Mean value properties on Sierpinski type fractals

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1. Introduction

For a domain Ω in Euclidean space, a continuous function u is harmonic $(\Delta u = 0)$ if and only if

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy = u(x)$$

if $B_r(x) \subset \Omega$ where $B_r(x)$ is the ball of radius r about x.

More generally, if u is not assumed harmonic but Δu is a continuous function, then

$$\lim_{r\to 0}\frac{1}{r^2}\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy - u(x)\right) = c_n\Delta u(x) \qquad (1)$$

for the appropriate dimensional constant c_n.

What are the fractal analogs of these results?
What are the fractal analogs of "balls" on which we do the averaging?
So if K is a fractal and $x \in K$, we would like to know that there is a sequence of sets $B_k(x)$ containing x with $\bigcap_k B_k(x) = \{x\}$ such that

$$\frac{1}{\mu(B_k(x))} \int_{B_k(x)} u(y)dy = u(x)$$

for every harmonic function u.

We call $B_k(x)$ the *k-level mean value neighborhood* of x.

Let $K = SG$ equipped with the standard harmonic structure and the standard self-similar measure.

If x is a nonboundary vertex point, we can easily answer the question with $B_k(x)$ being the two union of the k level neighboring cells of x.
What about the generic points?

More generally, for general u not assumed harmonic but belonging to $dom\Delta$, does an analogous formula of (1) still hold?

2. Mean value neighborhoods on Sierpinski gasket

Consider any cell $F_w(SG) = C_w$ with boundary points $F_w(q_i) = p_i$.
For any point $x \in C_w$ there exist coefficients $a_i(x)$ such that

$$h(x) = \sum_i a_i(x) h(p_i)$$

for any given harmonic function h.

Since constants are harmonic we must have

$$\sum_i a_i(x) = 1,$$

and by the maximum principle all $a_i(x) \geq 0$.

We can compute $\{a_i(x)\}$ for x any junction point by using the harmonic extension algorithm.

Let W denote the triangle in $\mathbb{R}^3$ with boundary points $(1,0,0),(0,1,0),(0,0,1)$.

So $\{a_i(x)\} \in W$ for any $x \in C_w$.

Of course not every point in W occurs in this way.

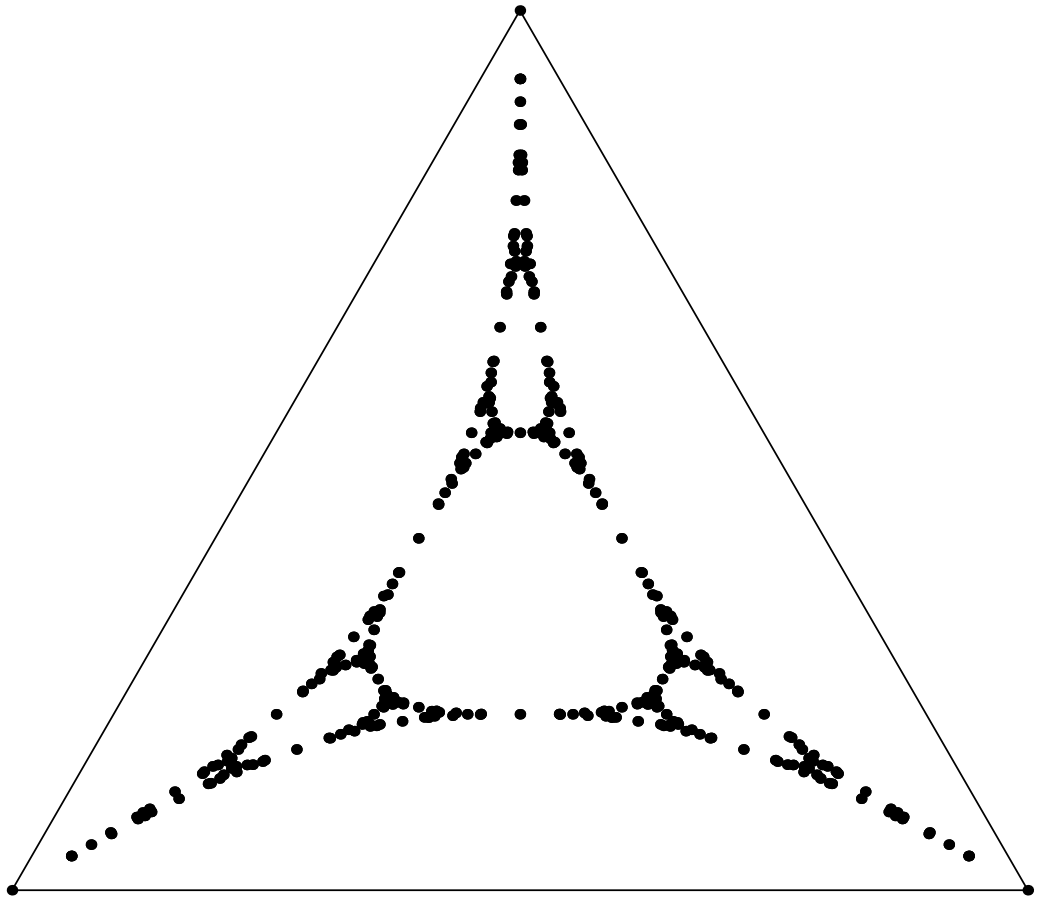

Figure 1.

Given a set $B \subset SG$, define

$$M_B(f) = \frac{1}{|B|} f d\mu$$

the *mean value* of function f on B.

By linearity,

$$M_B(h) = \sum_i a_i h(p_i)$$

for some coefficients $\{a_i\}$ for any harmonic function h.

We also have $\sum_i a_i = 1$ by considering $h \equiv 1$.

Idea: We hope that $\{a_i\} \in W$ if $C_w \subset B$. If this is true, we have a map

$$\{B\} \to W.$$

If we can show that the map is onto for some reasonable class of sets B, then we can get our first $B_0(x)$ for every $x \in C_w$, and then by zooming in get a full sequence $B_k(x)$.

Observation:

Note that $M_{C_w}(h) = \sum_i \frac{1}{3} h(p_i)$ so we hit the center point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of W by taking $B = C_w$.

If we take B to be the union of C_w and one of its neighboring cell of the same size, then

$$M_B(h) = h(p_i)$$

for the point of intersection.

So we can get the 3 vertices $(1,0,0),(0,1,0),(0,0,1)$ of W in this way.

By varying B continuously between the two we can hit a curve in W joining the center to any vertex.

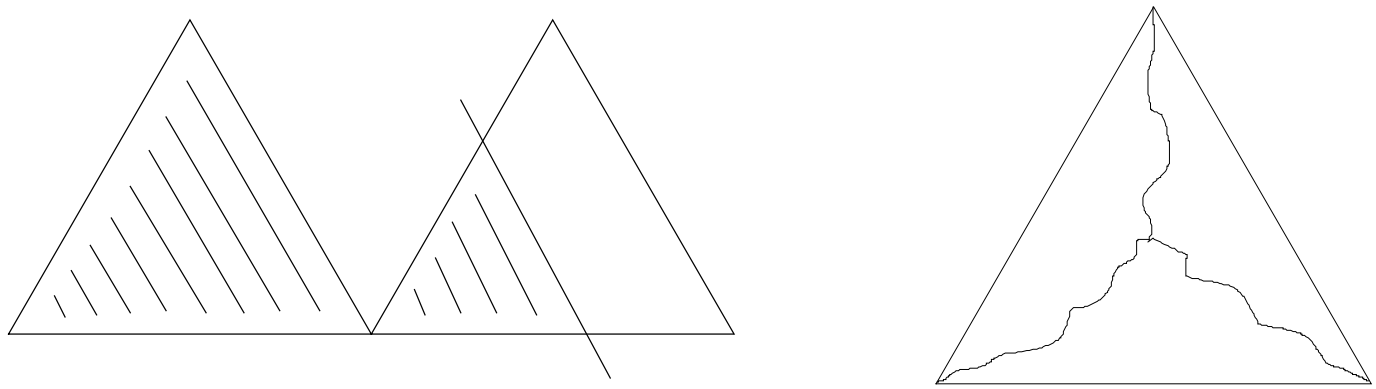

Figure 2.

This looks like the beginning of an argument to show that the mapping is onto.

By the observation, we should require B to be a subset of the union of C_w and its 3 neighbors, containing C_w.

Reasonable choice of B: connected, symmetric, depends only on the relative geometry of x and C_w, independent of the size of C_w and the location of C_w.

Denote by C_0, C_1, C_2 the three neighboring cells of the same size of C_w, intersecting C_w at p_i for each i. Write D_w the union of C_w and its 3 neighbors, i.e., $D_w = C_w \cup C_0 \cup C_1 \cup C_2$.

Consider a set B, $C_w \subset B \subset D_w$.

$$B = C_w \cup E_0 \cup E_1 \cup E_2,$$

where $E_i = B \cap C_i$, $i = 0, 1, 2$.

We restrict each E_i to be a triangle sub-domain of C_i, symmetric under the reflection symmetry that fixes p_i, containing p_i as one of its boundary points.

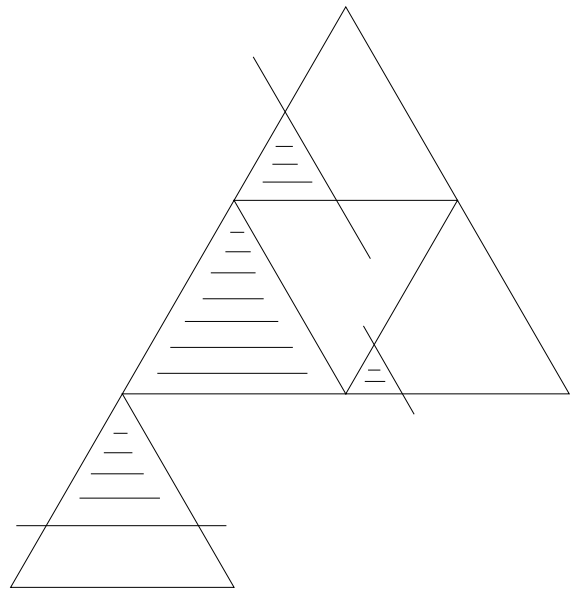

Figure 3.

Suppose the height of C_w is r, then the height of E_i should be $c_i r$ for $0 \leq c_i \leq 1$. Hence we can write the set $B = B(c_0, c_1, c_2)$.

For example, $B(0,0,0) = C_w$ and $B(1,1,1) = D_w$.

Denote by

$$\mathcal{B}^* = \{B(c_0, c_1, c_2) : 0 \leq c_i \leq 1\}$$

the family of all such sets.

Let $\mathcal{T}_w$ denote the map from $\mathcal{B}^*$ to π_W (the plane containing W) described before.

$\mathcal{T}_w$ can be viewed as a nonlinear vector valued function from $[0,1]^3$ to π_W. For simplicity, we may write $\mathcal{T}_w(\{c_i\}) = \{a_i\}$ for each set $B(c_0, c_1, c_2)$.

Fact 1. *The map $\mathcal{T}_w$ is independent of the particular choice of C_w.*

Hence we can drop the subscript w on $\mathcal{T}$ for simplicity.

The proof of Fact 1 benefits from the symmetric properties of E_i 's.

Fact 2. *There exists* $B \in \mathcal{B}^*$*, such that* $\mathcal{T}(B) \notin W$.

For example, a easy computation shows that $\mathcal{T}(0,1,1) = \{-\frac{1}{9}, \frac{5}{9}, \frac{5}{9}\}$.

However, if we can show that the image of the map $\mathcal{T}$ can cover the triangle W, things will still go well.

Theorem 1. *The map $\mathcal{T}$ from $\mathcal{B}^*$ to π_W fills out a region $\widetilde{W}$ which contains the triangle W.*

Sketch of the proof.

Step 1. Consider a subfamily $\mathcal{B}_1 = \{B(0,0,c_2) : 0 \leq c_2 \leq 1\}$ of $\mathcal{B}^*$. If we restrict the map $\mathcal{T}$ to $\mathcal{B}_1$, by varying c_2 continuously between 0 and 1 we can hit a curve (it is a line segment which follows from the symmetry of E_2) in W joining the center O to an vertex point N.

Step 2. Consider another subfamily $\mathcal{B}_2 = \{B(0, c, c) : 0 \leq c \leq 1\}$ of $\mathcal{B}^*$. If we restrict the map $\mathcal{T}$ to $\mathcal{B}_2$, by varying c continuously between 0 and 1 we can hit a curve (it is a line segment which follows from the symmetric effect of E_1 and E_2) in W joining the center O to an point Q. A directly computation shows that Q lies outside of the triangle W.

Step 3. $\mathcal{T}(\{B(0, c, 1) : 0 \leq c \leq 1\})$ is a curve located outside of the triangle W, joining N to Q. The intersection of this curve and W consists exactly only one point N.

Step 4. Fix a number $0 \leq y \leq 1$. Consider a subfamily $\mathcal{C}_y = \{B(0, c, y) : 0 \leq c \leq y\}$ of $\mathcal{B}^*$. If we restrict the map $\mathcal{T}$ on $\mathcal{C}_y$, by varying c continuously between 0 and y we can hit a curve Γ_y joining the two points $\mathcal{T}(B(0, 0, y))$ and $\mathcal{T}(B(0, y, y))$.
The first endpoint $\mathcal{T}(B(0, 0, y))$ lies on the line segment $\overline{ON}$ and the second endpoint $\mathcal{T}(B(0, y, y))$ lies on the line segment $\overline{OQ}$. Hence if we vary y continuously between 0 and 1, we can fill out the $1/6$ region of $\widetilde{W}$.

Then by exploiting the symmetry, we get the desired result. $\square$

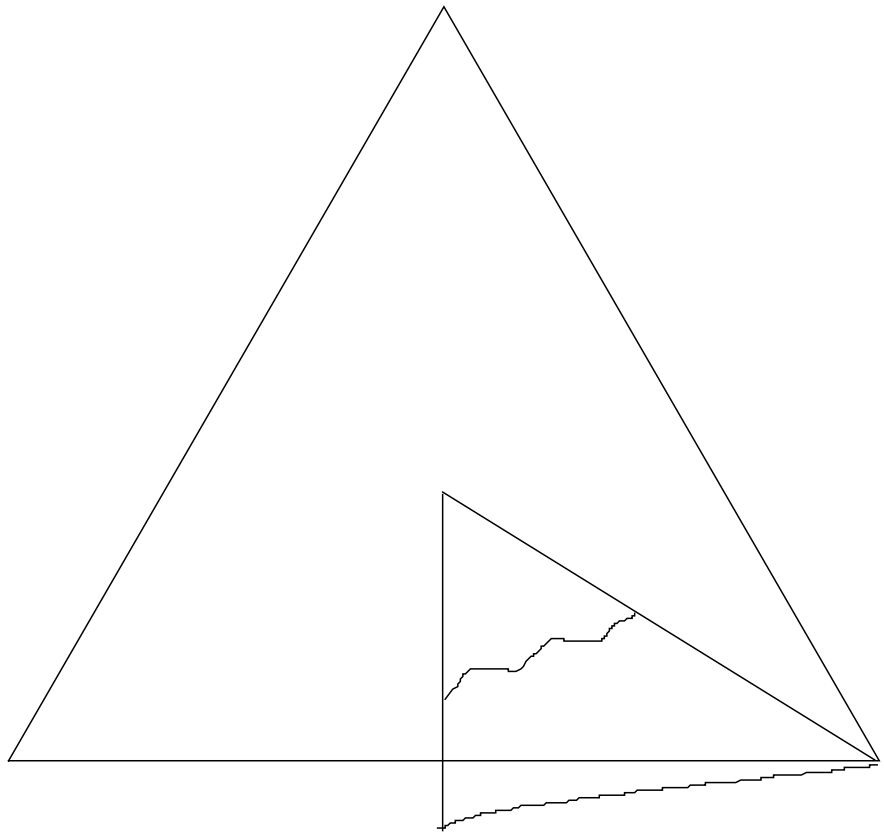

Figure 4.

Let $\mathcal{B}$ be a subfamily of $\mathcal{B}^*$ defined by

$$\begin{aligned}\mathcal{B} = \quad & \{B(0, c_2, c_3) : 0 \le c_2, c_3 \le 1\} \\ & \cup\{B(c_1, 0, c_3) : 0 \le c_1, c_3 \le 1\} \\ & \cup\{B(c_1, c_2, 0) : 0 \le c_1, c_2 \le 1\},\end{aligned}$$

i.e., $\mathcal{B}$ consisting of those elements B in $\mathcal{B}^*$ which have the decomposition form $B = C_w \cup E_1 \cup E_2$ or $B = C_w \cup E_1 \cup E_3$, or $B = C_w \cup E_2 \cup E_3$.

Remark of Theorem 1. Actually, we have proved that $\widetilde{W} = \mathcal{T}(\mathcal{B})$. Moreover, if we use $\mathcal{B}$ in stead of $\mathcal{B}^*$. Then the map

$$\mathcal{T} : \mathcal{B} \to \widetilde{W}$$

is one-to-one.

Hence for each $x \in C_w$, there exists a unique set $B \in \mathcal{B}$, $C_w \subset B \subset D_w$, such that $M_B(h) = h(x)$ for any harmonic function h. We call the set B associated to this C_w a *k level mean value neighborhood of* x where k is the length of w.

Given a point $x \in SG \setminus V_0$, let k_0 be the smallest value of k such that there exists a k level cell C_w containing x but not intersecting V_0. k_0 depends on the location of x in SG.

Then we can find a sequence of words $w^{(k)}$ of length k $(k \geq k_0)$ and a sequence of mean value neighborhoods $B_k(x)$ associated to $C_{w^{(k)}}$.

$\{B_k(x)\}_{k \geq k_0}$ forms a system of neighborhoods of the point x satisfying $\bigcap_{k \geq k_0} B_k(x) = \{x\}$.

3. Mean value properties of functions in $dom\Delta$

Given a point $x \in SG \setminus V_0$, we want to define c_B such that

$$M_B(u) - u(x) \approx c_B \Delta u(x)$$

for $u \in dom\Delta$.

Let v be a function satisfying $\Delta v \equiv 1$. Define c_B by

$$c_B = M_B(v) - v(x).$$

Note that the value of c_B is independent of which v, because any two differ by a harmonic function h and $M_B(h) - h(x) = 0$.

So we can choose

$$v = -\int_{SG} G(\cdot, y) d\mu(y).$$

(v vanishes on the boundary of SG, G is the Green's function).

c_B depends only on the relative geometry of B and C_w and the size of C_w, not on the location of x or C_w in SG. Moreover,

Theorem 2. *There exist two positive constants c_0 and c_1, such that*

$$c_0 \frac{1}{5^k} \leq c_B \leq c_1 \frac{1}{5^k}$$

for any k level mean value neighborhood B.

Given a point x and C_w a k level neighborhood of x, for any $u \in dom\Delta^2$, we can write

$$u = h^{(k)} + (\Delta u(x))v + R^{(k)}$$

on C_w, where $h^{(k)}$ is a harmonic function defined by

$$h^{(k)} + (\Delta u(x))v|_{\partial C_w} = u|_{\partial C_w}.$$

It is not hard to prove that

Fact 3. *The remainder satisfies*

$$R^{(k)} = O((\frac{3}{5} \cdot \frac{1}{5})^k)$$

on C_w*(also on* $B_k(x)$*).*

(This looks like the Taylor expansion of u at x.)

Proof of Fact 3.

It is easy to check that $\Delta_y R^{(k)}(y) = \Delta_y u(y) - \Delta_y u(x)$ and $R^{(k)}(y)$ vanishes on the boundary of C_w. Hence $R^{(k)}$ is given by the integral of $\Delta_y u(y) - \Delta_y u(x)$ on C_w against a scaled Green's function.

Since the scaling factor is $(\frac{1}{5})^k$, and

$$|\Delta_y u(y) - \Delta_y u(x)| \leq c(3/5)^k$$

(Δu satisfies the Holder condition with $\gamma = \frac{3}{5}$), we get

the desired result. $\square$

A more generalized version of Fact 3 is

Fact 3' *Let* $u \in dom\Delta$ *with* $g = \Delta u$ *satisfying the following Holder condition*

$$|g(y) - g(x)| \leq c\gamma^k, \qquad (0 < \gamma < 1)$$

for all $y \in C_w$*, then the remainder satisfies*

$$R^{(k)} = O((\gamma \cdot \frac{1}{5})^k)$$

on C_w*(also on* $B_k(x)$*).*

Using the Taylor expansion of u at x and Theorem 2, we have

$$\begin{aligned} & \frac{1}{c_{B_k(x)}}(M_{B_k(x)}(u) - u(x)) - \Delta u(x) \\ = & \frac{1}{c_{B_k(x)}}(M_{B_k(x)}(R^{(k)}) - R^{(k)}(x)) \\ = & \frac{1}{c_{B_k(x)}} O((\gamma \cdot \frac{1}{5})^k) = O(\gamma^k). \end{aligned}$$

Hence we get

$$\lim_{k\to\infty} \frac{1}{c_{B_k(x)}}(M_{B_k(x)}(u) - u(x)) = \Delta u(x),$$

which is a fractal analogous formula of (1).

4. Generalization on a class of Sierpinski type fractals

Let K be a $p.c.f.$ self-similar fractal. We assume that $\sharp V_0 = 3$ and all structures possess full $D3$ symmetry.

A) We can choose all $c_{i,j} = 1$ in defining the graph energy on Γ_0, and all the resistance renormalization factors r_i are the same number r.

B) We must have $\mu_1 = \mu_2 = \mu_3$, i.e., μ is the standard self-similar measure on K.

SG_3, SG_n, Hexagasket, 3-dimensional SG...

For the first question on how to find the mean value neighborhoods, it is done.

However, for the second question on how to estimate the c_B constants, it is still on working.

Thank you!