Spectral Properties of the Hata Tree

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March 20, 2016

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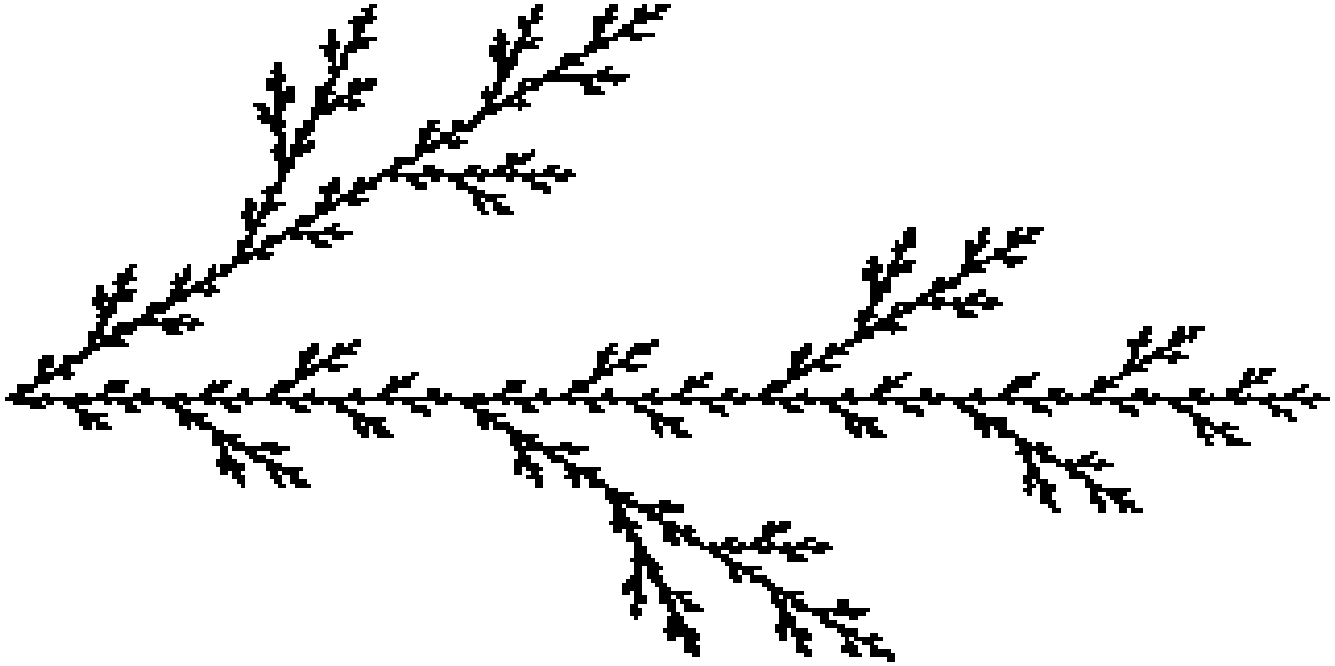
Let $\phi_0(z) = c\bar{z}$ and $\phi_1(z) = (1 - |c|^2)\bar{z} + |c|^2$, where $0 < |c| < 1$. The Hata tree is the unique compact set K satisfying $K = \phi_0(K) \cup \phi_1(K)$.



Let $\Phi(A) = \phi_0(A) \cup \phi_1(A)$. Let $V_0 = \{0, 1, c\}$ and $V_n = \Phi^n(V_0)$. The sets of vertices V_n can be given a graph structure in a natural way.

Definition

The probabilistic Laplacian P on a finite graph $G = (V, E)$ is a linear operator on $\mathbb{R}^V$ defined by

$$Pu(x) := \frac{1}{d_x} \sum_{x \sim y} (u(x) - u(y))$$

where $x \sim y$ if x and y share an edge and d_x denotes the degree of x. The normalized Laplacian N is defined by

$$Nu(x) := \sum_{x \sim y} (u(x) - \frac{1}{\sqrt{d_x d_y}} u(y)).$$

Let $P^{(n)}$ and $N^{(n)}$ denote the probabilistic and normalized Laplacians, respectively, on V_n.

Proposition

Let G be a graph. Let N be the normalized Laplacian on G and $D(N)$ its characteristic polynomial. Fix a vertex j in G and let $j_1, j_2, ..., j_k$ be its neighboring vertices. Then

$$D(N) = (1-\lambda)D(N_j) - \sum_{n=1}^{k} \frac{1}{d_j d_{j_n}} D(N_{j,j_n})$$

Proof: This follows by expanding the determinant

$$D(N) = \begin{vmatrix} 1-\lambda & -\sqrt{d_j d_{j_1}}^{-\frac{1}{2}} & -\sqrt{d_j d_{j_2}}^{-\frac{1}{2}} & \dots \\ -\sqrt{d_{j_1} d_j}^{-\frac{1}{2}} & 1-\lambda & -\sqrt{d_{j_1} d_{j_2}}^{-\frac{1}{2}} & \dots \\ -\sqrt{d_{j_2} d_j}^{-\frac{1}{2}} & -\sqrt{d_{j_2} d_{j_1}}^{-\frac{1}{2}} & 1-\lambda & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

We will need the polynomials

$$g^{(2)}(\lambda) = (1-\lambda)^5 - \frac{10}{9}(1-\lambda)^3 + \frac{2}{9}(1-\lambda)$$
$$g_u^{(2)}(\lambda) = (1-\lambda)^4 - \frac{2}{3}(1-\lambda)^2$$
$$g_w^{(2)}(\lambda) = (1-\lambda)^4 - \frac{7}{9}(1-\lambda)^2 + \frac{1}{9}$$
$$g_{uw}^{(2)}(\lambda) = (1-\lambda)^3 - \frac{1}{3}(1-\lambda)$$
$$g_{vw}^{(2)}(\lambda) = (1-\lambda)^3 - \frac{1}{3}(1-\lambda)$$
$$g_{uvw}^{(2)}(\lambda) = (1-\lambda)^2$$

and

$$g^{(3)}(\lambda) = (1-\lambda)^{11} - \frac{22}{9}(1-\lambda)^9 + \frac{170}{81}(1-\lambda)^7 - \frac{20}{27}(1-\lambda)^5 + \frac{22}{243}(1-\lambda)^3$$
$$g_u^{(3)}(\lambda) = (1-\lambda)^{10} - 2(1-\lambda)^8 + \frac{100}{81}(1-\lambda)^6 - \frac{2}{9}(1-\lambda)^4$$
$$g_w^{(3)}(\lambda) = (1-\lambda)^{10} - \frac{19}{9}(1-\lambda)^8 + \frac{125}{81}(1-\lambda)^6 - \frac{37}{81}(1-\lambda)^4 + \frac{11}{243}(1-\lambda)^2$$
$$g_{uw}^{(3)}(\lambda) = (1-\lambda)^9 - \frac{5}{3}(1-\lambda)^7 + \frac{67}{81}(1-\lambda)^5 - \frac{1}{9}(1-\lambda)^3$$
$$g_{vw}^{(3)}(\lambda) = (1-\lambda)^9 - \frac{5}{3}(1-\lambda)^7 + \frac{23}{27}(1-\lambda)^5 - \frac{11}{81}(1-\lambda)^3$$
$$g_{uvw}^{(3)}(\lambda) = (1-\lambda)^8 - \frac{11}{9}(1-\lambda)^6 + \frac{1}{3}(1-\lambda)^4$$

Proposition

Let $g^{(j)}, g_u^{(j)}, g_w^{(j)}, g_{uw}^{(j)}, g_{vw}^{(j)}, g_{uvw}^{(j)}$ for $j = 2, 3$ be defined as above. For $n \geq 4$, let

$$g^{(n)} = (1-\lambda)g^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_u^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g^{(n-1)}g_u^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g^{(n-1)}g^{(n-2)}g_{vw}^{(n-2)}$$

$$g_u^{(n)} = (1-\lambda)g^{(n-1)}g^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_u^{(n-1)}g^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g^{(n-1)}g_u^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g^{(n-1)}g^{(n-2)}g_{uvw}^{(n-2)}$$

$$g_w^{(n)} = (1-\lambda)g_w^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_{uw}^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_w^{(n-1)}g_u^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_w^{(n-1)}g^{(n-2)}g_{vw}^{(n-2)}$$

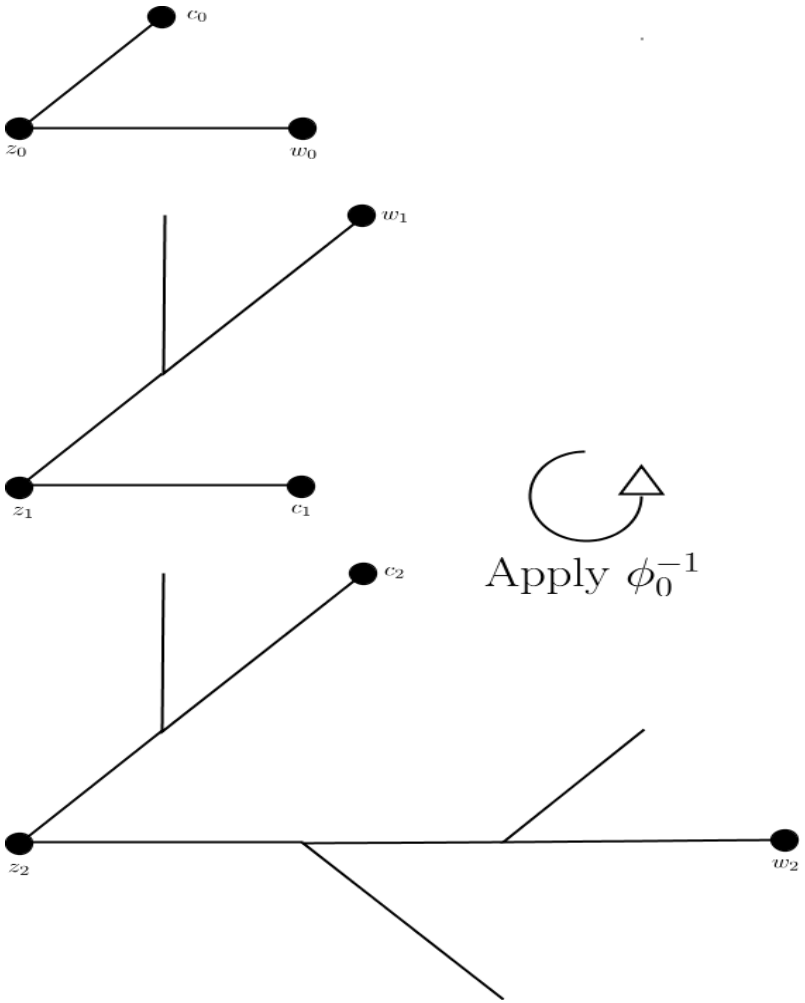
$$g_{uw}^{(n)} = (1-\lambda)g_w^{(n-1)}g^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_{uw}^{(n-1)}g^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_w^{(n-1)}g_u^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_w^{(n-1)}g^{(n-2)}g_{uvw}^{(n-2)}$$

$$g_{vw}^{(n)} = (1-\lambda)g_{vw}^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_{uvw}^{(n-1)}g^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_{vw}^{(n-1)}g_u^{(n-2)}g_w^{(n-2)} - \frac{1}{9}g_{vw}^{(n-1)}g^{(n-2)}g_{vw}^{(n-2)}$$

$$g_{uvw}^{(n)} = (1-\lambda)g_{vw}^{(n-1)}g^{(n-2)}g_{ww}^{(n-2)} - \frac{1}{9}g_{uvw}^{(n-1)}g^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_{vw}^{(n-1)}g_u^{(n-2)}g_{uw}^{(n-2)} - \frac{1}{9}g_{vw}^{(n-1)}g^{(n-2)}g_{uvw}^{(n-2)}$$

Then $D(P^{(n)}) = (1-\lambda)g^{(n)}g^{(n-1)} - \frac{1}{6}g_u^{(n)}g^{(n-1)} - \frac{1}{6}g^{(n)}g_u^{(n-1)}$.

Blow-Up of Graph Approximations to the Hata Tree



Decomposition of the spectrum of $P^{(n)}$

In this analysis it is more convenient to work with the eigenvectors of $P^{(n)}$.

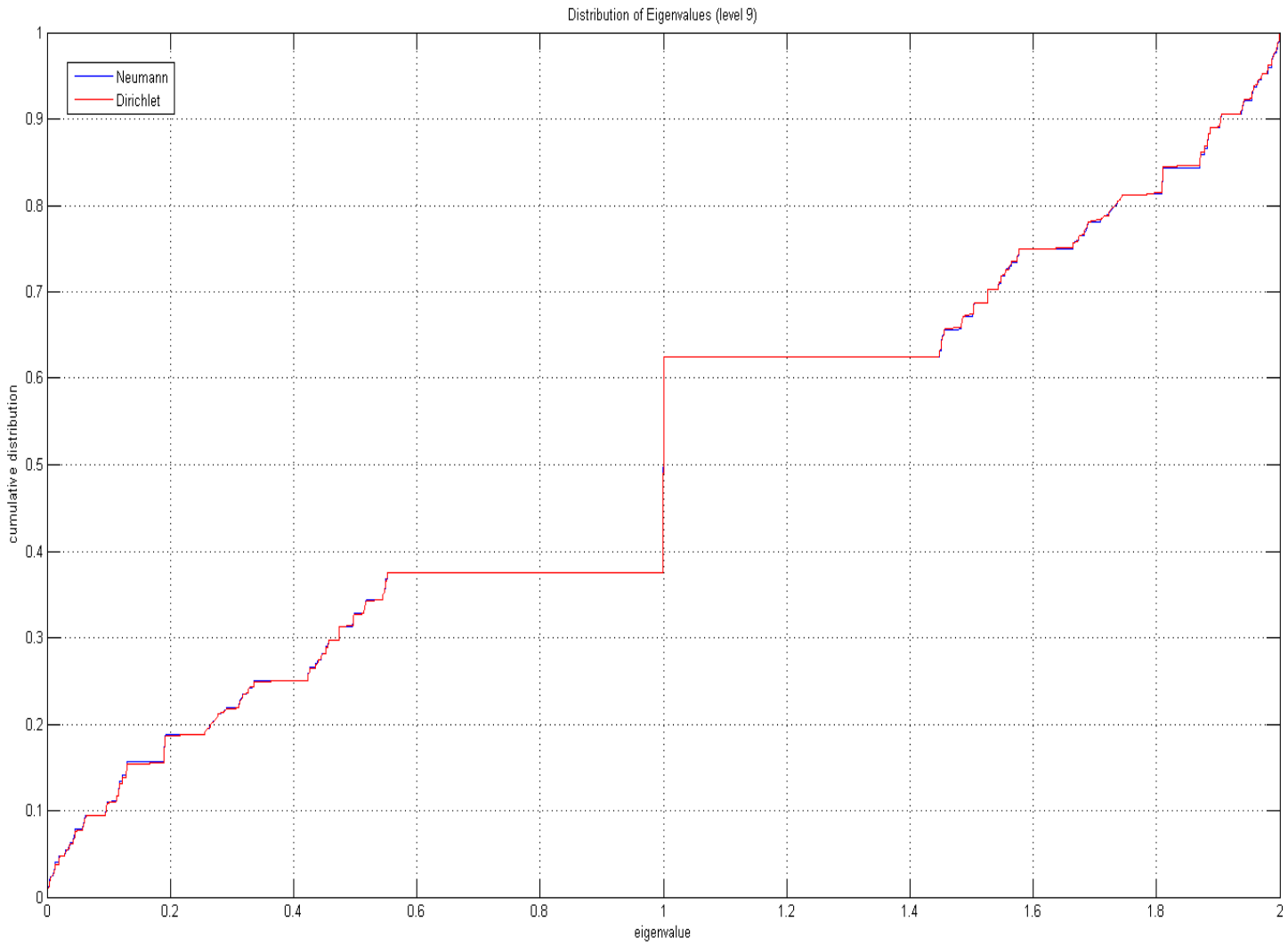
2^{n-1} eigenvalues of multiplicity one that are eigenvalues of multiplicity one of $P^{(n-1)}$.

2^n "new" eigenvalues of multplicity one that are not eigenvalues of $P^{(n-1)}$.

$2^{n-1}+1$ eigenvalues equal to one.

Total: $2^{n+1}+1$ eigenvalues

Distribution of Eigenvalues



Spectrum on the Infinite Blow-Up

We can define a Laplacian operator $P^{(\infty)}$ on the infinite blow-up in a pointwise manner. There is a natural extension of $P^{(n)}$ to the blow-up. It can be shown that the spectrum of $P^{(\infty)}$ is the limit of the spectrum of $P^{(n)}$.

Depending on the blow-up, we can restrict $P^{(n)}$ and $P^{(\infty)}$ to the interior of the lattice. The spectrum of this Dirichlet operator on the lattice is equal to the limit of the spectrum of the approximating operators.

Back to the Hata Tree

There is some work left in constructing the Green's function and Laplacian operator on the Hata tree itself and to make conclusions about its spectrum.

One complication is that resistances between any pair of points evolves as a Fibonacci sequence as opposed to being scaled by certain fixed factors.

In *Spectral properties of self-similar lattices and iteration of rational maps* (2003), Sabot works with dynamical systems that can be used to compute the spectrum of Laplacian operators on lattices that satisfy certain symmetry conditions.

The Hata tree does not satisfy these symmetry conditions. Nonetheless, it is possible to apply some aspects of the theory.

Let

$$A^{(0)} = \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix},$$

be an operator on V_0 and let $A^{(n)}$ be formed by placing a copy of $A^{(0)}$ on each cell of V_n isomorphic to V_0. In particular, we have

$$A^{(1)} = \begin{pmatrix} c & e & f & 0 & 0 \\ e & b & d & 0 & 0 \\ f & d & a+b & d & e \\ 0 & 0 & d & a & f \\ 0 & 0 & e & f & c \end{pmatrix}.$$

Define

$$\begin{aligned} T: \ \mathrm{Sym}_3 &\to \mathrm{Sym}_3 \\ Q &\mapsto (Q^{(1)})|_{\partial V_{-1}} \end{aligned}$$

Here, $Q^{(1)}$ denotes the operator on V_1 constructed in the same way as $A^{(1)}$. In matrix notation, we have

$$\begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix} \mapsto \begin{pmatrix} a^{(1)} & d^{(1)} & f^{(1)} \\ d^{(1)} & b^{(1)} & e^{(1)} \\ f^{(1)} & e^{(1)} & c^{(1)} \end{pmatrix}$$

where

$$a^{(1)} = c - \frac{af^2}{a^2+ab-d^2}, \quad b^{(1)} = b - \frac{ad^2}{a^2+ab-d^2}, \quad d^{(1)} = e - \frac{adf}{a^2+ab-d^2}, \quad e^{(1)} = \frac{d(-ae+df)}{a^2+ab-d^2}$$

$$c^{(1)} = \frac{-a^2c + cd^2 + f(-2de+bf) + a(-bc+e^2+f^2)}{-a^2-ab+d^2}, \quad f^{(1)} = \frac{f(-ae+df)}{a^2+ab-d^2}$$

It can be proved by induction that $(Q^{(n)})|_{\partial V_n} = T^n(Q)$.

In the case of the Hata tree, since there is no edge between 1 and c, we can write the map T as

$$T(a,b,c,d,e) = \left(c, b - \frac{ad^2}{a^2+ab-d^2}, c - \frac{ae^2}{a^2+ab-d^2}, e, -\frac{ade}{a^2+ab-d^2}\right).$$

Let

$$D(a,b,c,d,e) = abc - ae^2 - cd^2$$

be the determinant of the corresponding 3×3 matrix. Observe that $D(T^n(1-\lambda, 2-2\lambda, 1-\lambda, -1, -1))$ will give us the characteristic polynomial of the probabilistic Laplacian $P^{(n)}$.

It is possible to reduce this five dimensional dynamical system to two dimensions. In particular, let

$$\begin{aligned} c_{n+1} &= c_n - \frac{c_{n-1}e_n^2}{c_{n-1}^2 + c_{n-1}\left(c_{n-1} - e_{n-1}\frac{e_{n-2}^2 - e_{n-3}^2}{e_{n-3}e_{n-2}}\right) - e_{n-1}^2}, \\ e_{n+1} &= -\frac{c_{n-1}e_{n-1}e_n}{c_{n-1}^2 + c_{n-1}\left(c_{n-1} - e_{n-1}\frac{e_{n-2}^2 - e_{n-3}^2}{e_{n-3}e_{n-2}}\right) - e_{n-1}^2}. \end{aligned}$$

Then

$$D_n = c_{n-1}\left(c_{n-1} - e_{n-1}\frac{e_{n-2}^2 - e_{n-3}^2}{e_{n-3}e_{n-2}}\right)c_n - c_{n-1}e_n^2 - c_n e_{n-2}^2,$$

will be the characteristic polynomial of the corresponding Laplacian.

By the Sabot theory, we can construct a map R that is the analogue of T on some subset of a projective space that is isomorphic to a Lagrangian Grassmanian. This map can be used to write down a nice compact expression for the density of states (the limiting distribution of eigenvalues).

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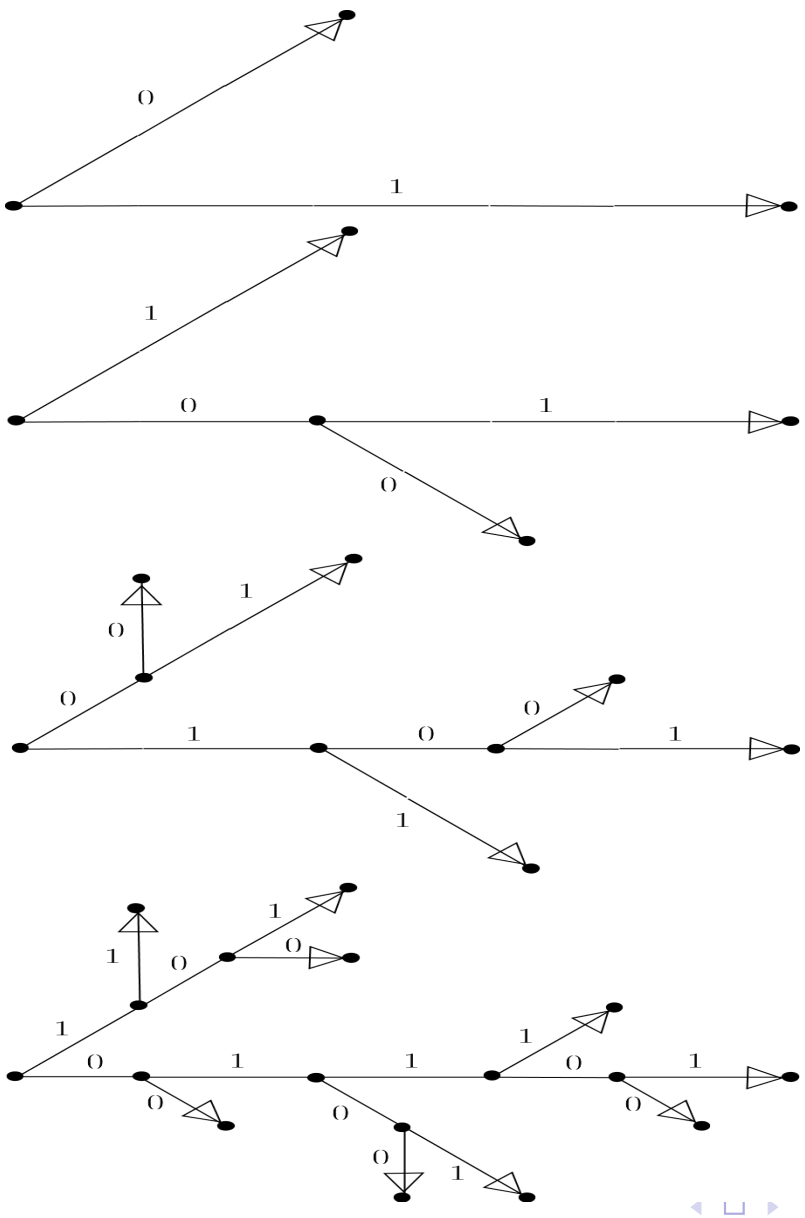
In *Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals* (1993), Kigami and Lapidus prove some results on the spectral asymptotics on a certain class of Laplacians on p.c.f. fractals, including the Hata tree.

It is possible to obtain similar results on the Hata tree by using an alternate construction.

Binary Functions and Orientations on V_n



Mixed Affine Nested Fractals

Let A be a finite set. For $a \in A$, let

$$\psi^a = \{\psi_i^a : i = 1, ..., m_a\}$$

denote a set of m_a similtudes in $\mathbb{C}$ that determines a unique fixed point S_a. Assume the set of fixed points E is the same for each $a \in A$.

To construct a composite fractal, we can define an address space T and an A-valued function U that will determine the location of cells in approximations to the fractal and the set of similtudes to be applied in the subsequent approximation, respectively. In particular, for $\mathbf{i} \in T_n$, let

$$(S)_{\mathbf{i}} = \psi_{\mathbf{i}(1)}^{U([\mathbf{i}]_0)} \circ \cdots \circ \psi_{\mathbf{i}(n)}^{U([\mathbf{i}]_{n-1})}(E),$$

and the corresponding mixed affine nested fractal is defined by

$$S = \cap_{n=0}^{\infty} \cup_{\mathbf{i} \in T_n} (S)_{\mathbf{i}}.$$

Define the set of similtudes $\psi^a = \{\psi_1^a, \psi_2^a, \psi_3^a\}$ on $\mathbb{C}$ such that

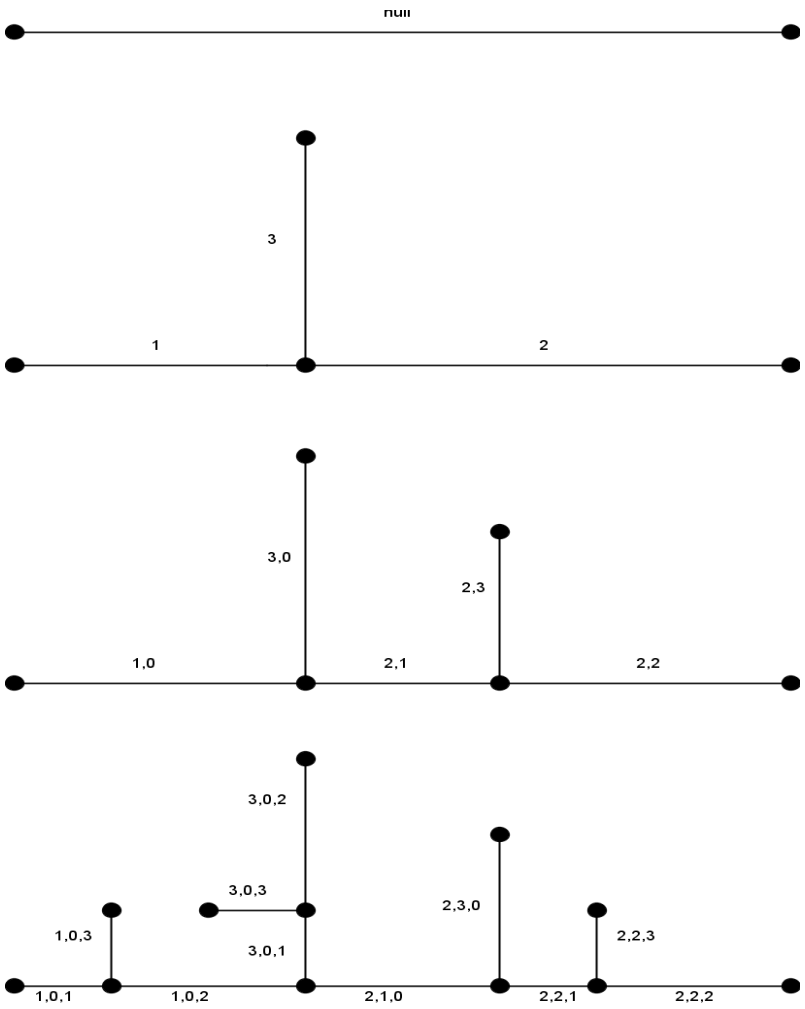
$$\psi_1^a = |c|^2 z; \quad \psi_2^a(z) = (1-|c|^2)(z-1)+1; \quad \psi_3^a = |c|^2 + i|c|(1-|c|^2)z.$$

Let ψ^e denote the set consisting of the identity map. Let $T_1 = \{\{1\},\{2\},\{3\}\}$, and define the $\{a, e\}$ valued function U by

$$U([\mathbf{i}]_m) = \begin{cases} a & : \text{if } [\mathbf{i}]_m(m) = 0 \text{ or } 2 \\ e & : \text{if } [\mathbf{i}]_m(m) = 1 \text{ or } 3 \end{cases}$$

Let T be the corresponding address space. ψ^a, ψ^e, T, and U determine a set S that is homeomorphic to "half" of a Hata tree.

The set S


null
3
1
2
3,0
2,3
1,0
2,1
2,2
3,0,2
3,0,3
2,3,0
1,0,3
3,0,1
2,2,3
1,0,1
1,0,2
2,1,0
2,2,1
2,2,2

The identity map in ψ^e is not a contraction map and thus resistances between successive approximations to S to not scale exactly. However, this can be remedied by jumping ahead to the second approximation.

Let $\psi^b = \{\psi_j^b : j = 1, ..., 5\}$, where $\psi_j^b = \psi_2^a \circ \psi_j^a$ for $j = 1, 2, 3$, $\psi_4^b = \psi_1^a$, and $\psi_5^b = \psi_3^a$. Using ψ^a and ψ^b, we can define two affine nested fractals S^o and S^e such that their approximations are homeomorphic to the odd and even approximations to the Hata tree, respectively.

More precisely, if we identify the vertex 0 in S^o and S^e, then K_n is homeomorphic to $S_n^e \cup S_n^o$ for $n \geq 1$.

By applying a multidimensional renewal theorem and the results/methods of Hambly and Nyberg on graph directed fractals, it is possible to obtain the spectral asymptotics for the eigenvalue counting functions.

Theorem

If ν_1 is non-lattice, then

$$\lim_{z\to\infty} N^x(z)z^{-d_s/2} = c_4(x)$$

$$\lim_{z\to\infty} N_0^x(z)z^{-d_s/2} = c_5(x)$$

where c_4, c_5 are constants depending on x. If ν_1 is lattice, then

$$\lim_{z\to\infty} N^x(z)z^{-d_s/2} - p_1^x(\log z) = 0$$

$$\lim_{z\to\infty} N_0^x(z)z^{-d_s/2} - p_2^x(\log z) = 0$$

where p_1^x, p_2^x are periodic functions depending on x.

Thank you!!