Existence and Uniqueness of Diffusions on $4N$- carpets

Ulysses Andrews
University of Connecticut

Talk Outline:

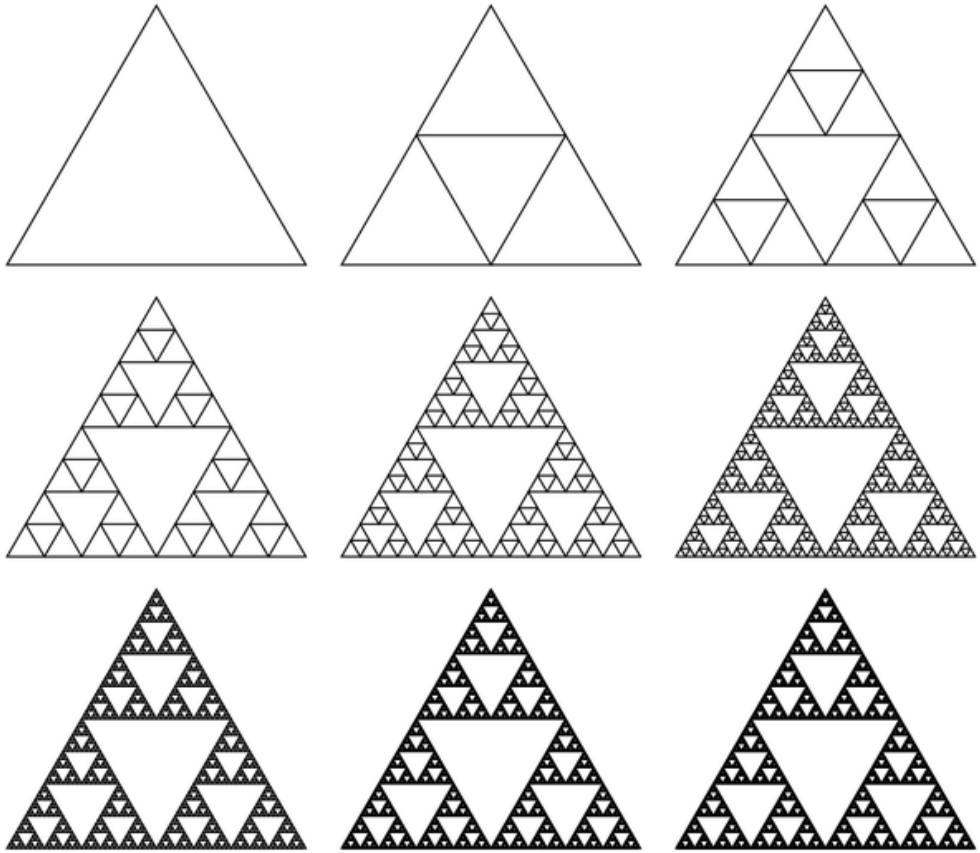
1. Background
2. The construction of $4N$-carpets, which we will call V.
3. Construct the Process on V.
4. Show that the process is a V-invariant, non-zero, local, regular, conservative Dirichlet form on V. Call the set of such Dirichlet forms $\mathfrak{E}$.
5. Show that $\mathfrak{E}$ contains at most one element.

Definition

1. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is a closed symmetric form on $L^2(E, m)$ that is Markovian. Write $\mathcal{F}$ for the domain.
2. A Dirichlet form is local if $\mathcal{E}(f, g) = 0$ whenever $f, g \in \mathcal{F}$ have disjoint compact supports.
3. A Dirchlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ is regular if E is a locally compact seperable metric space and m is a Radon measure on E with full support. $\mathcal{F} \cap C_c(E)$ is $\mathcal{E}_1$ dense in $\mathcal{F}$. $\mathcal{F} \cap C_c(E)$ is uniformly dense in $C_c(E)$.
4. If T_t is the semigroup associated to $(\mathcal{E}, \mathcal{F})$ then it is conservative if $T_t 1 = 1$ for all $t > 0$.

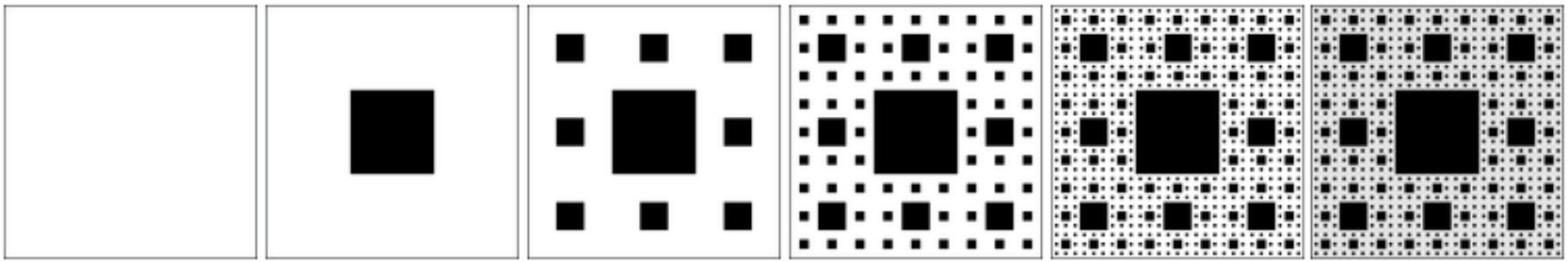
Barlow and Perkins 1988

Levels of the Sierpiński Gasket

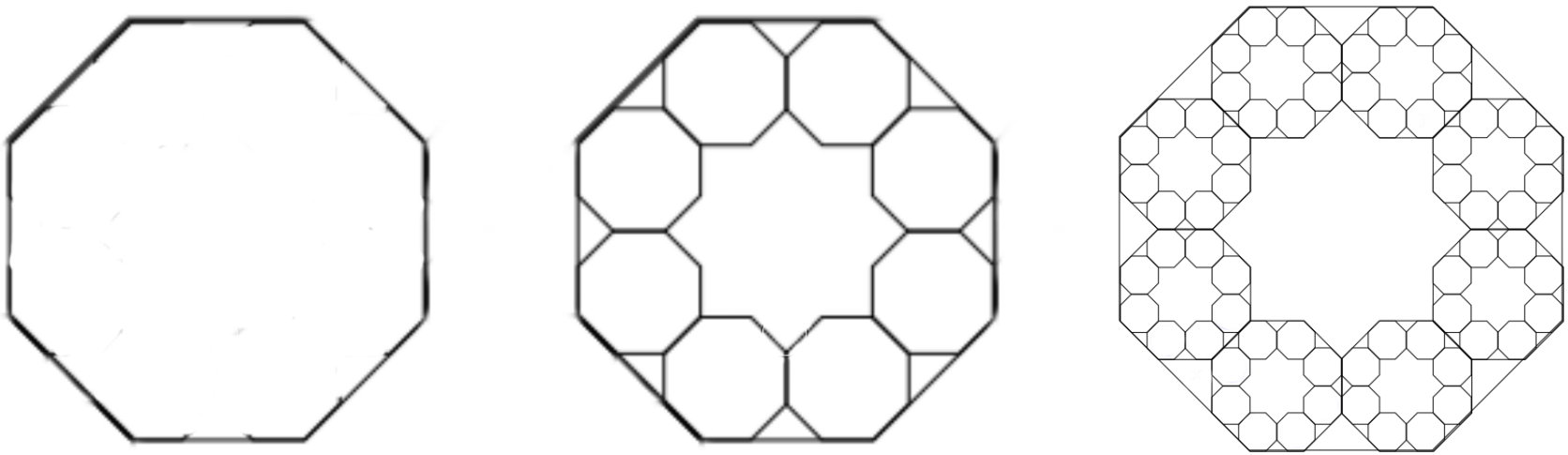


Barlow and Bass 1989-1999

Levels of the Sierpiński Carpet



Levels of the Octacarpet as an example of a $4N$-carpet.



Formally, the figure is the result of applying the iterated function system

$$F_i(x) = I_V^{-1}(x - q_i) + q_i \tag{1}$$

$$F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m} \quad w = (w_1, \ldots, w_m) \tag{2}$$

$$V_{n+1} = \cup_i F_i(V_n) \tag{3}$$

$$V = \cup_{|w|=0}^{\infty} F_w(V_i) \quad |w| = m \tag{4}$$

Constructing The Process on $4N$-carpets.

1. Begin with a normally reflecting Brownian motion W_t^n on V_n.
2. Prove a Uniform Elliptic Harnack inequality.
3. Consider an accelerated process $X_t^n = W_{a_n t}^n$

$$c_1(m_V \rho_V / l_V^2)^n \leq a_n \leq c_2(m_V \rho_V / l_V^2)^n \tag{5}$$

4. Show that the laws of X^n are tight.
5. If we denote the λ-resolvents by

$$U_n^\lambda f(x) = \mathbb{E}_n^x \int_0^\infty e^{-\lambda s} f(X_s^n) ds \tag{6}$$

   then the next step is to prove that they are tight as well.
6. Prove the existence of a subsequence n_j for which $U_{n_j}^\lambda f$ converges uniformly on V if f is continuous on V_0 and the law of X^{n_j} converges weakly for each x.

The resulting Dirichlet form for W^n is

$$\mathcal{E}_{W^n}(f,f)\int_{V_n}|\nabla f(x)|^2\mu_n(x), \tag{7}$$

the Dirichlet form for X^n is

$$\mathcal{E}_n(f,f)=a_n\int_{V_n}|\nabla f(x)|^2\mu_n(x), \tag{8}$$

both on $L^2(V,\mu_n)$.

Uniqueness of the Process
GSC proof by Barlow, Bass, Kumagai, Teplyaev 2009

Denote the semigroup of X by T_t. The limiting Dirichlet form is then given by

$$\mathcal{E}(f,f) = \sup_{t>0} \frac{1}{t}\langle f - T_t f, f\rangle. \tag{9}$$

The domain of $\mathcal{E}$, denoted $\mathcal{F}$, is the set of $f \in L^2(V,\mu)$ where the above is finite.

Theorem

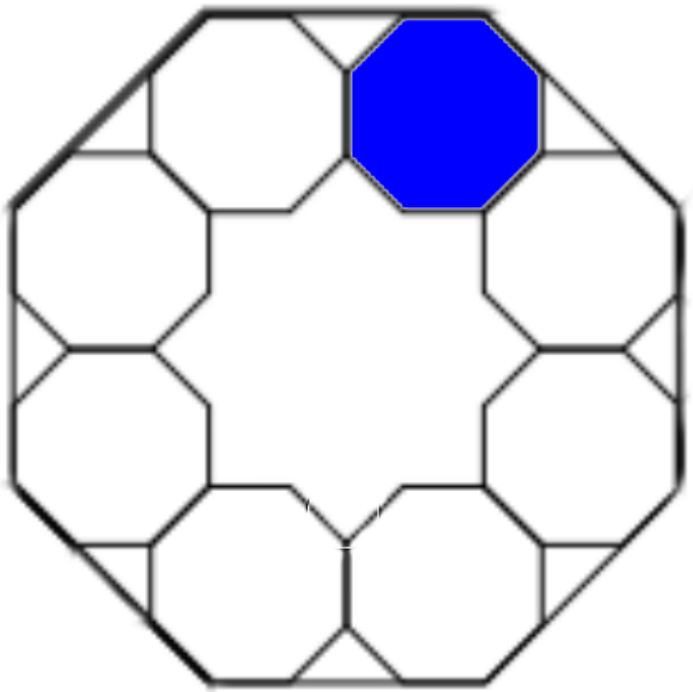
The Dirichlet form $\mathcal{E}$ is in $\mathfrak{E}$.

Proof Steps

1. Conservative: $T_t^{n_j} f \to T_t f$ uniformly for f continuous.
2. Regularity: the process is μ-symmetric Feller.
3. Local: this follows since the process is a diffusion.
4. Non-zero: by construction.
5. More notation will be needed to express invariance with respect to the symmetries of V.

Definition

We define the folding map $\varphi_S : V \to S$ for $S \in \mathcal{S}_n(V)$. Let $\overline{\varphi}_0 : [-1, 1] \to \mathbb{R}$ be defined by $\overline{\varphi}_0(x) = |x|$ for $|x| \leq 1$, and then extend the domain to $\mathbb{R}$ by periodicity, so that $\overline{\varphi}_0(x + 2n) = \overline{\varphi}_0(x)$ for $x \in \mathbb{R}, n \in \mathbb{Z}$. Denote the point of S closest to the origin by y. For $x \in V$ define $\varphi_S(x)$ to be the point for which the ith coordinate is given by $y_i + I_V^{-n}\overline{\varphi}_0(I_V^n(x_i - y_i))$.



$$\mathcal{S}_n = \mathcal{S}_n(V) = \{\mathcal{O} \cap V : \mathcal{O} \in \mathcal{O}_n(V)\}. \tag{10}$$

For $S \in \mathcal{S}_n$, $f : S \to \mathbb{R}$ and $g : V \to \mathbb{R}$ we define the unfolding and restriction operators by the following,

$$U_S f = f \circ \varphi_S, \quad R_S g = g|_S \tag{11}$$

Let $(\mathcal{E}, \mathcal{F})$ be a local regular Dirichlet form on $L^2(V, \mu)$. Let $S \in \mathcal{S}_n$. Write

$$\mathcal{E}^S(g, g) = \frac{1}{m_V^n} \mathcal{E}(U_S g, U_S g). \tag{12}$$

Define the domain of $\mathcal{E}^S$ to be $\mathcal{E}^S = \{g \,|\, g : S \to \mathbb{R}, U_S g \in \mathcal{F}\}$. Also we will write $\mu_S, \mu|_S$.

Definition

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(V, \mu)$. We say that $\mathcal{E}$ is a V-invariant Dirichlet form, i.e. that $\mathcal{E}$ is invariant with respect to all of the local symmetries of V if the following hold:

1. If $S \in \mathcal{S}_n(V)$, then $U_S R_S f \in \mathcal{F}$ for any $f \in \mathcal{F}$.
2. Let $n \geq 0$ and S_1, S_2 be any two elements of $\mathcal{S}_n$, and let Φ be any isometry of $\mathbb{R}^d$ which maps S_1 onto S_2. If $f \in \mathcal{F}^{S_2}$, then $f \circ \Phi \in \mathcal{F}^{S_1}$ and

$$\mathcal{E}^{S_1}(f \circ \Phi, f \circ \Phi) = \mathcal{E}^{S_2}(f, f). \tag{13}$$

3. For all $f \in \mathcal{F}$,

$$\mathcal{E}(f, f) = \sum_{S \in \mathcal{S}_n(V)} \mathcal{E}^S(R_s f, R_s f). \tag{14}$$

Definition (Elliptic Harnack Inequality)

We say that X satisfies the elliptic Harnack inequality (EHI) if there exists a constant c such that the following holds: for any ball $B(x, R)$ whenever u is a non-negative harmonic function on $B(x, R)$ then there is a quasi-continuous modification $\widetilde{u}$ of u which satisfies

$$\sup_{B(x,R/2)} \widetilde{u} \leq c \inf_{B(x,R/2)} \widetilde{u} \tag{15}$$

Definition (Volume Doubling)

Write $V(x,r) = \mu(B(x,r))$. We say that V satisfies volume doubling (VD) if there exists a constant c_1 such that

$$V(x,2R) \leq c_1 V(x,R) \quad \text{for all } x \in V, 0 \leq R \leq 1. \tag{16}$$

Definition (Heat Kernel Estimates)

Write h for the inverse of H. We say that $p_t(x,y)$ satisfies $HK(H;\eta_1,\eta_2,c_0)$ if for $x,y \in V, 0 < t \leq 1$,

$$p_t(x,y) \geq c_0^{-1} V(X,h(t))^{-1} \exp\left(-c_0 (H(d(x,y))/t)^{\eta_1}\right), \quad (17)$$

$$p_t(x,y) \leq c_0 V(X,h(t))^{-1} \exp\left(-c_0^{-1} (H(d(x,y))/t)^{\eta_2}\right). \quad (18)$$

A few constants that relate forthcoming bounds to the specific fractal on which we work.

$$\alpha = \log m_V / log l_V, \quad \beta_0 = \log(m_V \rho_V)/\log l_V. \tag{19}$$

Definition (Resistance Scaling)

We say that $\mathcal{E}$ satisfies the condition RES(H, c_1, c_2) if for all $x_0 \in V, r \in (0, I_V^{-1})$, we have

$$c_1 \frac{H(r)}{r^\alpha} \leq R_{\text{eff}}(B(x_0, r), B(x_0, 2r)^c) \leq c_2 \frac{H(r)}{r^\alpha}. \tag{20}$$

Conditions associated with H

1. $$C(t/s)^{\beta_0} \leq \frac{H(t)}{H(s)} \leq C'(t/s)^{\beta'} \text{ for } 0 < s \leq t < 1. \tag{21}$$

2. H satisfies the 'fast time growth' condition. Explicitly H satisfies 'time doubling':

$$H(2r) \leq cH(r) \text{ for } 0 \leq r \leq 1/2. \tag{22}$$

Theorem (Grigor'yan and Telcs 2012)

Let $H : [0,1] \to [0,\infty)$ be a strictly increasing function with $H(1) \in (0,\infty)$ that satisfies 21 and 22. Then the following are equivalent:

1. *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies VD, EHI, and $RES(H, c_1, c_2)$ for some $c_1, c_2 > 0$.*
2. *The Dirichlet form $\mathcal{E}, \mathcal{F})$ satisfies $HK(H; \eta_1, \eta_2, c_0)$ for some $\alpha, \eta_1, \eta_2, c_0 > 0$.*

A bit of notation...

$$G(f) = \sup_{0<r\leq 1} r^{-(\beta_0+\alpha)} \int_V \int_{B(x,r)} |f(x) - f(y)|^2 d\mu(x) d\mu(y) \quad (23)$$

Let $W = \{f \in L^2(V, \mu) : G(f) < \infty\}$.

Theorem (Kumagai, Sturm 2005)

Suppose p_t satisfies $HK(H, \eta_1, \eta_2, C_0)$, and H satisfies 22 and 21, then

$$C_1\mathcal{E}(f,f) \leq G(f) \leq C_2\mathcal{E}(f,f) \quad \text{for all } f \in W_H, \tag{24}$$

where the constants C_i depend only on the constants in 21,22, and in $HK(H; \eta_1, \eta_2, C_0)$. Further

$$\mathcal{F} = W \tag{25}$$

Definition

For $\mathcal{A}, \mathcal{B} \in \mathfrak{E}$ define

$$\sup(\mathcal{B}|\mathcal{A}) = \sup\left\{\frac{\mathcal{B}(f,f)}{\mathcal{A}(f,f)} : f \in W\right\}, \tag{26}$$

$$\inf(\mathcal{B}|\mathcal{A}) = \inf\left\{\frac{\mathcal{B}(f,f)}{\mathcal{A}(f,f)} : f \in W\right\} \tag{27}$$

$$h(\mathcal{A},\mathcal{B}) = \log\left(\frac{\sup(\mathcal{B}|\mathcal{A})}{\inf(\mathcal{B}|\mathcal{A})}\right); \tag{28}$$

Theorem

There exists a constant C_V, which depends only on V, such that if $\mathcal{A}, \mathcal{B} \in \mathfrak{E}$ then

$$h(\mathcal{A}, \mathcal{B}) \leq C_V \tag{29}$$

Theorem

Let $V \subset \mathbb{R}$ be a $4N$ carpet. Then, up to scalar multiples $\mathfrak{E}$ consists of at most one element. Furthermore, this element satisfies scale invariance.

The goal will be to prove that

$$h(\mathcal{A}, \mathcal{B}) = \log\left(\frac{\sup(\mathcal{B}|\mathcal{A})}{\inf(\mathcal{B}|\mathcal{A})}\right) = 0 \tag{30}$$

This implies that the Dirichlet forms only differ by a non-zero multiple.

We've shown earlier that $\mathfrak{E}$ is non-empty. Let $\mathcal{A}, \mathcal{B} \in \mathfrak{E}$, and $\lambda = \inf(\mathcal{B}|\mathcal{A})$. Let $\delta > 0$ and $\mathcal{C} = (1+\delta)\mathcal{B} - \lambda\mathcal{A}$. $\mathcal{C}$ is a local regular Dirichlet form on $L^2(\nu, \mu)$ and $\mathcal{C} \in \mathfrak{E}$. Since

$$\frac{\mathcal{C}(f,f)}{\mathcal{A}(f,f)} = (1+\delta)\frac{\mathcal{B}(f,f)}{\mathcal{A}(f,f)} - \lambda, \quad f \in W, \tag{31}$$

we have

$$\sup(\mathcal{C}|\mathcal{A}) = (1+\delta)\sup(\mathcal{B}|\mathcal{A}) - \lambda \tag{32}$$

$$\inf(\mathcal{C}|\mathcal{A}) = (1+\delta)\inf(\mathcal{B}|\mathcal{A}) - \lambda = \delta\lambda \tag{33}$$

From which it follows that for any $\delta > 0$, we have

$$e^{h(\mathcal{A},\mathcal{C})} = \frac{(1+\delta)\sup(\mathcal{B}|\mathcal{A}) - \lambda}{\delta\lambda} \geq \frac{1}{\delta}(e^{h(\mathcal{A},\mathcal{B})} - 1) \tag{34}$$

If $h(\mathcal{A}, \mathcal{B}) > 0$ the quantity is not bounded as $\delta \to 0$. This contradicts the previous theorem. It must therefore be the case that $h(\mathcal{A}, \mathcal{B}) = 0$.

I would like to thank the organizers for their work in putting together this conference. I would also like to thank Professor Teplyaev and Professor Rogers for their guidance.