Energy on fractals and related questions: about the use of differential 1-forms on the Sierpinski Gasket and other fractals

Part 3

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Early (physics) results on spectral analysis on fractals

- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984)
- Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983)**17** (1984)

Early results on diffusions on fractals

Sheldon Goldstein, *Random walks and diffusions on fractals.* Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., 8, Springer

Summary: we investigate the asymptotic motion of a random walker, which at time $\mathbf{n}$ is at $\mathbf{X(n)}$, on certain 'fractal lattices'. For the 'Sierpiński lattice' in dimension $\mathbf{d}$ we show that, as $\mathbf{L} \to \infty$, the process $\mathbf{Y_L(t) \equiv X([(d+3)^L t])/2^L}$ converges in distribution to a diffusion on the Sierpin'ski gasket, a Cantor set of Lebesgue measure zero. The analysis is based on a simple 'renormalization group' type argument, involving self-similarity and 'decimation invariance'. In particular,

$$|\mathbf{X(n)}| \sim \mathbf{n}^{\gamma},$$

where $\gamma = (\ln 2)/\ln(d+3)) \leqslant 2$.

Shigeo Kusuoka, *A diffusion process on a fractal.* Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 1987.

- M.T. Barlow, E.A. Perkins, *Brownian motion on the Sierpinski gasket.* (1988)
- M. T. Barlow, R. F. Bass, *The construction of Brownian motion on the Sierpiński carpet.* Ann. Inst. Poincaré Probab. Statist. (1989)
- S. Kusuoka, *Dirichlet forms on fractals and products of random matrices.* (1989)
- T. Lindstrøm, *Brownian motion on nested fractals.* Mem. Amer. Math. Soc. **420**, 1989.
- J. Kigami, *A harmonic calculus on the Sierpiński spaces.* (1989)
- J. Béllissard, *Renormalization group analysis and quasicrystals,* Ideas and methods in quantum and statistical physics (Oslo, 1988) Cambridge Univ. Press, 1992.
- M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket.* (1992)
- J. Kigami, *Harmonic calculus on p.c.f. self–similar sets.* Trans. Amer. Math. Soc. **335** (1993)
- J. Kigami and M. L. Lapidus, *Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals.* Comm. Math. Phys. **158** (1993)

Main classes of fractals considered

- $[0, 1]$
- Sierpiński gasket
- nested fractals
- p.c.f. self-similar sets, possibly with various symmetries
- finitely ramified self-similar sets, possibly with various symmetries
- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)
- metric measure Dirichlet spaces, possibly with heat kernel estimates (MMD+HKE)

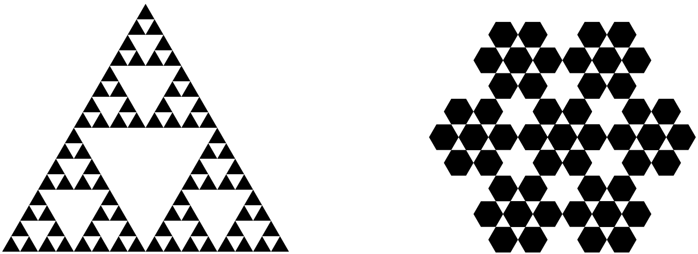


Figure: Sierpiński gasket and Lindstrøm snowflake (nested fractals), p.c.f., finitely ramified)

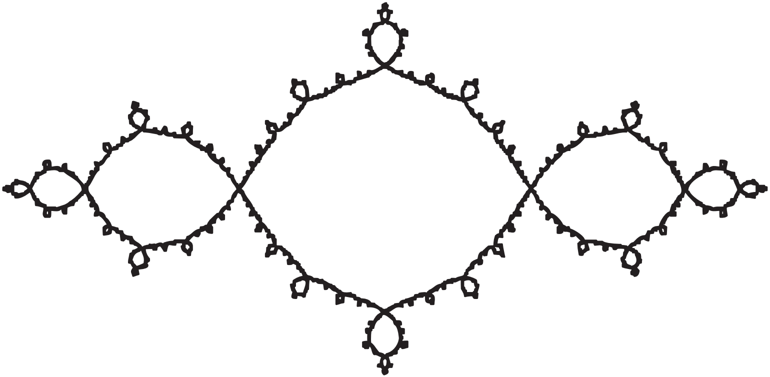


Figure: The basilica Julia set, the Julia set of $z^2 - 1$ and the limit set of the basilica group of exponential growth (Grigorchuk, Żuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda et al., Rogers-T.).

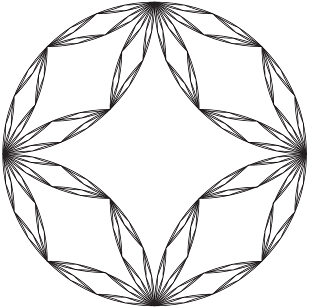


Figure: Diamond fractals, non-p.c.f., but finitely ramified

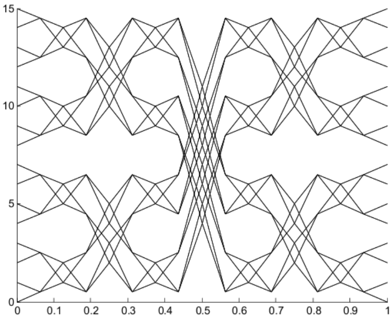


Figure: Laakso Spaces (Ben Steinhurst), infinitely ramified

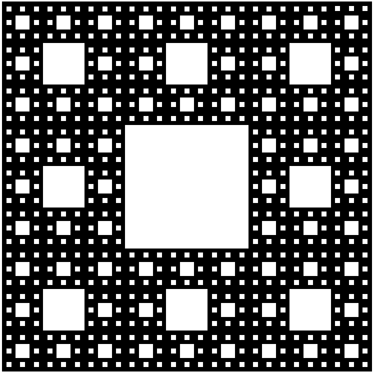


Figure: Sierpiński carpet, infinitely ramified

Existence, uniqueness, heat kernel estimates

Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y|^2}{4t}\right)$$

$$\textbf{distance} \sim \sqrt{\textbf{time}}$$

"Einstein space–time relation for Brownian motion"

De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;

Li-Yau (1986) type estimates on a geodesically complete Riemannian manifold with **Ricci** $\geqslant \mathbf{0}$:

$$\mathbf{p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)}$$

$$\textbf{distance} \sim \sqrt{\textbf{time}}$$

Brownian motion on $\mathbb{R}^{\mathbf{d}}$: $\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| = \mathbf{ct^{1/2}}$.

Anomalous diffusion: $\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| = \mathbf{o(t^{1/2})}$, or (in regular enough situations),

$$\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| \approx \mathbf{t^{1/d_w}}$$

with $\mathbf{d_w} > \mathbf{2}$.

Here $\mathbf{d_w}$ is the so-called **walk dimension** (should be called **"walk index"** perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$\text{distance} \sim (\text{time})^{\frac{1}{d_w}}$$

$$d_H = \text{Hausdorff dimension}$$

$$\frac{1}{\gamma} = d_w = \text{"walk dimension" } (\gamma\text{=diffusion index})$$

$$\frac{2d_H}{d_w} = d_S = \text{"spectral dimension" (diffusion dimension)}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

Theorem (Barlow, Bass, Kumagai (2006)).

Under natural assumptions on the MMD (geodesic Metric Measure space with a regular symmetric conservative Dirichlet form), the sub-Gaussian **heat kernel estimates are stable under rough isometries**, *i.e. under maps that preserve distance and energy up to scalar factors.*

Gromov-Hausdorff + energy

Theorem. (Barlow, Bass, Kumagai, T. (1989–2010).) On any fractal in the class of generalized Sierpiński carpets (includes cubes in $\mathbb{R}^{\mathbf{d}}$) there exists a unique, up to a scalar multiple, local regular Dirichlet form that is invariant under the local isometries.

Therefore there there is a unique corresponding symmetric Markov process and a unique Laplacian. Moreover, the Markov process is Feller and its transition density satisfies sub-Gaussian heat kernel estimates.

Remark: intrinsic uniqueness is proved by Steinhurst for the Laakso spaces (to appear in Potential Analysis) and for non-self-similar Sierpinski carpets (work in progress)

Main difficulties for the Sierpinski carpet:

If it is not a cube in $\mathbb{R}^n$, then

- $\mathbf{d_S < d_H}$, $\mathbf{d_w > 2}$
- the energy measure and the Hausdorff measure are mutually singular;
- the domain of the Laplacian is not an algebra;
- if $\mathbf{d(x, y)}$ is the (Euclidean-induced) shortest path metric, then $\mathbf{d(x, \cdot)}$ is not in the domain of the Dirichlet form (not of finite energy) and so methods of Differential geometry seem to be not applicable;**
- Lipschitz functions are not of finite energy;**
- in fact, we can not compute any non-constant functions of finite energy;
- Fourier and complex analysis methods seem to be not applicable.

** see recent papers by Koskela and Zhou and by Hinz, Kelleher, T

Theorem. (Grigor'yan and Telcs, also [BBK])

On a MMD space the following are equivalent

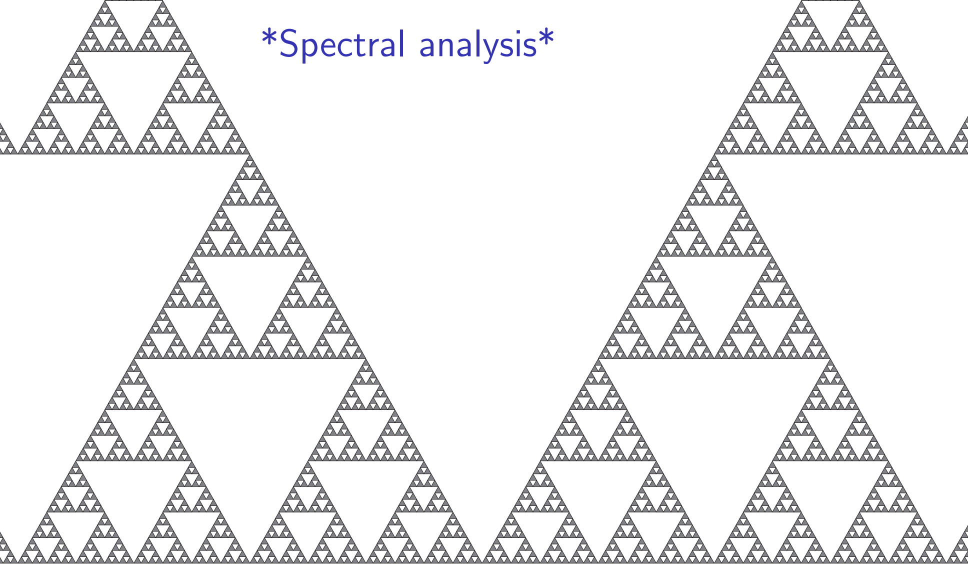
- **(VD)**, **(EHI)** and **(RES)**
- **(VD)**, **(EHI)** and **(ETE)**
- **(PHI)**
- **(HKE)**

and the constants in each implication are effective.

Abbreviations: Metric Measure Dirichlet spaces, Volume Doubling, Elliptic Harnack Inequality, Exit Time Estimates, Parabolic Harnack Inequality, Heat Kernel Estimates.

Remark: recent improvements in Grigor'yan and Hu, Heat kernels and Green functions on metric measure spaces, to appear in Canad. J. Math.

Spectral analysis



A part of an infinite Sierpiński gasket.

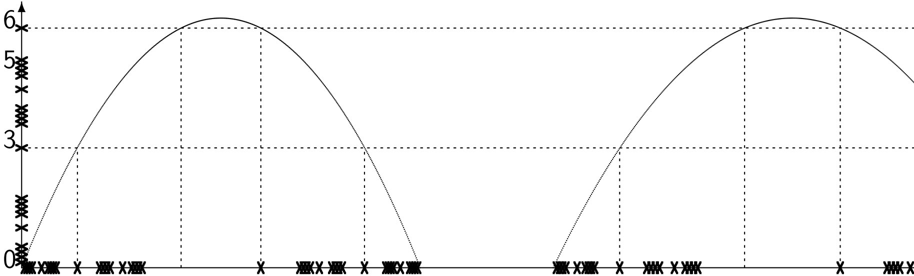
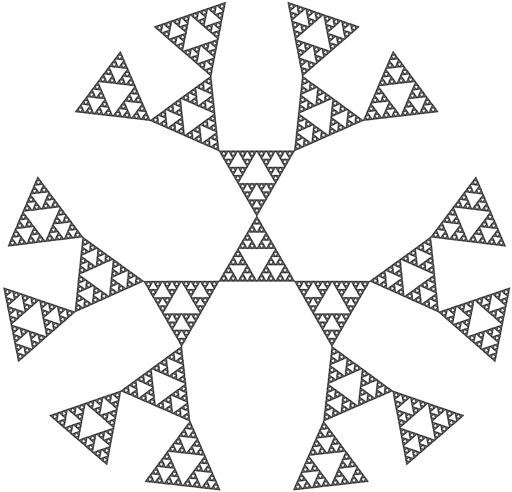
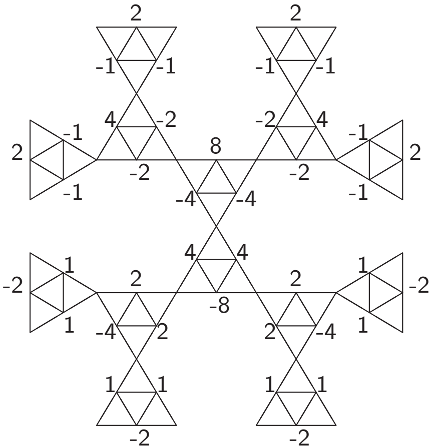


Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

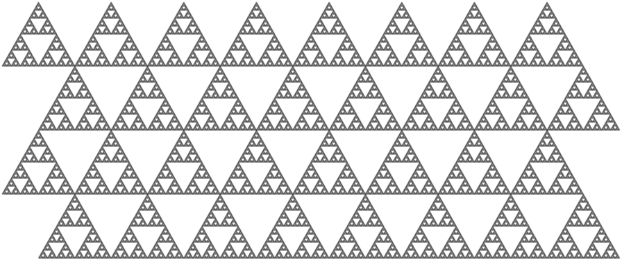
Theorem. (Béllissard 1988, T. 1998, Quint 2009)
On the infinite Sierpiński gasket the spectrum of the Laplacian consists of a **dense set of eigenvalues** $\mathfrak{R}^{-1}(\Sigma_0)$ **of infinite multiplicity** and a **singularly continuous component of spectral multiplicity one supported on** $\mathfrak{R}^{-1}(\mathcal{J}_R)$.

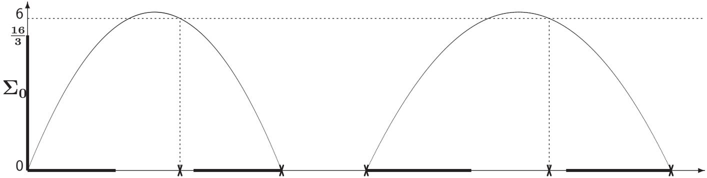


The Tree Fractafold.



An eigenfunction on the Tree Fractafold.





Theorem. (Strichartz, T. 2010) The Laplacian on the periodic triangular lattice finitely ramified Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum. The **absolutely continuous spectrum is $\mathfrak{R}^{-1}[0, \frac{16}{3}]$.** The **pure point spectrum** consists of two infinite series of eigenvalues of infinite multiplicity. The spectral resolution is given in the main theorem.

Recent results on spectral analysis and applications

G. Derfel, P. J. Grabner, and F. Vogl: Laplace Operators on Fractals and Related Functional Equations, J. Phys. A, 45(46), (2012), 463001

N. Kajino, *Spectral asymptotics for Laplacians on self-similar sets*. J. Funct. Anal. 258 (2010)

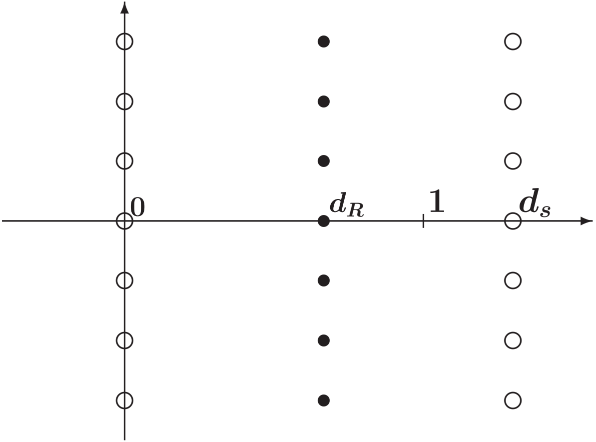
N. Kajino, T, *Spectral gap sequence and on-diagonal oscillation of heat kernels*, work in progress

Joe Chen, R. Strichartz, *Spectral asymptotics and heat kernels on three-dimensional fractal sponges*

J. F.-C. Chan, S.-M. Ngai, T, *One-dimensional wave equations defined by fractal Laplacians*

U. Freiberg, L. Rogers, T, *Eigenvalue convergence of second order operators on the real line*

B. Steinhurst, T, *Spectral Analysis and Dirichlet Forms on Barlow-Evans Fractals* arXiv:1204.5207



$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

Poles (white circles) of the spectral zeta function of the Sierpiński gasket.

See work of Grabner et al on relation to complex analysis and of Steinhurst et al on applications to Casimir energy.

Remark: what are dimensions of the Sierpiński gasket?

- $\frac{\log 3}{\log \frac{5}{3}} \approx \mathbf{2.15} =$ Hausdorff dimension in effective resistance metric
- $\mathbf{2} =$ geometric, linear dimension
- $\frac{\log 3}{\log 2} \approx \mathbf{1.58} =$ usual Hausdorff (Minkowsky, box, self-similarity) dimension in Euclidean coordinates (geodesic metric)
- $\frac{2 \log 3}{\log 5} \approx \mathbf{1.37} =$ usual spectral dimension
- **... ... ...** $=$
  there are several Lyapunov exponent type dimensions related to harmonic functions and harmonic coordinates (Kajino, Ionescu-Rogers-T)
- $\mathbf{1} =$ topological dimension, martingale dimension
- $\frac{2 \log 2}{\log 5} \approx \mathbf{0.86} =$ polynomial spectral co-dimension (Grabner)?

*conclusion

perhaps
analysis and probability on fractals
analysis on metric measure spaces
sub-Riemannian geometry
converge

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Thank you for your attention :-)