U \subseteq X$.

A pair $(\mathcal{E}, \mathcal{F})$ is called a symmetric **Dirichlet Form** on $L_2(X, \mu)$ if

(DF1) $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is a nonnegative definite bilinear form on a dense subspace $\mathcal{F} \subset L_2(X, \mu)$.

(DF2) $(\mathcal{E}, \mathcal{F})$ is closed i.e. $(\mathcal{F}, \mathcal{E}_1)$ is a Hilbert space, where $\mathcal{E}_1(f, g) = \mathcal{E}(f, g) + \langle f, g \rangle_{L_2}$.

(DF3) **Markov property** $u \in \mathcal{F}$ implies that $\tilde{u} = \max\{\min\{u, 0\}, 1\} \in \mathcal{F}$ and $\mathcal{E}(\tilde{u}) \le \mathcal{E}(u)$.

(DF4) $\mathcal{C} := C_c(X) \cap \mathcal{F}$ is uniformly dense in compactly supported continuous functions $C_c(X)$ and dense in $\mathcal{F}$ with respect to the Hilbert space norm $\mathcal{E}_1(f)^{1/2}$.

Dirichlet algebra $\mathcal{C} := C_c(X) \cap \mathcal{F}$ is an $\mathbb{R}$-algebra of bounded functions, since

$$\mathcal{E}(fg)^{1/2} \leq \|f\|_{L_\infty(X,\mu)}\, \mathcal{E}(g)^{1/2} + \|g\|_{L_\infty(X,\mu)}\, \mathcal{E}(f)^{1/2}, \quad f, g \in \mathcal{C},$$

Energy Measures $f \in \mathcal{C}$ we may define a nonnegative Radon measure $\Gamma(f)$ on X by

$$\int \varphi \, d\Gamma(f) = \mathcal{E}(\varphi f, f) - \frac{1}{2}\mathcal{E}(\varphi, f^2), \quad \varphi \in \mathcal{C}.$$

Now let m be an energy dominant measure for $(\mathcal{E}, \mathcal{F})$. For simplicity we use the symbol $\Gamma(f)$ also to denote the density $d\Gamma(f)/dm$. Set

$$\mathcal{A} := \{f \in \mathcal{F}_{\text{loc}} \cap C(X) \mid \Gamma(f) \in L_\infty(X, m)\}. \tag{3}$$

The **intrinsic metric** or **Carnot-Caratheodory metric** induced by $(\mathcal{E}, \mathcal{F})$ and m is defined by

$$d_{\Gamma,m}(x, y) := \sup\{f(x) - f(y) \;:\; f \in \mathcal{A} \text{ with } \Gamma(f) \leq m\} \tag{4}$$

Let $C_0(X)$ denote the space of continuous functions on X that vanish at infinity and consider the space

$$\mathcal{A}_0 := \{f \in \mathcal{F}_{\mathrm{loc}} \cap C_0(X) \mid \Gamma(f) \in L_\infty(X,m)\}. \tag{5}$$

Set $\mathcal{A}_0^1 := \{f \in \mathcal{A}_0 : \Gamma(f) \leq m\}$.

Theorem

Let $(\mathcal{E},\mathcal{F})$ be a strongly local Dirichlet form on $L_2(X,\mu)$, let m be an energy dominant measure for $(\mathcal{E},\mathcal{F})$ and assume there exists a point separating coordinate sequence for $(\mathcal{E},\mathcal{F})$ with respect to m. Then $\mathcal{A}_0^1$ is **compact** *in $C_0(X)$ if and only if $d_{\Gamma,m}$* **induces the original topology**.

If the reference measure μ itself is energy dominant, by **Stollmann/Sturm**: If d and $d_{\Gamma,\mu}$induce the same topology, then $(X, d_{\Gamma,\mu})$ is a length space.

However, a look at the proofs and their background reveals that this conclusion remains valid for any energy dominant measure m, provided a point separating coordinate sequence exists.

Corollary

Let $(\mathcal{E}, \mathcal{F})$ be a strongly local Dirichlet form on $L_2(X, \mu)$, let m be an energy dominant measure for $(\mathcal{E}, \mathcal{F})$, and assume there exists a point separating coordinate sequence for $(\mathcal{E}, \mathcal{F})$ with respect to m. If $\mathcal{A}_0^1$ is compact in $C_0(X)$, then the **metric space $(X, d_{\Gamma,m})$ is a length space**.

Resistance form

For any set X, a pair $(\mathcal{E}, \mathcal{F})$ is called a **resistance form** on X if

(RF1) $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is a nonnegative definite symmetric bilinear form on a vector space $\mathcal{F}$ of real valued functions on X, and $\mathcal{E}(u) = 0$ if and only if u is constant on X,

(RF2) $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space; here $\sim$ is the equivalence relation on $\mathcal{F}$ given by $u \sim v$ if $u - v$ is constant on X,

(RF3) **Markov property**, $\mathcal{E}(\tilde{u}) \leq \mathcal{E}(u)$,

(RF4) **Resistance metric**

$$d_R(x, y) := \sup\left\{|u(x) - u(y)|^2 \;:\; u \in \mathcal{F},\ \mathcal{E}(u) \leq 1\right\}$$

(RF5) $\mathcal{F}$ separates the points of X.

Confer with Kigami 2012 Memoir.

Theorem

Suppose $(\mathcal{E}, \mathcal{F})$ *is a local* **resistance form** *on* X *such that* (X, d_R) *is compact, let* m *be an energy dominant measure for* $(\mathcal{E}, \mathcal{F})$*, and assume there exists a coordinate sequence for* $(\mathcal{E}, \mathcal{F})$ *with respect to* m*. Then the topologies induced by* d_R *and* $d_{\Gamma,m}$ *coincide, and* $(X, d_{\Gamma,m})$ *is a length space.*

Further, the space $(X, d_{\Gamma,m})$ *is compact and therefore* **complete and geodesic**.

Differential forms on Dirichlet spaces

Let X be a locally compact second countable Hausdorff space and m be a Radon measure on X with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on $L_2(X, m)$ and write again $\mathcal{C} := C_0(X) \cap \mathcal{F}$.

The space $\mathcal{C}$ is a normed space with

$$\|f\|_{\mathcal{C}} := \mathcal{E}_1(f)^{1/2} + \sup_{x \in X} |f(x)|.$$

We equip the space $\mathcal{C} \otimes \mathcal{C}$ with a bilinear form, determined by

$$\langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} = \int_X bd \, d\Gamma(a, c).$$

This bilinear form is nonnegative definite, hence it defines a seminorm on $\mathcal{C} \otimes \mathcal{C}$.

$\mathcal{H}$: the Hilbert space obtained by factoring out zero seminorm elements and completing.

These notions are mostly due to Fabio Cipriani, Jean-Luc Sauvageot

We call $\mathcal{H}$ the *space of differential* 1*-forms* associated with $(\mathcal{E}, \mathcal{F})$.

The space $\mathcal{H}$ can be made into a $\mathcal{C}$-$\mathcal{C}$-bimodule by setting

$$a(b \otimes c) := (ab) \otimes c - a \otimes (bc) \quad \text{and} \quad (b \otimes c)d := b \otimes (cd)$$

and extending linearly.

$\mathcal{C}$ acts on both sides by uniformly bounded operators.

we can introduce a derivation operator by defining $\partial : \mathcal{C} \to \mathcal{H}$ by $\partial a := a \otimes \mathbf{1}$.

$\|\partial a\|^2 \leq 2\mathcal{E}(a)$ and the **Leibniz rule** holds,

$$\partial(ab) = a\partial b + b\partial a, \quad a, b \in \mathcal{C}.$$

Co-Differential on Dirichlet spaces

The operator ∂ extends to a closed unbounded linear operator from $L_2(X, m)$ into $\mathcal{H}$ with domain $\mathcal{F}$. Let ∂^* denote its adjoint, such that

$$\langle \partial^* \omega, g \rangle_{\mathcal{H}} = \langle \omega, \partial g \rangle_{L_2(X,m)} \tag{6}$$

Let $\mathcal{C}^*$ be the dual space of the normed space $\mathcal{C}$. Then ∂^* defines a bounded linear operator from $\mathcal{H}$ into $\mathcal{C}^*$.

Define the Hilbert space

$$\mathbb{H} := L_2(X, m) \oplus \mathcal{H}$$

with the natural scalar product

$$\langle (f, \omega), (g, \eta) \rangle_{\mathbb{H}} := \langle f, g \rangle_{L_2(X,m)} + \langle \omega, \eta \rangle_{\mathcal{H}} .$$

Put $dom\,\mathbb{D} := \mathcal{F} \oplus dom\,\partial^*$ and define an unbounded linear operator $\mathbb{D} : \mathbb{H} \to \mathbb{H}$ by

$$\mathbb{D}(f,\omega) := (\partial^*\omega, \partial f), \quad (f,\omega) \in dom\,\mathbb{D}.$$

To $\mathbb{D}$ we refer as the **Dirac operator** associated with $(\mathcal{E}, \mathcal{F})$. In matrix notation its definition reads

$$\mathbb{D} = \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix}.$$

Lemma

The operator $(\mathbb{D}, dom\,\mathbb{D})$ *is self-adjoint on* $\mathbb{H}$.

Square of the Dirac operator

The square of $\mathbb{D}$ is given by

$$\mathbb{D}^2 = \begin{pmatrix} \partial^*\partial & 0 \\ 0 & \partial\partial^* \end{pmatrix}.$$

$(L, dom\ L)$: the infinitesimal $L_2(X, m)$-generator of $(\mathcal{E}, \mathcal{F})$.
Then $L = \partial^*\partial$.
Set

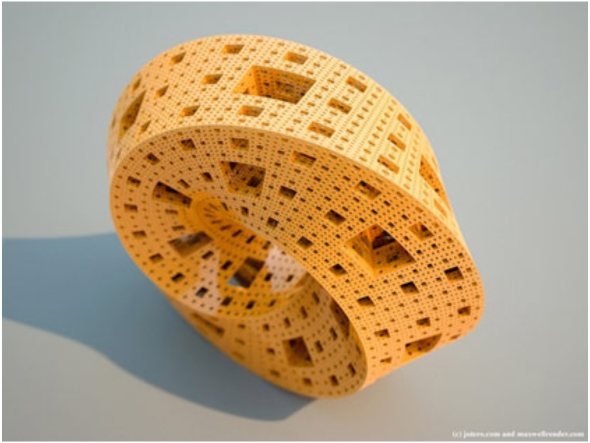
$$dom\ \Delta_1 := \{\omega \in dom\ \partial^* : \partial^*\omega \in \mathcal{F}\}$$

and $\Delta_1\omega := \partial\partial^*$, $\omega \in dom\ \Delta_1$.

Lemma

The operator $(\mathbb{D}^2, dom\ L \oplus dom\ \Delta_1)$ *is self-adjoint on* $\mathbb{H}$.

If our fractal is topologically 1-dimensional, then $\mathbb{D}^2$ is the ***Hodge Laplacian.*** (see Tep+Hinz)



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GAFA Geometric And Functional Analysis

MODULUS AND POINCARÉ INEQUALITIES ON NON-SELF-SIMILAR SIERPIŃSKI CARPETS

John M. Mackay, Jeremy T. Tyson and Kevin Wildrick

Abstract. A carpet is a metric space homeomorphic to the Sierpiński carpet. We characterize, within a certain class of examples, non-self-similar carpets supporting curve families of nontrivial modulus and supporting Poincaré inequalities. Our results yield new examples of compact doubling metric measure spaces supporting Poincaré inequalities: these examples have no manifold points, yet embed isometrically as subsets of Euclidean space.

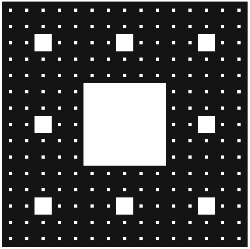

Figure 2: $S_{(1/3,1/5,1/7,\dots)}$

Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data

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Abstract

The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension d_s and walk dimension d_w associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

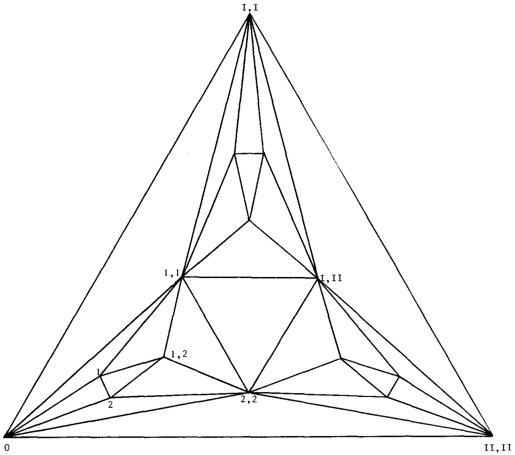

Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

Englert et al. ***Metric space-time as fixed point of the renormalization group equations on fractal structures***

Lemma

If the generator L of $(\mathcal{E}, \mathcal{F})$ has pure point spectrum

$$-Lf = \sum_{j=1}^{\infty} \lambda_j \langle f, \varphi_j \rangle_{L_2(X,\mu)} \varphi_j, \quad f \in dom\, L,$$

then $\mathbb{D}$ admits the spectral representation

$$\mathbb{D}v = \sum_{j=1}^{\infty} \lambda_j^{1/2} \langle v, v_j \rangle_{\mathbb{H}} v_j - \sum_{j=1}^{\infty} \lambda_j^{1/2} \langle v, w_j \rangle_{\mathbb{H}} w_j, \quad v \in dom\, \mathbb{D},$$

$$v_j = \frac{1}{\sqrt{2}}(\varphi_j, \lambda_j^{-1/2} \partial \varphi_j) \quad and \quad w_j = \frac{1}{\sqrt{2}}(\varphi_j, -\lambda_j^{-1/2} \partial \varphi_j)$$

In general, any eigenvalue of $\mathbb{D}$ may be of infinite multiplicity and $\ker \mathbb{D}$ consists of the 0-eigenvalue and 1-forms which are orthogonal to the image of ∂.

Corollary

Assume that L has pure point spectrum with spectral representation as above. Then the operator $(\mathbb{D}^2, dom\, \mathbb{D}^2)$ *admits the spectral representation*

$$\mathbb{D}^2 v = \sum_{i=0,1} \sum_{j=1}^{\infty} \lambda_j \langle v, v_{j,i} \rangle_{\mathbb{H}} v_{j,i}, \quad v \in dom\, \mathbb{D}^2.$$

Where $v_{j,0} = (\phi_j, 0)$ *and* $v_{j,1} = (0, w_j)$.
Again, any eigenfunction of $\mathcal{D}$ *may be of infinite multiplicity.*

Appendix: Spectral Triples

Definition

A (possibly kernel degenerate) *spectral triple* for an involutive algebra A is a triple (A, H, D) where H is a Hilbert space and $(D, dom\ D)$ a self-adjoint operator on H such that

(i) there is a faithful $*$-representation $\pi : A \to L(H)$,

(ii) there is a dense $*$-subalgebra A_0 of A such that for all $a \in A_0$ the commutator $[D, \pi(a)]$ is well defined as a bounded linear operator on H,

(iii) the operator $(1 + D)^{-1}$ is compact on $(ker\ D)^{\perp}$.

Spectral Triples

$$\mathbb{A}_0 := \{f \in \mathcal{C} : \Gamma(f) \in L_\infty(X,\mu)\} \qquad (7)$$

$$\mathbb{A} := \mathrm{clos}_{C_0(X)}(\mathcal{A}_0). \qquad (8)$$

Theorem

Let $(\mathcal{E},\mathcal{F})$ be a regular symmetric Dirichlet form on $L_2(X,\mu)$ and assume μ is energy dominant for $(\mathcal{E},\mathcal{F})$. Then

(i) *There is a faithful representation $\pi : \mathbb{A} \to L(\mathbb{H})$,*

(ii) *For any $a \in \mathbb{A}_0$ the commutator $[\mathbb{D}, \pi(a)]$ is a bounded linear operator on $\mathbb{H}$,*

(iii) *If the $L_2(X,\mu)$-generator L of $(\mathcal{E},\mathcal{F})$ has discrete spectrum then $(1+\mathbb{D})^{-1}$ is compact on $(\ker \mathbb{D})^\perp$, and $(\mathbb{A},\mathbb{H},\mathbb{D})$ is a spectral triple for $\mathbb{A}$.*

$$d_D(x,y) := \sup\{a(x) - a(y) \mid a \in A_0 \text{ is such that } \|[D,a]\| \le 1\}$$

defines a metric in the wide sense on X, (a version of) the *Connes distance.*

Theorem

Let $(\mathcal{E},\mathcal{F})$ be a strongly local Dirichlet form on $L_2(X,\mu)$, let m be energy dominant for $(\mathcal{E},\mathcal{F})$ and assume there exists a point separating coordinate sequence for $(\mathcal{E},\mathcal{F})$ with respect to m. Let $\mathbb{D}$, $\mathbb{A}_0$ and $\mathbb{A}$ be as above. Then

$$d_{\mathbb{D}}(x,y) := \sup\{a(x) - a(y) \mid a \in \mathbb{A}_0 \textit{ is such that } \|[\mathbb{D},a]\| \le 1\}$$

is a metric in the wide sense on X and $d_{\mathbb{D}} \le d_{\Gamma,\mu}$. If X is compact then $d_{\mathbb{D}} = d_{\Gamma,\mu}$.

Gelfand transform

We can use the theory of Banach algebras and the Gelfand transform to to consider Dirichlet forms on measurable spaces where the topology is not defined in advanced.

That is if we start off with a measure space $(X, \mathcal{X}, \mu)$, and a non-negative bilinear form $\mathcal{E}$ with suitable domain, then we use the relationship between the Gelfand transform and Daniell-Stone integration theory to extend $\mathcal{E}$ to a closable form on a space which has X embedded.